HILBERT SERIES AND APPLICATIONS TO GRADED RINGS

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Received 23 July 2001 and in revised form 14 March 2002

This paper contains a number of practical remarks on Hilbert series that we expect to be useful in various contexts. We use the fractional Riemann-Roch formula of Fletcher and Reid to write out explicit formulas for the Hilbert series $P(t)$ in a number of cases of interest for singular surfaces (see [Lemma 2.1](#)) and 3-folds. If X is a $\mathbb{Q}$ -Fano 3-fold and $S \in |-K_X|$ a $K3$ surface in its anticanonical system (or the general elephant of X), polarised with $D = \mathcal{O}_S(-K_X)$, we determine the relation between $P_X(t)$ and $P_{S,D}(t)$. We discuss the denominator $\prod(1-t^{a_i})$ of $P(t)$ and, in particular, the question of how to choose a reasonably small denominator. This idea has applications to finding $K3$ surfaces and Fano 3-folds whose corresponding graded rings have small codimension. Most of the information about the anticanonical ring of a Fano 3-fold or $K3$ surface is contained in its Hilbert series. We believe that, by using information on Hilbert series, the classification of $\mathbb{Q}$ -Fano 3-folds is too close. Finding $K3$ surfaces are important because they occur as the general elephant of a $\mathbb{Q}$ -Fano 3-fold.

2000 Mathematics Subject Classification: 14Q10, 14Q15, 32S25, 13A02.

1. Introduction. We work with graded rings $R = \bigoplus_{n \geq 0} R_n$ that are finitely generated over an algebraically closed field k of characteristic 0 and satisfy $R_0 = k$. The *Hilbert function* of R is the numerical function $P_n = \dim R_n$ for $n \geq 0$; the *Hilbert series* $P(t)$ or $P_R(t)$ of R is the formal power series defined by $P(t) = \sum P_n t^n$. It is elementary and well known that $P(t)$ is a rational function of t . In fact, if $x_1, \dots, x_d$ are homogeneous elements of weight $\text{wt } x_i = a_i$ generating R (or more generally, generating a subring over which R is finite), then $\prod(1-t^{a_i})P(t) = Q(t)$ is a polynomial.

2. Fractional Riemann-Roch formula

2.1. Surfaces with Du Val singularities. We use the definitions and notation of Reid [\[9\]](#) for singularities. If S is a projective surface with Du Val singularities and D a Weil divisor on S , then some multiple rD is Cartier, and there is a formula [\[9, Theorem 9.1\]](#)

$$\chi(S, \mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - DK_S) + \sum_P c_P(D), \quad (2.1)$$

where $c_P(D) \in \mathbb{Q}$ is a fractional contribution due to the singularity of S and

$\mathcal{O}_S(D)$ at P . Here, $DK_S \in \mathbb{Z}$ is the intersection number with the canonical class, and $D^2 \in \mathbb{Q}$ the self-intersection of the $\mathbb{Q}$ -Cartier divisor D . Moreover, $c_P(D)$ can be written as a sum

$$c_P(D) = - \sum_{(r,a) \in \mathcal{B}} \frac{a(r-a)}{2r} \quad (2.2)$$

over a *basket* $\mathcal{B} = \{(r,a)\}$ with $0 < a < r$ and each a is coprime to r . Here, the basket appears simply as a list of combinatorial data for computing the right-hand side of (2.2); Reid [9, Section 9] interprets (2.1) and (2.2) as the singularity of S , and $\mathcal{O}_S(D)$ at P has the same effect on $\chi(S, \mathcal{O}_S(D))$ as a basket $\mathcal{B}$ of virtual cyclic quotient singularities of type $(1/r)(1, -1)$, at which $\mathcal{O}_S(D)$ is locally isomorphic to the eigensheaf of ε^a (see [9, Chapter III] for definition).

It follows from this interpretation and the proof of [9, Theorem 9.1] that for $n \in \mathbb{Z}$,

$$c_P(nD) = - \sum_{(r,a) \in \mathcal{B}} \frac{\overline{na}(r - \overline{na})}{2r}, \quad (2.3)$$

where, for each $(r,a) \in \mathcal{B}$, the bar stands for the smallest positive residue modulo r .

LEMMA 2.1. *Let S be a surface with Du Val singularities and D a Weil divisor. We assume that $H^i(S, \mathcal{O}_S(nD)) = 0$ for all $i > 0$ and for all $n \geq 1$. Then, the graded ring $R(S, D) = \bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nD))$ has Hilbert series*

$$\begin{aligned} P_S(t) &= \frac{1 + (\chi - 1)t}{1 - t} + \frac{t + t^2}{2(1 - t)^3} D^2 - \frac{t}{2(1 - t)^2} DK_S \\ &\quad - \sum_{(r,a) \in \mathcal{B}} \frac{\sum_{n=1}^{r-1} \overline{an}(r - \overline{an})t^n}{2r(1 - t^r)}. \end{aligned} \quad (2.4)$$

Here, $\chi = \chi(\mathcal{O}_S)$.

PROOF. The first three terms of (2.4) expand out as

$$\begin{aligned} 1 + \chi t + \chi t^2 + \dots, \quad & \frac{(t + 2t^2 + 3t^3 + \dots)D^2}{2}, \\ & - \frac{(t + 2t^2 + 3t^3 + \dots)DK_S}{2}, \end{aligned} \quad (2.5)$$

respectively, corresponding to the sum over nD of first three terms of (2.1). For each element of the basket, the denominator $1 - t^r$ has the effect of repeating the contribution $-\overline{na}(r - \overline{na})/2r$ from (2.3) periodically over the intervals $[0, r], [r, 2r], \dots$ (zero at the endpoints), giving the last term of (2.1). $\square$

2.2. 3-folds with canonical singularities. Let X be a projective 3-fold with canonical singularities and A a Weil divisor. To use the formulas of [9, Section 10], assume that, at every singular point $P \in X$, we have $\mathcal{O}_X(A) \cong \mathcal{O}_X(lK_X)$ for

some l (possibly depending on P). Then, [9, Theorem 10.2] states that

$$\chi(X, A) = \chi(\mathcal{O}_X) + \frac{1}{6}A^3 - \frac{1}{4}A^2K_X + \frac{1}{12}A(K_X^2 + c_2) + \sum c_Q(A), \quad (2.6)$$

where $c_Q(A)$ is a sum $\sum_{\mathcal{B}} c(r, a, l)$ taken over a basket $\mathcal{B} = \{(r, a, l)\}$, with a and l coprime to r , and the contributions are

$$c(r, a, l) = -\frac{r^2-1}{12r}l + \sum_{i=1}^{l-1} \frac{\overline{ai}(r-\overline{ai})}{2r} \quad (2.7)$$

$$= \frac{r^2-1}{12r}(r-l) - \sum_{i=l}^{r-1} \frac{\overline{ai}(r-\overline{ai})}{2r}. \quad (2.8)$$

The interpretation here is that $c(r, a, l)$ is the contribution from a singularity of type $(1/r)(a, 1, -1)$ at which A is locally the ε^{al} eigensheaf.

We can write out the other terms in the Hilbert series by analogy with Lemma 2.1 (see the following section). The contribution made by each element $(r, a, l) \in \mathcal{B}$ to the Hilbert series is thus

$$\sum_{n=1}^{\infty} c(r, a, nl)t^n = \frac{1}{1-t^r} \sum_{n=1}^{r-1} c(r, a, nl)t^n. \quad (2.9)$$

REMARK 2.2. The two formulas (2.7) and (2.8) are equal because

$$\sum_{i=1}^{r-1} \overline{ai}(r-\overline{ai}) = \sum_{i=1}^{r-1} i(r-i) = \frac{1}{6}r(r^2-1); \quad (2.10)$$

moreover, they also hold for l not in $[0, r]$ (the expression only depends on l modulo r).

In dealing with a single divisor A , we may assume that l is coprime to r ; but the proof of [9, Section 10] is valid for any l . Thus, we can use formulas (2.7) and (2.8) for the contribution $c(r, a, nl)$ for any n in calculating $\chi(nA)$.

3. Fano 3-folds and $K3$ surfaces

3.1. Fano 3-folds. Let X be a Fano 3-fold, that is, a 3-fold with canonical singularities and ample anticanonical class $A = -K_X$. Standard use of vanishing gives

$$P_n = h^0(X, nA) = \chi(\mathcal{O}(nA)), \quad \text{for } n \geq 0. \quad (3.1)$$

Now, $\chi(\mathcal{O}_X) = 1$ and the basket of X is $\mathcal{B} = \{(r, a, -1)\}$, and by [9, Corollary 10.3], we have

$$\frac{1}{12}Ac_2 = -\frac{1}{12}K_Xc_2 = 2 - \sum_{\mathcal{B}} \frac{r^2-1}{12r}. \quad (3.2)$$

Using all this, formulas (2.6) and (2.7) specialise to give

$$P_n = 2n + 1 + \frac{1}{12}n(n+1)(2n+1)A^3 + \sum_{\mathfrak{B}} \left(-\frac{r^2-1}{12r}n + c(r, a, -n) \right). \quad (3.3)$$

Now, consider $P_n - P_{n-1}$. To handle the second term, we use

$$n(n+1)(2n+1) - (n-1)(n)(2n-1) = 6n^2. \quad (3.4)$$

By (2.8), the bracketed expression inside the sum over $\mathfrak{B}$ equals

$$-\sum_{i=-n}^{r-1} \frac{\overline{ai}(r-\overline{ai})}{2r}, \quad (3.5)$$

and the difference from n to $n-1$ is just one term of the sum. We obtain

$$P_n - P_{n-1} = 2 + \frac{1}{2}n^2A^3 - \sum_{(r,a,-1) \in \mathfrak{B}} \frac{\overline{an}(r-\overline{an})}{2r}. \quad (3.6)$$

Since $P_n - P_{n-1}$ is the coefficient of t^n in $(1-t)P_X(t)$, arguing as in Lemma 2.1 gives the following result.

COROLLARY 3.1. *The Hilbert series of a Fano 3-fold X is*

$$P_X(t) = \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{2(1-t)^4}A^3 - \sum_{(r,a,-1) \in \mathfrak{B}} \frac{\sum_{n=1}^{r-1} \overline{an}(r-\overline{an})t^n}{2r(1-t)(1-tr)}. \quad (3.7)$$

3.2. $K3$ surfaces. For a $K3$ surface S with Du Val singularities and a Weil divisor D , Lemma 2.1 specialises to give the Hilbert series of $P_n = h^0(S, nD)$ in the form

$$P_S(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{2(1-t)^3}D^2 - \sum_{(r,a) \in \mathfrak{B}} \frac{\sum_{n=1}^{r-1} \overline{an}(r-\overline{an})t^n}{2r(1-tr)}. \quad (3.8)$$

COROLLARY 3.2. *If X is a Fano 3-fold polarised by $A = -K_X$ and $S \in |-K_X|$ a $K3$ surface polarised by $D = A|_S$, then*

$$P_S(t) = (1-t)P_X(t). \quad (3.9)$$

This result follows, of course, from the restriction exact sequence

$$0 \longrightarrow \mathbb{O}_X((n-1)A) \longrightarrow \mathbb{O}_X(nA) \longrightarrow \mathbb{O}_S(nD) \longrightarrow 0. \quad (3.10)$$

The point, however, is that the corollary gives a formula for the Hilbert series $P_X(t)$ of the Fano 3-fold in terms of simpler data for a $K3$ surface. This formula is valid even if there is no $K3$ surface $S \in |-K_X|$, for example, if $P_1(X) = 0$ so that $|-K_X| = \emptyset$. Compare Corti, Pukhlikov, and Reid [5, Remark 7.2.3].

4. Applications. In [2], we studied polarised $K3$ surfaces S in terms of their graded rings $R(S, D)$. We also studied Fano 3-folds X in a similar way. The first step in our strategy consisted of finding suitable weights a_0, a_1 , and a_2 so that

$$(1 - t^{a_0})(1 - t^{a_1})(1 - t^{a_2})P(t) \quad (4.1)$$

is a polynomial with positive coefficients and a_0, a_1 , and a_2 are “fairly small.” This is a combinatorial analogue of finding a polynomial subring $k[x_0, x_1, x_2] \subset R(S, D)$ over which $R(S, D)$ is a finite-free module of “fairly small rank.” If $\mathcal{B} = \{(r, a)\}$ is the basket of S , then each $(1 - t^r)$ appears in the denominator so that a first necessary condition for this is that each r divides some a_i .

We have the obvious bound $\sum n \leq 19$ for the number and types of Du Val singularities A_n, D_n , and E_n on a $K3$ surface S . This implies the bound $\sum (r - 1) \leq 19$ for the basket $\mathcal{B} = \{(r, a)\}$ on a $K3$ surface (see [6, Theorem III.9.20] and [7, Theorem II.8.21]). For a Fano 3-fold X , a similar bound on the basket $\{(r, a, -1)\}$ is provided by an argument of Kawamata [8] who first proves that $Ac_2 > 0$; then (3.2) implies

$$\sum \left(r - \frac{1}{r} \right) < 24. \quad (4.2)$$

Thus, there are only finitely many possibilities for the basket $\mathcal{B}$. An easy calculation shows that in either case, at most, 5 distinct values of r occur.

In [2], we develop a procedure based on these ideas to find all possible Hilbert series for $K3$ surfaces and Fano 3-folds with graded ring of given codimension. We give explicit lists [1] of codimension 3 and 4 cases (comparable to the lists of hypersurfaces and codimension 2 in Fletcher [7]) and, in most cases, settle the question of the existence of the varieties.

The *numerical data* of a $K3$ surface S with D is $P_1 = h^1(S, D)$, where $0 \leq P_1 \leq 3 + \text{codim} S$ and the basket $\mathcal{B} = \{(r, a)\}$. We can rewrite D^2 from the formula of P_1 in terms of P_1 and the basket $\mathcal{B}$,

$$D^2 = 2(P_1 - 2) + \sum \frac{a(r - a)}{r}. \quad (4.3)$$

We produced the lists of codimension 3 and 4 [1] by searching all possible $K3$ surfaces of given numerical data. We give an example below to show how this search is carried out. More details of applications are given in the Singapore paper [3].

EXAMPLE 4.1. A typical example is

$$D^2 = -2 + \frac{1}{2} + \frac{1}{2} + \frac{3}{4} + \frac{4}{5}. \quad (4.4)$$

That is, the numerical data are the basket $\mathcal{B} = \{(2, 1), (2, 1), (4, 1), (5, 1)\}$ and $P_1 = 1$ (see formula (4.3)). This is #39 in the codimension 3 list [1]. Since $P_1 = 1$,

there is a generator of degree 1, say x . From formula (3.8),

$$P_S(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{2(1-t)^3} \frac{11}{20} - 2 \times \frac{t}{2 \cdot 2(1-t^2)} - \frac{3t+4t^2+3t^3}{2 \cdot 4(1-t^4)} - \frac{4t+6t^2+6t^3+4t^4}{2 \cdot 5(1-t^5)}. \quad (4.5)$$

To kill its denominators, we must have at least two more generators, say t_1 and v , whose degrees are divisible by 4 and 5, respectively. Therefore, the smallest choice of (a_1, a_2, a_3) is $(1, 4, 5)$. After simplifying, we obtain

$$(1-t)(1-t^4)(1-t^5)P_S(t) = t^{10} + t^8 + t^7 + 2t^6 + t^5 + 2t^4 + t^3 + t^2 + 1. \quad (4.6)$$

This looks like the Hilbert series of an Artinian ring with further generators y, z, t_2 of degrees 2, 3, 4 so that the possible candidates S is in the weighted projective space $\mathbb{P}(1, 2, 3, 4, 4, 5)$ (see [2, 6, 7] for definition). To find the structure of its resolution, multiply (3.6) again by $(1-t^2)(1-t^3)(1-t^4)$,

$$\begin{aligned} & (1-t)(1-t^2)(1-t^3)(1-t^4)^2(1-t^5)P_S(t) \\ &= 1 - t^6 - t^7 - 2t^8 - t^9 + t^{10} + 2t^{11} + t^{12} + t^{13} - t^{19}. \end{aligned} \quad (4.7)$$

From here, we read off the shape of the resolution of $R(S, D)$ over $A = k[x, y, z, t_1, t_2, v]$, namely,

$$\begin{aligned} 0 \longrightarrow A(-19) &\xrightarrow{P^t} A(-13) \oplus A(-12) \oplus A(-11) \oplus A(-11) \oplus A(-10) \\ &\xrightarrow{M} A(-9) \oplus A(-8) \oplus A(-8) \oplus A(-7) \oplus A(-6) \xrightarrow{P} A \longrightarrow R \longrightarrow 0, \end{aligned} \quad (4.8)$$

where P is a 5×1 vector and M is a 5×5 skew-symmetric matrix. In other words, we expect 5 relations in degrees 6, 7, 8, 8, and 9, and 5 syzygies in degrees 10, 11, 11, 12, and 13. Note that $n = 1 + 2 + 3 + 4 + 4 + 5$ corresponds to the canonical class of $\mathbb{P}(1, 2, 3, 4, 4, 5)$ and hence the canonical divisor K_S of S is $\mathbb{O}(19 - n)$, which is trivial. The shape of the polynomial, together with the Buchsbaum-Eisenbud theorem on Gorenstein rings in codimension 3 (see [4]), gives us the equations of the relations as the Paffian of a 5×5 skew-symmetric matrix M with degrees

$$\begin{pmatrix} 0 & 2 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 & 5 \\ 2 & 3 & 0 & 4 & 5 \\ 3 & 4 & 4 & 0 & 6 \\ 4 & 5 & 5 & 6 & 0 \end{pmatrix}. \quad (4.9)$$

ACKNOWLEDGMENTS. This work derives mainly from the Ph.D. thesis [2], University of Warwick. I would like to thank both Miles Reid and Alessio Corti for their help and encouragement at different times.

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