α-COMPACTNESS IN SMOOTH TOPOLOGICAL SPACES

CHUN-KEE PARK, WON KEUN MIN, and MYEONG HWAN KIM

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We introduce the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined by Demirci (1997) and obtain some of their structural properties.

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1. Introduction. Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang’s fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar et al. [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some properties of them.

In this paper, we define the smooth α-closure and smooth α-interior of a fuzzy set and investigate some of their properties. In fact, the smooth α-closure and smooth α-interior of a fuzzy set coincide with the smooth closure and smooth interior of a fuzzy set defined in [3] when α = 0. We also introduce the concepts of several types of α-compactness using smooth α-closure and smooth α-interior of a fuzzy set and investigate some of their properties.

2. Preliminaries. In this section, we give some notations and definitions which are to be used in the sequel. Let X be a set and let I = [0, 1] be the unit interval of the real line. Let I^X denote the set of all fuzzy sets of X. Let 0_X and 1_X denote the characteristic functions of φ and X, respectively.

A smooth topological space (s.t.s.) [6] is an ordered pair (X, τ), where X is a nonempty set and τ : I^X → I is a mapping satisfying the following conditions:

(1) τ(0_X) = τ(1_X) = 1;
(2) for all A, B ∈ I^X, τ(A ∩ B) ≥ τ(A) ∧ τ(B);
(3) for every subfamily {A_i : i ∈ J} ⊆ I^X, τ(∪_i ∈ J A_i) ≥ ∨_i ∈ J τ(A_i).

Then the mapping τ : I^X → I is called a smooth topology on X. The number τ(A) is called the degree of openness of A.

A mapping τ* : I^X → I is called a smooth cotopology [6] if and only if the following three conditions are satisfied:

(1) τ*(0_X) = τ*(1_X) = 1;
(2) for all A, B ∈ I^X, τ*(A ∪ B) ≥ τ*(A) ∧ τ*(B);
(3) for every subfamily \( \{A_i : i \in J\} \subseteq I^X, \tau^*(\cap_{i \in J} A_i) \geq \wedge_{i \in J} \tau^*(A_i) \).

If \( \tau \) is a smooth topology on \( X \), then the mapping \( \tau^* : I^X \rightarrow I \), defined by \( \tau^*(A) = \tau(A^c) \) where \( A^c \) denotes the complement of \( A \), is a smooth cotopology on \( X \). Conversely, if \( \tau^* \) is a smooth cotopology on \( X \), then the mapping \( \tau : I^X \rightarrow I \), defined by \( \tau(A) = \tau^*(A^c) \), is a smooth topology on \( X \) [6].

For the s.t.s. \((X, \tau)\) and \( \alpha \in [0, 1] \), the family \( \tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\} \) defines a Chang's fuzzy topology (CFT) on \( X \) [2]. The family of all closed fuzzy sets with respect to \( \tau_\alpha \) is denoted by \( \tau^*_\alpha \) and we have \( \tau^*_\alpha = \{A \in I^X : \tau^*(A) \geq \alpha\} \).

For \( A \in I^X \) and \( \alpha \in [0, 1] \), the \( \alpha \)-closure (resp., \( \alpha \)-interior) of \( A \), denoted by \( \text{cl}_\alpha(A) \) (resp., \( \text{int}_\alpha(A) \)), is defined by \( \text{cl}_\alpha(A) = \cap \{K \in \tau^*_\alpha : A \subseteq K\} \) (resp., \( \text{int}_\alpha(A) = \cup \{K \in \tau_\alpha : K \subseteq A\} \)).

Demicici [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows.

Let \((X, \tau)\) be an s.t.s. and \( A \in I^X \). Then the \( \tau \)-smooth closure (resp., \( \tau \)-smooth interior) of \( A \), denoted by \( \bar{A} \) (resp., \( A^o \)), is defined by \( \bar{A} = \cap \{K \in I^X : \tau(K) > 0, A \subseteq K\} \) (resp., \( A^o = \cup \{K \in I^X : \tau(K) > 0, K \subseteq A\} \)).

Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \( f : X \rightarrow Y \) is called smooth continuous with respect to \( \tau \) and \( \sigma \) [6] if and only if \( \tau(f^{-1}(A)) \geq \sigma(A) \) for every \( A \in I^Y \). A function \( f : X \rightarrow Y \) is called weakly smooth continuous with respect to \( \tau \) and \( \sigma \) [6] if and only if \( \sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0 \) for every \( A \in I^Y \).

A function \( f : X \rightarrow Y \) is smooth continuous with respect to \( \tau \) and \( \sigma \) if and only if \( \tau^*(f^{-1}(A)) \geq \sigma^*(A) \) for every \( A \in I^Y \). A function \( f : X \rightarrow Y \) is weakly smooth continuous with respect to \( \tau \) and \( \sigma \) if and only if \( \sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0 \) for every \( A \in I^Y \) [6].

A function \( f : X \rightarrow Y \) is called smooth open (resp., smooth closed) with respect to \( \tau \) and \( \sigma \) [6] if and only if \( \tau(A) \leq \sigma(f(A)) \) (resp., \( \tau^*(A) \leq \sigma^*(f(A)) \)) for every \( A \in I^X \).

A function \( f : X \rightarrow Y \) is called smooth preserving (resp., strict smooth preserving) with respect to \( \tau \) and \( \sigma \) [5] if and only if \( \sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B)) \) (resp., \( \sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B)) \)) for every \( A, B \in I^Y \).

If \( f : X \rightarrow Y \) is a smooth preserving function (resp., a strict smooth preserving function) with respect to \( \tau \) and \( \sigma \), then \( \sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B)) \) (resp., \( \sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B)) \)) for every \( A, B \in I^Y \) [5].

A function \( f : X \rightarrow Y \) is called smooth open preserving (resp., strict smooth open preserving) with respect to \( \tau \) and \( \sigma \) [5] if and only if \( \tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B)) \) (resp., \( \tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B)) \)) for every \( A, B \in I^X \).

3. Smooth \( \alpha \)-closure and smooth \( \alpha \)-interior. In this section, we introduce the concepts of smooth \( \alpha \)-closure and smooth \( \alpha \)-interior of a fuzzy set in smooth topological spaces and investigate some properties of them.
**Definition 3.1.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A \in I^X\). The \(\tau\)-smooth \(\alpha\)-closure (resp., \(\tau\)-smooth \(\alpha\)-interior) of \(A\), denoted by \(\overline{A}_\alpha\) (resp., \(A^o_\alpha\)), is defined by \(\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha \tau^*(A), A \subseteq K\}\) (resp., \(A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha \tau(A), K \subseteq A\}\)).

**Theorem 3.2.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A, B \in I^X\). Then
(a) \(\tau^*(\overline{A}_\alpha) \geq \alpha \tau^*(A)\),
(b) \(\tau(A^o_\alpha) \geq \alpha \tau(A)\),
(c) \(A \subseteq B\) and \(\tau^*(A) \leq \tau^*(B)\) \(\Rightarrow \overline{A}_\alpha \subseteq \overline{B}_\alpha\),
(d) \(A \subseteq B\) and \(\tau(B) \leq \tau(A)\) \(\Rightarrow A^o_\alpha \subseteq B^o_\alpha\).

**Proof.** (a) and (b) follow directly from Definition 3.1.
(c) If \(A \subseteq B\) and \(\tau^*(A) \leq \tau^*(B)\), then \(K \in \{K \in I^X : \tau^*(K) > \alpha \tau^*(B), B \subseteq K\} \Rightarrow K \in \{K \in I^X : \tau^*(K) > \alpha \tau^*(A), A \subseteq K\}\). Hence \(\overline{A}_\alpha \subseteq \overline{B}_\alpha\).
(d) The proof is similar to the proof of (c).

**Theorem 3.3.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A \in I^X\). Then
(a) \((\overline{A}_\alpha)^c = (A^c)^o_\alpha\),
(b) \(\overline{A}_\alpha = ((A^c)^o_\alpha)^c\),
(c) \((A^o_\alpha)^c = \overline{(A^c)}^c_\alpha\),
(d) \(A^o_\alpha = ((A^c)^o_\alpha)^c\).

**Proof.** (a) From Definition 3.1, we have

\[
(\overline{A}_\alpha)^c = (\cap\{K \in I^X : \tau^*(K) > \alpha \tau^*(A), A \subseteq K\})^c = \cup\{K^c : K \in I^X, \tau(K^c) = \tau^*(K) > \alpha \tau^*(A) = \alpha \tau(A^c), K^c \subseteq A^c\} = \cup\{U \in I^X : \tau(U) > \alpha \tau(A^c), U \subseteq A^c\} = (A^c)^o_\alpha.
\]

(b), (c), and (d) are easily obtained from (a).

**Theorem 3.4.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A, B \in I^X\). Then
(a) \((\overline{0_X}_\alpha) = 0_X\),
(b) \(A \subseteq \overline{A}_\alpha\),
(c) \(\overline{A}_\alpha \subseteq (\overline{A}_\alpha)^o_\alpha\),
(d) \(\overline{A}_\alpha \cap \overline{B}_\alpha \subseteq (\overline{A \cup B}_\alpha)\).

**Proof.** (a) and (b) are easily obtained from Definition 3.1. (c) follows directly from (b).
(d) For every \(A, B \in I^X\), we have

\[
(\overline{A \cup B}_\alpha) = \cap\{K \in I^X : \tau^*(K) > \alpha \tau^*(A \cup B), A \cup B \subseteq K\} \supseteq \cap\{K \in I^X : \tau^*(K) > \alpha \tau^*(A) \land \alpha \tau^*(B), A \cup B \subseteq K\}.
\]
\[= \cap \{ K \in I^X : \tau^*(K) > \alpha \tau^*(A) \] or \[\tau^*(K) > \alpha \tau^*(B), \ A \subseteq K, \ B \subseteq K \} \]
\[= \cap \{ K \in I^X : (\tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \} \] or \[\tau^*(K) > \alpha \tau^*(B), \ A \subseteq K, \ B \subseteq K \} \]
\[\subseteq \cap \{ \{ K \in I^X : \tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \} \] or \[\tau^*(K) > \alpha \tau^*(B), \ A \subseteq K, \ B \subseteq K \} \]
\[\cup \{ \{ K \in I^X : \tau^*(K) > \alpha \tau^*(B), \ B \subseteq K \} \]
\[= [ \cap \{ K \in I^X : \tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \} \] or \[\tau^*(K) > \alpha \tau^*(B), \ A \subseteq K, \ B \subseteq K \} \]
\[= \overline{A}_\alpha \cap \overline{B}_\alpha. \] (3.2)

**Theorem 3.5.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A, B \in I^X\). Then

(a) \((1_X)_\alpha = 1_X\),

(b) \(A^o_\alpha \subseteq A\),

(c) \((A^o_\beta)^o_\alpha \subseteq A^o_\alpha\),

(d) \((A \cap B)^o_\alpha \subseteq A^o_\alpha \cup B^o_\alpha\).

**Proof.** The proof is similar to the proof of Theorem 3.4. \(\Box\)

**Theorem 3.6.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A \in I^X\). Then

(a) \(\tau^*(A) > 0 \Rightarrow \overline{A}_\alpha = A\),

(b) \(\tau(A) > 0 \Rightarrow A^o_\alpha = A\).

**Proof.** (a) Let \(\tau^*(A) > 0\). Then \(A \in \{ K \in I^X : \tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \}\). By Definition 3.1, \(\overline{A}_\alpha \subseteq A\). By Theorem 3.4, \(A \subseteq \overline{A}_\alpha\). Hence \(\overline{A}_\alpha = A\).

(b) Let \(\tau(A) > 0\). Then \(A \in \{ K \in I^X : \tau(K) > \alpha \tau(A), \ K \subseteq A \}\). By Definition 3.1, \(A \subseteq \overline{A}_\alpha\). By Theorem 3.5, \(A^o_\alpha \subseteq A\). Hence \(A^o_\alpha = A\). \(\Box\)

**Remark 3.7.** Let \((X, \tau)\) be an s.t.s., \(\alpha_1, \alpha_2 \in [0,1)\) with \(\alpha_1 \leq \alpha_2\), and \(A \in I^X\). Then \(\overline{A}_{\alpha_1} \subseteq \overline{A}_{\alpha_2}\) and \(A^o_{\alpha_2} \subseteq A^o_{\alpha_1}\).

**Theorem 3.8.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A \in I^X\). Then

(a) \(\overline{A}_\alpha = \cap_{\beta > \alpha \tau^*(A)} \text{cl}_\beta (A)\),

(b) \(A^o_\alpha = \cup_{\beta > \alpha \tau(A)} \text{int}_\beta (A)\).

**Proof.** (a) For each \(x \in X\), we have

\[\overline{A}_\alpha(x) = [ \cap \{ K \in I^X : \tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \} ](x)\]
\[= \inf \{ K(x) : K \in I^X, \ \tau^*(K) > \alpha \tau^*(A), \ A \subseteq K \}\]
\[= \inf_{\beta > \alpha \tau^*(A)} \inf \{ K(x) : K \in I^X, \ \tau^*(K) \geq \beta, \ A \subseteq K \}\]
\[ \inf_{\beta > \alpha \tau^* (A)} [\bigcap \{ K \in I^X : \tau^* (K) \geq \beta, \, A \subseteq K \}] (x) \]
\[ \inf_{\beta > \alpha \tau^* (A)} \text{cl}_{\beta} (A) (x) \]
\[ = [ \bigcap_{\beta > \alpha \tau^* (A)} \text{cl}_{\beta} (A)] (x). \]

(3.3)

Hence, \( \overline{A}_\alpha = \cap_{\beta > \alpha \tau^* (A)} \text{cl}_{\beta} (A). \)

(b) The proof is similar to that of (a).

**Remark 3.9.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0,1)\), and \(A \in I^X\). From Theorems 3.4, 3.5, and 3.8, we easily obtain the following:

(a) if there exists a \(\beta \in (\alpha \tau^* (A), 1]\) such that \(A = \text{cl}_{\beta} (A)\), then \(A = \overline{A}_\alpha\);

(b) if there exists a \(\beta \in (\alpha \tau (A), 1]\) such that \(A = \text{int}_{\beta} (A)\), then \(A = A_{\alpha}^0\).

**Definition 3.10.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0,1)\). A function \(f : X \to Y\) is called smooth \(\alpha\)-preserving (resp., strict smooth \(\alpha\)-preserving) with respect to \(\tau\) and \(\sigma\) if and only if \(\sigma (A) \geq \alpha \sigma (B)\) \(\iff\) \(\tau (f^{-1} (A)) \geq \alpha \tau (f^{-1} (B))\) (resp., \(\sigma (A) > \alpha \sigma (B)\) \(\iff\) \(\tau (f^{-1} (A)) > \alpha \tau (f^{-1} (B))\)) for every \(A, B \in I^Y\).

A function \(f : X \to Y\) is called smooth open \(\alpha\)-preserving (resp., strict smooth open \(\alpha\)-preserving) with respect to \(\tau\) and \(\sigma\) if and only if \(\tau (A) \geq \alpha \tau (B) \Rightarrow \sigma (f (A)) \geq \alpha \sigma (f (B))\) (resp., \(\tau (A) > \alpha \tau (B) \Rightarrow \sigma (f (A)) > \alpha \sigma (f (B))\)) for every \(A, B \in I^Y\).

**Theorem 3.11.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0,1)\). If \(f : X \to Y\) is a smooth \(\alpha\)-preserving function (resp., a strict smooth \(\alpha\)-preserving function) with respect to \(\tau\) and \(\sigma\), then \(\sigma^* (A) \geq \alpha \sigma^* (B) \iff \sigma (A^c) \geq \alpha \sigma (B^c)\)

\[ \iff \tau (f^{-1} (A^c)) \geq \alpha \tau (f^{-1} (B^c)) \]
\[ \iff \tau ((f^{-1} (A))^c) \geq \alpha \tau ((f^{-1} (B))^c) \]
\[ \iff \tau^* (f^{-1} (A)) \geq \alpha \tau^* (f^{-1} (B)) \]

(3.4)

for every \(A, B \in I^Y\).

The proof is similar when \(f : X \to Y\) is a strict smooth \(\alpha\)-preserving function with respect to \(\tau\) and \(\sigma\).
Theorem 3.12. Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is injective and strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, then $f(\overline{A_\alpha}) \subseteq (f(A))_\alpha$ for every $A \in I^X$.

Proof. For every $A \in I^X$, we have

$$f^{-1}(\overline{f(A)_\alpha}) = f^{-1}[\cap \{ U \in I^Y : \sigma^*(U) > \alpha \sigma^*(f(A)), f(A) \subseteq U \}]$$

$$\supseteq f^{-1}[\cap \{ U \in I^Y : \tau^*(f^{-1}(U)) > \alpha \tau^*(A), f^{-1}(A) \subseteq f^{-1}(U) \} ]$$

$$= \cap \{ f^{-1}(U) \in I^X : \tau^*(f^{-1}(U)) > \alpha \tau^*(A), f^{-1}(A) \subseteq f^{-1}(U) \}$$

$$\supseteq \cap \{ K \in I^X : \tau(K) > \alpha \tau(f^{-1}(A)), f^{-1}(A) \subseteq K \}$$

$$= \overline{\overline{f^{-1}(A)}_\alpha}.$$  (3.5)

Hence, $f(\overline{A_\alpha}) \subseteq (f(A))_\alpha$.  \(\square\)

Theorem 3.13. Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, then

(a) $\overline{(f^{-1}(A))_\alpha} \subseteq f^{-1}(\overline{A_\alpha})$ for every $A \in I^Y$,

(b) $f^{-1}(A_\alpha^0) \subseteq (f^{-1}(A))_\alpha^0$ for every $A \in I^Y$.

Proof. (a) For every $A \in I^Y$, we have

$$f^{-1}(\overline{A_\alpha}) = f^{-1}[\cap \{ U \in I^Y : \sigma^*(U) > \alpha \sigma^*(A), A \subseteq U \}]$$

$$\supseteq f^{-1}[\cap \{ U \in I^Y : \tau^*(f^{-1}(U)) > \alpha \tau^*(f^{-1}(A)), f^{-1}(A) \subseteq f^{-1}(U) \} ]$$

$$= \cap \{ f^{-1}(U) \in I^X : \tau^*(f^{-1}(U)) > \alpha \tau^*(f^{-1}(A)), f^{-1}(A) \subseteq f^{-1}(U) \}$$

$$\supseteq \cap \{ K \in I^X : \tau(K) > \alpha \tau(f^{-1}(A)), f^{-1}(A) \subseteq K \}$$

$$= \overline{\overline{(f^{-1}(A))_\alpha}}.$$  (3.6)

(b) For every $A \in I^Y$, we have

$$f^{-1}(A_\alpha^0) = f^{-1}[\cup \{ U \in I^Y : \sigma(U) > \alpha \sigma(A), U \subseteq A \}]$$

$$
\subseteq f^{-1}[\cup \{ U \in I^Y : \tau(f^{-1}(U)) > \alpha \tau(f^{-1}(A)), f^{-1}(U) \subseteq f^{-1}(A) \} ]$$

$$= \cup \{ f^{-1}(U) \in I^X : \tau(f^{-1}(U)) > \alpha \tau(f^{-1}(A)), f^{-1}(U) \subseteq f^{-1}(A) \}$$

$$\subseteq \cup \{ K \in I^X : \tau(K) > \alpha \tau(f^{-1}(A)), K \subseteq f^{-1}(A) \}$$

$$= (f^{-1}(A))_\alpha^0.$$  (3.7)

\(\square\)
**Theorem 3.14.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \(\alpha \in [0, 1)\). If a function \(f : X \to Y\) is strict smooth open \(\alpha\)-preserving with respect to \(\tau\) and \(\sigma\), then \(f(A^\alpha_{\alpha}) = (f(A))_{\alpha}^\alpha\) for every \(A \in I^X\).

**Proof.** For every \(A \in I^X\), we have
\[
f(A^\alpha_{\alpha}) = f\left[\bigcup \{U \in I^X : \tau(U) > \alpha \tau(A), \, U \subseteq A\}\right]
\subseteq f\left[\bigcup \{U \in I^X : \sigma(f(U)) > \alpha \sigma(f(A)), \, f(U) \subseteq f(A)\}\right]
= \bigcup \{f(U) \in I^Y : \sigma(f(U)) > \alpha \sigma(f(A)), \, f(U) \subseteq f(A)\}
= \bigcup \{K \in I^Y : \sigma(K) > \alpha \sigma(f(A)), \, K \subseteq f(A)\}
= (f(A))_{\alpha}^\alpha.
\]

4. Types of smooth \(\alpha\)-compactness. In this section, we introduce the concepts of several types of smooth \(\alpha\)-compactness in smooth topological spaces and investigate some properties of them.

**Definition 4.1** [5]. An s.t.s. \((X, \tau)\) is called smooth compact if and only if for every family \(\{A_i : i \in J\}\) in \(\{A \in I^X : \tau(A) > 0\}\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} A_i = 1_X\).

**Theorem 4.2** [4]. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and \(f : X \to Y\) a surjective weakly smooth continuous function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is smooth compact, then so is \((Y, \sigma)\).

**Definition 4.3.** Let \(\alpha \in [0, 1)\). An s.t.s. \((X, \tau)\) is called smooth nearly \(\alpha\)-compact if and only if for every family \(\{A_i : i \in J\}\) in \(\{A \in I^X : \tau(A) > 0\}\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} (\overline{A_i})_{\alpha} = 1_X\).

**Definition 4.4.** Let \(\alpha \in [0, 1)\). An s.t.s. \((X, \tau)\) is called smooth almost \(\alpha\)-compact if and only if for every family \(\{A_i : i \in J\}\) in \(\{A \in I^X : \tau(A) > 0\}\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} (\overline{A_i})_{\alpha} = 1_X\).

**Definition 4.5.** Let \(\alpha \in [0, 1)\). An s.t.s. \((X, \tau)\) is called smooth \(\alpha\)-regular if and only if each fuzzy set \(A \in I^X\) satisfying \(\tau(A) > 0\) can be written as \(A = \bigcup \{K \in I^X : \tau(K) \geq \tau(A), \, \overline{K}_{\alpha} \subseteq A\}\).

**Definition 4.6.** A smooth topology \(\tau : I^X \to I\) on \(X\) is called monotonic increasing (resp., monotonic decreasing) if and only if \(A \subseteq B \Rightarrow \tau(A) \leq \tau(B)\) (resp., \(A \subseteq B \Rightarrow \tau(A) \geq \tau(B)\)) for every \(A, B \in I^X\).

**Theorem 4.7.** Let \((X, \tau)\) be an s.t.s., \(\alpha \in [0, 1)\), and \(\tau\) a monotonic decreasing smooth topology. If \((X, \tau)\) is smooth compact, then \((X, \tau)\) is smooth nearly \(\alpha\)-compact.

**Proof.** Let \((X, \tau)\) be a smooth compact s.t.s. Then for every family \(\{A_i : i \in J\}\) in \(\{A \in I^X : \tau(A) > 0\}\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such
that \( \cup_{i \in J_0} A_i = 1_X \). Since \( \tau(A_i) > 0 \) for each \( i \in J \), we have \( A_i = (A_i)^{\alpha}_\sigma \) for each \( i \in J \) by Theorem 3.6. Since \( \tau \) is monotonic decreasing and \( A_i \subseteq (A_i)^{\alpha}_\sigma \) for each \( i \in J \), we have \( \tau(A_i) \geq \tau((A_i)^{\alpha}_\sigma) \) for each \( i \in J \). Hence from Theorem 3.2, we have \( A_i = (A_i)^{\alpha}_\sigma \subseteq ((A_i)^{\alpha}_\sigma)^{\alpha}_\sigma \) for each \( i \in J \). Thus \( 1_X = \cup_{i \in J_0} A_i \subseteq \cup_{i \in J_0} ((A_i)^{\alpha}_\sigma)^{\alpha}_\sigma \), that is, \( \cup_{i \in J_0} ((A_i)^{\alpha}_\sigma)^{\alpha}_\sigma = 1_X \). Hence \((X, \tau)\) is smooth nearly \( \alpha \)-compact. \( \square \)

**Theorem 4.8.** Let \( \alpha \in [0,1) \). Then a smooth nearly \( \alpha \)-compact s.t.s. \((X, \tau)\) is smooth almost \( \alpha \)-compact.

**Proof.** Let \((X, \tau)\) be a smooth nearly \( \alpha \)-compact s.t.s. Then for every family \( \{A_i : i \in J\} \subseteq \{A \in I^X : \tau(A) > 0\} \) covering \( X \), there exists a finite subset \( J_0 \) of \( J \) such that \( \cup_{i \in J_0} (A_i)^{\alpha}_\sigma = 1_X \). Since \( (A_i)^{\alpha}_\sigma \subseteq (A_i)^{\alpha}_\sigma \) for each \( i \in J \) by Theorem 3.5, \( 1_X = \cup_{i \in J_0} (A_i)^{\alpha}_\sigma \subseteq \cup_{i \in J_0} (A_i)^{\alpha}_\sigma \). Hence \((X, \tau)\) is smooth almost \( \alpha \)-compact. \( \square \)

**Theorem 4.9.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \( \alpha \in [0,1) \), and \( f : X \to Y \) a surjective, weakly smooth continuous, and strict smooth \( \alpha \)-preserving function with respect to \( \tau \) and \( \sigma \). If \((X, \tau)\) is smooth almost \( \alpha \)-compact, then so is \((Y, \sigma)\).

**Proof.** Let \( \{A_i : i \in J\} \) be a family in \( \{A \in I^Y : \sigma(A) > 0\} \) covering \( Y \), that is, \( \cup_{i \in J} A_i = 1_Y \). Then \( 1_Y = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i) \). Since \( f \) is weakly smooth continuous, \( \tau(f^{-1}(A_i)) > 0 \) for each \( i \in J \). Since \((X, \tau)\) is smooth almost \( \alpha \)-compact, there exists a finite subset \( J_0 \) of \( J \) such that \( \cup_{i \in J_0} (f^{-1}(A_i))^{\alpha}_\sigma = 1_X \). From the surjectivity of \( f \) we have \( 1_Y = f(1_X) = f(\cup_{i \in J_0} (f^{-1}(A_i))^{\alpha}_\sigma) = \cup_{i \in J_0} f((f^{-1}(A_i))^{\alpha}_\sigma) \). Since \( f : X \to Y \) is strict smooth \( \alpha \)-preserving with respect to \( \tau \) and \( \sigma \), from Theorem 3.13 we have \( (f^{-1}(A_i))^{\alpha}_\sigma \subseteq f^{-1}((A_i)^{\alpha}_\sigma) \) for each \( i \in J \). Hence we have \( 1_Y = \cup_{i \in J_0} f((f^{-1}(A_i))^{\alpha}_\sigma) \subseteq \cup_{i \in J_0} f(f^{-1}((A_i)^{\alpha}_\sigma)) = \cup_{i \in J_0} (A_i)^{\alpha}_\sigma \), that is, \( \cup_{i \in J_0} (A_i)^{\alpha}_\sigma = 1_Y \). Thus \((Y, \sigma)\) is smooth almost \( \alpha \)-compact. \( \square \)

We obtain the following corollary from Theorems 4.8 and 4.9.

**Corollary 4.10.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \( \alpha \in [0,1) \), and \( f : X \to Y \) a surjective, weakly smooth continuous, and strict smooth \( \alpha \)-preserving function with respect to \( \tau \) and \( \sigma \). If \((X, \tau)\) is smooth nearly \( \alpha \)-compact, then \((Y, \sigma)\) is smooth almost \( \alpha \)-compact.

**Theorem 4.11.** Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \( \alpha \in [0,1) \), and \( f : X \to Y \) a surjective, weakly smooth continuous, strict smooth \( \alpha \)-preserving, and strict smooth open \( \alpha \)-preserving function with respect to \( \tau \) and \( \sigma \). If \((X, \tau)\) is smooth nearly \( \alpha \)-compact, then so is \((Y, \sigma)\).

**Proof.** Let \( \{A_i : i \in J\} \) be a family in \( \{A \in I^Y : \sigma(A) > 0\} \) covering \( Y \), that is, \( \cup_{i \in J} A_i = 1_Y \). Then \( 1_Y = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i) \). Since \( f \) is weakly smooth continuous, \( \tau(f^{-1}(A_i)) > 0 \) for each \( i \in J \). Since \((X, \tau)\) is smooth nearly \( \alpha \)-compact, there exists a finite subset \( J_0 \) of \( J \) such that \( \cup_{i \in J_0} ((f^{-1}(A_i))^{\alpha}_\sigma = 1_X \).
From the surjectivity of $f$ we have $1_Y = f(1_X) = f(\cup_{i \in J_0} ((f^{-1}(A_i))^\alpha)\alpha) = \cup_{i \in J_0} f( ((f^{-1}(A_i))^\alpha)\alpha)$. Since $f : X \to Y$ is strict smooth open $\alpha$-preserving with respect to $\tau$ and $\sigma$, from Theorem 3.14 we have $f( ((f^{-1}(A_i))^\alpha)\alpha) \subseteq (f((f^{-1}(A_i))^\alpha))\alpha$ for each $i \in J$. Since $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, from Theorem 3.13 we have $(f^{-1}(A_i))^\alpha \subseteq f^{-1}((A_i)^\alpha)$ for each $i \in J$. Hence, we have

$$1_Y = \cup_{i \in J_0} f( ((f^{-1}(A_i))^\alpha)\alpha) \subseteq \cup_{i \in J_0} (f((f^{-1}(A_i))^\alpha))\alpha \subseteq \cup_{i \in J_0} (f(f^{-1}(A_i))^\alpha)\alpha \subseteq \cup_{i \in J_0} (A_i)^\alpha.$$

Hence, $\cup_{i \in J_0} (A_i)^\alpha = 1_Y$. Thus $(Y, \sigma)$ is smooth nearly $\alpha$-compact.

**Theorem 4.12.** Let $\alpha \in [0, 1)$. Then a smooth almost $\alpha$-compact smooth $\alpha$-regular s.t.s. $(X, \tau)$ is smooth compact.

**Proof.** Let $\{A_i : i \in J\}$ be a family in $\{A \in I^X : \sigma(A) > 0\}$ covering $X$, that is, $\cup_{i \in J} A_i = 1_X$. Since $(X, \tau)$ is smooth $\alpha$-regular, $A_i = \cup_{i \in J} \{K_i \subseteq I^X : \tau(K_i) \geq \tau(A_i), (K_i)^\alpha \subseteq A_i\}$ for each $i \in J$. Since $\cup_{i \in J} A_i = \cup_{i \in J} [\cup_{i \in J} A_i] = 1_X$ and $(X, \tau)$ is smooth almost $\alpha$-compact, there exists a finite subfamily $\{K_i : i \in L, l \in L\}$ such that $\cup_{i \in L} (K_i)^\alpha = 1_X$. Since for each $l \in L$ there exists $i \in L$ such that $(K_i)^\alpha \subseteq A_i$, we have $\cup_{i \in L} A_i = 1_X$, where $J_0$ is a finite subset of $J$. Hence $(X, \tau)$ is smooth compact.

We obtain the following corollary from Theorems 4.8 and 4.12.

**Corollary 4.13.** Let $\alpha \in [0, 1)$. Then a smooth nearly $\alpha$-compact smooth $\alpha$-regular s.t.s. $(X, \tau)$ is smooth compact.

**References**

Chun-Kee Park: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
EC-mail address: ckipark@kangw.on.ac.kr

Won Keun Min: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
EC-mail address: wkmin@cc.kangw.on.ac.kr

Myeong Hwan Kim: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
EC-mail address: kimw@kangw.on.ac.kr
Submit your manuscripts at http://www.hindawi.com