A CONVOLUTION PRODUCT OF $(2j)$ th DERIVATIVE OF DIRAC'S DELTA IN r AND MULTIPLICATIVE DISTRIBUTIONAL PRODUCT BETWEEN r^{-k} AND $\nabla(\triangle^j \delta)$

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The purpose of this paper is to obtain a relation between the distribution $\delta^{(2j)}(r)$ and the operator $\triangle^j \delta$ and to give a sense to the convolution distributional product $\delta^{(2j)}(r) * \delta^{(2s)}(r)$ and the multiplicative distributional products $r^{-k} \cdot \nabla(\triangle^j \delta)$ and $(r-c)^{-k} \cdot \nabla(\triangle^j \delta)$.

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1. Introduction. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space $\mathbb{R}^n$.

We call $\varphi(x)$ the C^∞ -functions with compact support defined from $\mathbb{R}^n$ to $\mathbb{R}$. Let

$$r^2 = x_1^2 + x_2^2 + \dots + x_n^2 \quad (1.1)$$

and consider the functional r^λ defined by

$$(r^\lambda, \varphi) = \int_{\mathbb{R}^n} r^\lambda \varphi(x) dx \quad (1.2)$$

(see [5, page 71]), where λ is a complex number and $dx = dx_1 dx_2 \dots dx_n$.

For $\text{Re}(\lambda) > -n$, this integral converges and is an analytic function of λ . Analytic continuation to $\text{Re}(\lambda) \leq -n$ can be used to extend the definition of (r^λ, φ) .

Calling Ω_n to the hypersurface area of the unit sphere imbedded in the n -Euclidean space, we find in [5, page 71] that

$$(r^\lambda, \varphi) = \Omega_n \int_0^\infty r^{\lambda+n-1} S_\varphi(r) dr, \quad (1.3)$$

where

$$S_\varphi(r) = \frac{1}{\Omega_n} \int_\Omega \varphi dw \quad (1.4)$$

and dw is the hypersurface element of the unit sphere.

$S_\varphi(r)$ is the mean value of $\varphi(x)$ on the sphere of radius r (cf. [5, page 71]). The functional r^λ [5, pages 72–73] has a simple pole at

$$\lambda = -n - 2j, \quad j = 0, 1, \dots, \quad (1.5)$$

and from [5, page 99], the Laurent series expansion of r^λ in a neighbourhood of $\lambda = -n - 2j$, $j = 0, 1, 2, \dots$, is

$$\begin{aligned} r^\lambda &= \frac{\Omega_n}{(2j)!} \delta^{(2j)}(r) \frac{1}{\lambda + n + 2j} \\ &+ \Omega_n r^{-2j-n} + \Omega_n (\lambda + n + 2j) r^{-2j-n} \ln(r) + \dots \end{aligned} \quad (1.6)$$

In (1.6), r^{-2j-n} is not the value of the functional r^λ at $\lambda = -n - 2j$ (in fact, it has a pole at this point) but is the value of the regular part of the Laurent expansion of r^λ at this point.

From [6, page 366, formula (3.4)], we know that the neutrix product of r^{-k} and $\nabla \delta$ on $\mathbb{R}^m$ exists and, furthermore,

$$r^{-2k} \circ \nabla \delta = -\frac{1}{2^{k+1}(k+1)!(m+2) \cdots (m+2k)} \sum_{i=0}^m (x_i \Delta^{k+1} \delta), \quad (1.7)$$

$$r^{1-2k} \circ \nabla \delta = 0, \quad (1.8)$$

where k is a positive integer, m is the dimension of the space, Δ^j is the iterated Laplacian operator defined by (1.10), and ∇ is the operator defined by

$$\nabla = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n} = \sum_{i=1}^n \frac{\partial}{\partial x_i}. \quad (1.9)$$

In (1.7) and (1.8), by the symbol $\circ$ we mean “neutrix product” which is defined by Li in [6, page 363, Definition 1.4, formula (1.11)].

The purpose of this paper is to obtain a relation between the distribution $\delta^{(2j)}(r)$ and the operator $\Delta^j \delta$ and to give a sense to convolution distributional product $\delta^{(2j)}(r) * \delta^{(2s)}(r)$ and the multiplicative distributional products $r^{-k} \cdot \nabla(\Delta^j \delta)$ and $(r - c)^{-k} \cdot \nabla(\Delta^j \delta)$ which are showed in Sections 2, 3.1, 3.2, and 3.3. Here, Δ^j is defined by

$$\Delta^j = \left\{ \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right\}^j, \quad (1.10)$$

and ∇ is the operator defined by (1.9).

We observed that relation (2.3) cannot be deduced from the formula

$$\delta^{(n+2j-1)}(r) = a_{j,n} \triangle^j \delta \quad (1.11)$$

which appear in [1], where

$$a_{j,n} = \frac{2^n \pi^{(n-1)/2} (-1)^{n+2j-1} \Gamma(n/2 + j + 1/2)}{j!} \quad (1.12)$$

with n dimension of the space.

Our formulae (3.7) and (3.18) result in a generalization of the neutrix product (1.7) and (1.8), respectively, due to Li (cf. [6, page 366, formula 3.4]).

To obtain our results, we need the following formulae:

$$(\delta^{(k)}(r-c), \varphi) = (-1)^k \Omega_n \left[\frac{\partial^k}{\partial r^k} (r^{n-1} S_\varphi(r)) \right]_{r=c} \quad (1.13)$$

(see [3, page 58, formula (II, 2, 5)]), where

$$(\delta^{(k)}(r-c), \varphi) = \int \delta^{(k)}(r-c) \varphi dx = \frac{(-1)^k}{c^{n-1}} \int_{O_c} \frac{\partial^k}{\partial r^k} (\varphi r^{n-1}) dO_c \quad (1.14)$$

(see [5, page 231, formula (10)]), O_c is the sphere $r-c=0$, and dO_c is the Euclidean element of area of it;

$$\text{Res}_{\lambda=-n-2j} (r^\lambda, \varphi) = \frac{\Omega_n}{2^j j! n(n+2) \cdots (n+2j-2)} (\triangle^j \delta, \varphi) \quad (1.15)$$

(see [5, pages 72-73]), where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad (1.16)$$

$$\Gamma(z+k) = z(z+1) \cdots (z+k-1) \Gamma(z) \quad (1.17)$$

(see [4, page 3, formula (2)])

$$\Gamma(z)(1-z) = \frac{\pi}{\text{sen}(z\pi)} \quad (1.18)$$

(see [4, page 3, formula (6)])

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (1.19)$$

(see [4, page 5, formula (5)]), and

$$\text{Res}_{\mu=-k, k=1,2,\dots} (x_+^\mu, \varphi) = \frac{\varphi^{(k-1)}(0)}{(k-1)!} \quad (1.20)$$

(see [5, page 49]), where x_+^μ is the functional defined by

$$(x_+^\mu, \varphi) = \int_0^\infty x^\mu \varphi(x) dx \quad (1.21)$$

(see [5, page 48]), which is analytic for $\operatorname{Re}(\mu) > -1$ and can be analytically continued to the entire μ plane except for the point $\mu = -1, -2, \dots$ where it has simple poles.

2. The relation between the distribution $\delta^{(2j)}(r)$ and the operator $\Delta^j \delta$. In this section, we want to obtain a formula that relates the distribution $\delta^{(2j)}(r)$ to the operator $\Delta^j \delta$.

From (1.12) and considering formula (1.13), the residue of (r^λ, φ) at $\lambda = -n - 2j$ for nonnegative integer j is given by

$$\operatorname{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \frac{\Omega_n \Gamma(n/2)}{2^{2j} j! \Gamma(n/2 + j)} (\Delta^j \delta, \varphi), \quad (2.1)$$

where Δ^j is defined by (1.10) and Ω_n by (1.16), with n the dimension of the space and $j = 0, 1, 2, \dots$

From [5, page 72], S_φ is an even function of the simple variable r in K , where K is the space of infinitely differentiable functions with bounded support. Then, the $S_\varphi(r)$, where integral (1.3) represents the application of $\Omega_n x_+^\mu$ (with $\mu = \lambda + n - 1$) to x_+^μ , is defined by (1.6).

Using the Laurent series expansion of r^λ in a neighbourhood of $\lambda = -n - 2j$, $j = 0, 1, 2, \dots$, from (1.6), we have

$$\delta^{(2j)}(r) = \frac{(2j)!}{\Omega_n} \lim_{\lambda \rightarrow -n-2j} (\lambda + n + 2j) r^\lambda. \quad (2.2)$$

From (2.2) and using (2.1), we obtain the following formula:

$$\begin{aligned} \delta^{(2j)}(r) &= \frac{(2j)!}{\Omega_n} \lim_{\lambda \rightarrow -n-2j} (\lambda + n + 2j) r^\lambda \\ &= \frac{(2j)!}{\Omega_n} \operatorname{Res}_{\lambda \rightarrow -n-2j} r^\lambda = \frac{(2j)! \Gamma(n/2)}{2^{2j} j! \Gamma(n/2 + j)} \Delta^j \delta. \end{aligned} \quad (2.3)$$

Using (1.17), formula (2.3) can be rewritten in the following form:

$$\delta^{(2j)}(r) = \frac{(2j)!}{j!} \frac{1}{2^j j! n(n+2) \cdots (n+2j-2)} \Delta^j \delta. \quad (2.4)$$

3. Applications of the basic formula (2.3). In this section, we want to give a sense to the convolution distributional product of the form $\delta^{(2j)}(r) * \delta^{(2s)}(r)$ and the distributional products $r^{-k} \cdot \nabla(\Delta^j \delta)$ and $(r-c)^{-k} \cdot \nabla(\Delta^j \delta)$.

3.1. The convolution distributional product of the form $\delta^{(2j)}(r) * \delta^{(2s)}(r)$. In this section, we designate $*$ the convolution.

We know from (2.3) that the following formula is true:

$$\delta^{(2j)}(r) = \frac{(2j)!\Gamma(n/2)}{2^{2j}j!\Gamma(n/2+j)}\Delta^j\delta. \quad (3.1)$$

From (3.1), $\delta^{(2j)}(r)$ is a finite linear combination of δ and its derivatives, in consequence, we conclude that $\delta^{(2j)}(r)$ is a distribution of the class O'_c , where O'_c [7, page 244] is the space of rapidly decreasing distributions. Therefore, using the formula

$$\Delta^t \delta * \Delta^s \delta = \Delta^{t+s} \delta \quad (3.2)$$

[2, page 75, formula (26)], where Δ^t is the iterated Laplacian operator defined by (1.10), we obtain the following formula:

$$\delta^{(2j)}(r) * \delta^{(2s)}(r) = b_{j,s,n} \delta^{(2(j+s))}(r), \quad (3.3)$$

where

$$b_{j,s,n} = \frac{(j+s)!(2s)!(2j)!\Gamma(n/2)\Gamma(n/2+j+s)}{j!(2(j+s))!s!\Gamma(n/2+s)\Gamma(n/2+j)}. \quad (3.4)$$

In particular, letting $j = s = 0$ in (3.3), we have

$$\delta(r) * \delta(r) = \delta(r), \quad (3.5)$$

where $r = \sqrt[2]{x_1^2 + x_2^2 + \cdots + x_n^2}$.

3.2. The multiplicative distributional product of $r^{-k} \cdot \nabla(\Delta^j \delta)$. To give a sense to the multiplicative distributional product of

$$r^{-k} \cdot \nabla(\Delta^j \delta), \quad (3.6)$$

we must study the cases $r^{-2k} \cdot \nabla(\Delta^j \delta)$ and $r^{1-2k} \cdot \nabla(\Delta^j \delta)$ where ∇ is the operator defined by (1.9) and Δ^j is the iterated Laplacian operator defined by (1.10).

Our formulae (3.7) and (3.18) result in a generalization of the neutrix product (1.7) and (1.8), respectively, due to Li (cf. [6, page 366, formula (3.4)]).

THEOREM 3.1. *Let k be a positive integer and let j be a nonnegative integer, then the formula*

$$r^{-2k} \cdot \nabla (\triangle^j \delta) = \frac{-(n+2j)(2j)!}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2(k+j))} \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+1} \delta \quad (3.7)$$

is valid if $k \neq n/2, n/2+1, \dots, n/2+s, s=0, 1, \dots$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is defined by (1.10).

PROOF. Using formula (2.4), we have

$$\begin{aligned} r^{-2k} \cdot \nabla (\triangle^j \delta) &= r^{-2k} \cdot \nabla \left(\frac{2^j n(n+2) \cdots (n+2j-2)j!}{(2j)!} \delta^{(2j)}(r) \right) \\ &= \frac{2^j n(n+2) \cdots (n+2j-2)j!}{(2j)!} r^{-2k} \cdot \nabla \delta^{(2j)}(r) \end{aligned} \quad (3.8)$$

if $k \neq n/2, n/2+1, \dots, n/2+s, s=0, 1, \dots$

Now, using the properties

$$\frac{\partial}{\partial x_j} \delta^{(k)}(P) = \frac{\partial P}{\partial x_j} \delta^{(k+1)}(P) \quad (3.9)$$

(see [5, page 232]) for

$$P = P(x_1, x_2, \dots, x_n) = r = \sqrt[2]{x_1^2 + x_2^2 + \cdots + x_n^2} \quad (3.10)$$

and using formula (1.9), we have

$$\nabla \delta^{(2j)}(r) = \sum_{i=1}^n \delta^{(2j+1)}(r) \frac{x_i}{r}. \quad (3.11)$$

From (3.8) and (3.11), we have

$$r^{-2k} \cdot \nabla \delta^{(2j)}(r) = \sum_{i=1}^n x_i \left(r^{-2k-1} \delta^{(2j+1)}(r) \right). \quad (3.12)$$

On the other hand, using formula (2.2), we have

$$\begin{aligned} r^{-2k-1} \cdot \delta^{(2j+1)}(r) &= r^{-2k-1} \cdot \frac{\partial}{\partial r} \delta^{(2j)}(r) \\ &= \frac{(2j)!(-n-2j)}{(2(k+j+1))!} \delta^{(2(k+j+1))}(r). \end{aligned} \quad (3.13)$$

From (3.13) and using formula (2.4), we have

$$\begin{aligned} r^{-2k-1} \cdot \delta^{(2j+1)}(r) &= \frac{(2j)!(-1)(n+2j)}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2(k+j+1)-2)} \triangle^{k+j+1} \delta. \end{aligned} \quad (3.14)$$

Therefore, from (3.13) and using (3.14), we obtain

$$\begin{aligned} r^{-2k} \cdot \nabla(\triangle^j \delta) &= \left(\sum_{i=1}^n x_i \right) r^{-2k-1} \cdot \delta^{(2j+1)}(r) \\ &= \frac{-(n+2j)(2j)!}{(k+j+1)!2^{k+j+1}n(n+2) \cdots (n+2(k+j))} \\ &\quad \times \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+1} \delta. \end{aligned} \quad (3.15)$$

if $k \neq n/2, n/2+1, \dots, n/2+s, s=0, 1, \dots$

Formula (3.15) coincides with formula (3.7). Theorem 3.1 and formula (3.7) generalize the neutrix product $r^{-2k} \circ \nabla \delta$ given by Li [6, page 366, Theorem 3.4, formula (3.4)].

In fact, letting $j=0$ in (3.7) and using that

$$\triangle^0 \delta = \delta, \quad (3.16)$$

we have

$$r^{-2k} \cdot \nabla \delta = -\frac{1}{(k+1)!2^{k+1}(n+2) \cdots (n+2k)} \left(\sum_{i=1}^n x_i \right) \triangle^{k+1} \delta. \quad (3.17)$$

Formula (3.17) coincides with formula (1.7). $\square$

THEOREM 3.2. *Let k be a positive integer and let j be a nonnegative integer, then the formula*

$$r^{1-2k} \cdot \nabla(\triangle^j \delta) = 0 \quad (3.18)$$

is valid if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2, s=0, 1, \dots$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is defined by (1.10).

PROOF. Using formulae (2.4), (3.9), and (3.11), we have

$$\begin{aligned} r^{1-2k} \cdot \nabla(\triangle^j \delta) &= r^{1-2k} \cdot \nabla \left(\frac{2^j j! n(n+2) \cdots (n+2j-1)}{(2j)!} \delta^{(2j)}(r) \right) \\ &= \frac{2^j j! n(n+2) \cdots (n+2j-1)}{(2j)!} \left(\sum_{i=1}^n x_i \right) r^{-2k} \cdot \delta^{(2j+1)}(r) \end{aligned} \quad (3.19)$$

if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2, s=0, 1, \dots$

On the other hand, using formula (2.2) and the properties

$$\Gamma(\beta+1) = \beta\Gamma(\beta) \quad (\text{see (1.17)}), \quad (3.20)$$

we have

$$\begin{aligned}
 r^{-2k} \cdot \delta^{(2j+1)}(r) &= r^{-2k} \cdot \frac{\partial}{\partial r} \delta^{(2j)}(r) \\
 &= -\frac{(n+2j)(2j)!}{\Omega_n} \lim_{\beta \rightarrow 0} \beta r^{\beta-n-2j-2k-1} \\
 &= -\frac{(n+2j)(2j)!}{\Omega_n} \lim_{\beta \rightarrow 0} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} r^{\beta-n-2j-2k-1}.
 \end{aligned} \tag{3.21}$$

Now, using that r^λ is regular at the points $\lambda = -n - 2(j-k) - 1$, $j = 0, 1, \dots$, $k = 1, 2, \dots$, and the properties

$$\lim_{\beta \rightarrow 0} \frac{1}{\Gamma(\beta)} = 0 \tag{3.22}$$

(see (1.17)), we have

$$r^{-2k} \cdot \delta^{(2j+1)}(r) = 0. \tag{3.23}$$

From (3.19) and using (3.23), we obtain

$$r^{1-2k} \cdot \nabla(\triangle^j \delta) = 0 \tag{3.24}$$

if $k \neq n/2, n/2 + 1/2, \dots, n/2 + s + 1/2$, $s = 0, 1, \dots$

Formula (3.24) coincides with formula (3.18). $\square$

Theorem 3.2 and formula (3.15) generalized the Neutrix Product $r^{1-2k} \cdot \nabla \delta$ given by Li [6, page 366, formula (3.4), Theorem 3.1]. In fact, letting $j = 0$ in (3.18) and using (3.16), we obtain

$$r^{1-2k} \cdot \nabla \delta = 0. \tag{3.25}$$

Formula (3.25) coincides with formula (1.8).

3.3. The multiplicative distributional product of $(r-c)^{-k} \cdot \nabla(\triangle^j \delta)$. To give a sense to the multiplicative distributional product of $(r-c)^{-k} \cdot \nabla(\triangle^j \delta)$, we must study the cases $(r-c)^{-2k} \cdot \nabla(\triangle^j \delta)$ and $(r-c)^{1-2k} \cdot \nabla(\triangle^j \delta)$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is the iterated Laplacian operator defined by (1.10).

THEOREM 3.3. Let k be a positive integer and let j be a nonnegative integer, then the formula

$$\begin{aligned} & (r-c)^{-2k} \cdot \nabla (\triangle^j \delta) \\ &= \sum_{l \geq 0} \binom{2k+2l-1}{2l} c^{2l} \cdot \left[\frac{-(n+2j)(2j)!}{(k+j+l+1)! 2^{k+j+l+1} n(n+2) \cdots (n+2(k+j+l))} \right] \\ & \quad \times \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+l+1} \delta \end{aligned} \quad (3.26)$$

is valid if $k \neq n/2, n/2+1, \dots, n/2+s, s=0, 1, \dots$, and

$$\begin{aligned} & (r-c)^{1-2k} \cdot \nabla (\triangle^j \delta) \\ &= \sum_{t \geq 1} \binom{2k+2t-1}{2t-1} c^{2t-1} \cdot \left[\frac{-(n+2j)(2j)!}{(k+j+t)! 2^{k+j+t} n(n+2) \cdots (n+2(k+j+t))} \right] \\ & \quad \times \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+t} \delta \end{aligned} \quad (3.27)$$

if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2, s=0, 1, \dots$ where $(r-c)^{-k}$ is defined by formula (3.29).

PROOF. Using the formula

$$(1+z)^\lambda = \sum_{l \geq 0} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)} \frac{z^l}{l!} \quad (3.28)$$

if $|z| < 1$ [4, Volume I, page 101, formulae (1), (2), and (4)], we have

$$(r-c)^\lambda = \sum_{l \geq 0} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)} \frac{(-c)^l}{l!} r^{\lambda-l} \quad (3.29)$$

if $c < r$.

Letting $\lambda = -h$, $h = 1, 2, \dots$, and $h \neq -n, -n-2, \dots, -n-2s, s = 0, 1, 2, \dots$ in (3.29) and using the formula

$$\frac{\Gamma(\lambda+1)}{l! \Gamma(\lambda+1-l)} = \frac{(-1)^l \Gamma(-\lambda+l)}{l! \Gamma(-\lambda)} = (-1)^l \binom{-\lambda+l-1}{l}, \quad (3.30)$$

we have

$$(r-c)^{-h} = \sum_{l \geq 0} \binom{h+l-1}{l} \frac{c^l}{l!} r^{-h-l} \quad (3.31)$$

if $c < r$ and $h \neq -n, -n-2, \dots, -n-2s, s = 0, 1, 2, \dots$

Now, letting $h = 2k$ in (3.31), we have

$$\begin{aligned}
 (r-c)^{-2k} \cdot \nabla(\triangle^j \delta) &= \sum_{l \geq 0} \binom{2k+l-1}{l} c^l (r^{-2k-l} \cdot \nabla(\triangle^j \delta)) \\
 &= \sum_{l \geq 0} \binom{2k+2l-1}{2l} c^{2l} (r^{-2(k+l)} \cdot \nabla(\triangle^j \delta)) \\
 &\quad + \sum_{t \geq 1} \binom{2k+2t-2}{2t-1} c^{2t-1} (r^{1-2(t+k)} \cdot \nabla(\triangle^j \delta)).
 \end{aligned} \tag{3.32}$$

From (3.32), using (3.7), (3.19), and (3.25), we obtain the formula

$$\begin{aligned}
 &(r-c)^{-2k} \cdot \nabla(\triangle^j \delta) \\
 &= \sum_{l \geq 0} \binom{2k+2l-1}{2l} c^{2l} (r^{-2(k+l)} \cdot \nabla(\triangle^j \delta)) \\
 &= \sum_{l \geq 0} \binom{2k+2l-1}{2l} c^{2l} \cdot \left[\frac{-(n+2j)(2j)!}{(k+j+l+1)! 2^{k+j+l+1} n(n+2) \cdots (n+2(k+j+l))} \right] \\
 &\quad \times \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+l+1} \delta
 \end{aligned} \tag{3.33}$$

if $k \neq n/2, n/2+1, \dots, n/2+s$, $s = 0, 1, \dots$. Formula (3.33) coincides with (3.26).

Similarly, letting $h = 2k-1$ in (3.31) and using (3.8) and (3.18), we have

$$\begin{aligned}
 &(r-c)^{1-2k} \cdot \nabla(\triangle^j \delta) \\
 &= \sum_{l \geq 0} \binom{2k+l-2}{l} c^l (r^{1-2k-l} \cdot \nabla(\triangle^j \delta)) \\
 &= \sum_{l \geq 0} \binom{2k+2l}{2l} c^{2l} (r^{1-2(k+l)} \cdot \nabla(\triangle^j \delta)) \\
 &\quad + \sum_{t \geq 1} \binom{2k+2t-3}{2t-1} c^{2t-1} (r^{-2(k+t-1)} \cdot \nabla(\triangle^j \delta)) \\
 &= \sum_{t \geq 1} \binom{2k+2t-3}{2t-1} c^{2t-1} (r^{-2(k+t-1)} \cdot \nabla(\triangle^j \delta)) \\
 &\quad \cdot \left[\frac{-(n+2j)(2j)!}{(k+j+t)! 2^{k+j+t} n(n+2) \cdots (n+2(k+j+t))} \right] \left(\sum_{i=1}^n x_i \right) \triangle^{k+j+t} \delta
 \end{aligned} \tag{3.34}$$

if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2$, $s = 0, 1, \dots$. Formula (3.31) coincides with (3.27). $\square$

It is clear that letting $c = 0$ in (3.26) and (3.27), we obtain formulae (3.7) and (3.18), respectively.

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