A NOTE ON SOME APPLICATIONS OF α -OPEN SETS

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The object of this note is to introduce and study topological properties of α -derived, α -border, α -frontier, and α -exterior of a set using the concept of α -open sets. Moreover, we study some further properties of the well-known notions of α -closure and α -interior. We also obtain a new decomposition of α -continuous functions.

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1. Introduction. The notion of α -open set (originally called α -sets) in topological spaces was introduced by Njåstad [2] in 1965. Since then, it has been widely investigated in the literature. For these sets, we introduce the notions of α -derived, α -border, α -frontier, and α -exterior of a set and show that some of their properties are analogous to those for open sets. Also, we give some additional properties of α -closure and α -interior of a set due to Njåstad [2].

Throughout this paper, (X, τ) (simply X) always mean topological spaces. A subset A of (X, τ) is called α -open [2] if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$. The complement of an α -open set is called α -closed. The intersection of all α -closed sets containing A is called the α -closure of A , denoted by $\text{Cl}_\alpha(A)$. A subset A is also α -closed if and only if $A = \text{Cl}_\alpha(A)$. We denote the family of α -open sets of (X, τ) by τ^α . It is shown in [2] (see also [4]) that each of $\tau \subset \tau^\alpha$ and τ^α is a topology on X .

2. Applications of α -open sets

DEFINITION 2.1. Let A be a subset of a space X . A point $x \in A$ is said to be α -limit point of A if for each α -open set U containing x , $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all α -limit points of A is called an α -derived set of A and is denoted by $D_\alpha(A)$.

THEOREM 2.2. For subsets A, B of a space X , the following statements hold:

- (1) $D_\alpha(A) \subset D(A)$, where $D(A)$ is the derived set of A ;
- (2) if $A \subset B$, then $D_\alpha(A) \subset D_\alpha(B)$;
- (3) $D_\alpha(A) \cup D_\alpha(B) \subset D_\alpha(A \cup B)$ and $D_\alpha(A \cap B) \subset D_\alpha(A) \cap D_\alpha(B)$;
- (4) $D_\alpha(D_\alpha(A)) \setminus A \subset D_\alpha(A)$;
- (5) $D_\alpha(A \cup D_\alpha(A)) \subset A \cup D_\alpha(A)$.

PROOF. (1) It suffices to observe that every open set is α -open.

(3) Follows by (2).

(4) If $x \in D_\alpha(D_\alpha(A)) \setminus A$ and U is an α -open set containing x , then $U \cap (D_\alpha(A) \setminus \{x\}) \neq \emptyset$. Let $y \in U \cap (D_\alpha(A) \setminus \{x\})$. Then, since $y \in D_\alpha(A)$ and $y \in U$, $U \cap (A \setminus \{y\}) \neq \emptyset$. Let $z \in U \cap (A \setminus \{y\})$. Then, $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap (A \setminus \{x\}) \neq \emptyset$. Therefore, $x \in D_\alpha(A)$.

(5) Let $x \in D_\alpha(A \cup D_\alpha(A))$. If $x \in A$, the result is obvious. So, let $x \in D_\alpha(A \cup D_\alpha(A)) \setminus A$, then, for α -open set U containing x , $U \cap (A \cup D_\alpha(A) \setminus \{x\}) \neq \emptyset$. Thus, $U \cap (A \setminus \{x\}) \neq \emptyset$ or $U \cap (D_\alpha(A) \setminus \{x\}) \neq \emptyset$. Now, it follows similarly from (4) that $U \cap (A \setminus \{x\}) \neq \emptyset$. Hence, $x \in D_\alpha(A)$. Therefore, in any case, $D_\alpha(A \cup D_\alpha(A)) \subset A \cup D_\alpha(A)$. $\square$

In general, the converse of (1) may not be true and the equality does not hold in (3) of [Theorem 2.2](#).

EXAMPLE 2.3. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$. Thus, $\tau^\alpha = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Take the following:

- (i) $A = \{c\}$. Then, $D(A) = \{b\}$ and $D_\alpha(A) = \emptyset$. Hence, $D(A) \not\subset D_\alpha(A)$;
- (ii) $C = \{a\}$ and $E = \{b, c\}$. Then, $D_\alpha(C) = \{b, c\}$ and $D_\alpha(E) = \emptyset$. Hence, $D_\alpha(C \cup E) \neq D_\alpha(C) \cup D_\alpha(E)$.

THEOREM 2.4. For any subset A of a space X , $\text{Cl}_\alpha(A) = A \cup D_\alpha(A)$.

PROOF. Since $D_\alpha(A) \subset \text{Cl}_\alpha(A)$, $A \cup D_\alpha(A) \subset \text{Cl}_\alpha(A)$. On the other hand, let $x \in \text{Cl}_\alpha(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each α -open set U containing x intersects A at a point distinct from x ; so $x \in D_\alpha(A)$. Thus, $\text{Cl}_\alpha(A) \subset A \cup D_\alpha(A)$, which completes the proof. $\square$

COROLLARY 2.5. A subset A is α -closed if and only if it contains the set of its α -limit points.

DEFINITION 2.6. A point $x \in X$ is said to be an α -interior point of A if there exists an α -open set U containing x such that $U \subset A$. The set of all α -interior points of A is said to be α -interior of A [1] and denoted by $\text{Int}_\alpha(A)$.

THEOREM 2.7. For subsets A, B of a space X , the following statements are true:

- (1) $\text{Int}_\alpha(A)$ is the largest α -open set contained in A ;
- (2) A is α -open if and only if $A = \text{Int}_\alpha(A)$;
- (3) $\text{Int}_\alpha(\text{Int}_\alpha(A)) = \text{Int}_\alpha(A)$;
- (4) $\text{Int}_\alpha(A) = A \setminus D_\alpha(X \setminus A)$;
- (5) $X \setminus \text{Int}_\alpha(A) = \text{Cl}_\alpha(X \setminus A)$;
- (6) $X \setminus \text{Cl}_\alpha(A) = \text{Int}_\alpha(X \setminus A)$;
- (7) $A \subset B$, then $\text{Int}_\alpha(A) \subset \text{Int}_\alpha(B)$;
- (8) $\text{Int}_\alpha(A) \cup \text{Int}_\alpha(B) \subset \text{Int}_\alpha(A \cup B)$;
- (9) $\text{Int}_\alpha(A) \cap \text{Int}_\alpha(B) \supset \text{Int}_\alpha(A \cap B)$.

PROOF. (4) If $x \in A \setminus D_\alpha(X \setminus A)$, then $x \notin D_\alpha(X \setminus A)$ and so there exists an α -open set U containing x such that $U \cap (X \setminus A) = \emptyset$. Then, $x \in U \subset A$ and hence $x \in \text{Int}_\alpha(A)$, that is, $A \setminus D_\alpha(X \setminus A) \subset \text{Int}_\alpha(A)$. On the other hand, if $x \in \text{Int}_\alpha(A)$, then $x \notin D_\alpha(X \setminus A)$ since $\text{Int}_\alpha(A)$ is α -open and $\text{Int}_\alpha(A) \cap (X \setminus A) = \emptyset$. Hence, $\text{Int}_\alpha(A) = A \setminus D_\alpha(X \setminus A)$.

$$(5) X \setminus \text{Int}_\alpha(A) = X \setminus (A \setminus D_\alpha(X \setminus A)) = (X \setminus A) \cup D_\alpha(X \setminus A) = \text{Cl}_\alpha(X \setminus A). \quad \square$$

DEFINITION 2.8. $b_\alpha(A) = A \setminus \text{Int}_\alpha(A)$ is said to be the α -border of A .

THEOREM 2.9. For a subset A of a space X , the following statements hold:

- (1) $b_\alpha(A) \subset b(A)$ where $b(A)$ denotes the border of A ;
- (2) $A = \text{Int}_\alpha(A) \cup b_\alpha(A)$;
- (3) $\text{Int}_\alpha(A) \cap b_\alpha(A) = \emptyset$;
- (4) A is an α -open set if and only if $b_\alpha(A) = \emptyset$;
- (5) $b_\alpha(\text{Int}_\alpha(A)) = \emptyset$;
- (6) $\text{Int}_\alpha(b_\alpha(A)) = \emptyset$;
- (7) $b_\alpha(b_\alpha(A)) = b_\alpha(A)$;
- (8) $b_\alpha(A) = A \cap \text{Cl}_\alpha(X \setminus A)$;
- (9) $b_\alpha(A) = D_\alpha(X \setminus A)$.

PROOF. (6) If $x \in \text{Int}_\alpha(b_\alpha(A))$, then $x \in b_\alpha(A)$. On the other hand, since $b_\alpha(A) \subset A$, $x \in \text{Int}_\alpha(b_\alpha(A)) \subset \text{Int}_\alpha(A)$. Hence, $x \in \text{Int}_\alpha(A) \cap b_\alpha(A)$, which contradicts (3). Thus, $\text{Int}_\alpha(b_\alpha(A)) = \emptyset$.

$$(8) b_\alpha(A) = A \setminus \text{Int}_\alpha(A) = A \setminus (X \setminus \text{Cl}_\alpha(X \setminus A)) = A \cap \text{Cl}_\alpha(X \setminus A).$$

$$(9) b_\alpha(A) = A \setminus \text{Int}_\alpha(A) = A \setminus (A \setminus D_\alpha(X \setminus A)) = D_\alpha(X \setminus A). \quad \square$$

EXAMPLE 2.10. Consider the topological space (X, τ) given in [Example 2.3](#). If $A = \{a, b\}$, then $b_\alpha(A) = \emptyset$ and $b(A) = \{b\}$. Hence, $b(A) \not\subset b_\alpha(A)$, that is, in general, the converse [Theorem 2.9](#)(1) may not be true.

DEFINITION 2.11. $\text{Fr}_\alpha(A) = \text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A)$ is said to be the α -frontier of A .

THEOREM 2.12. For a subset A of a space X , the following statements hold:

- (1) $\text{Fr}_\alpha(A) \subset \text{Fr}(A)$ where $\text{Fr}(A)$ denotes the frontier of A ;
- (2) $\text{Cl}_\alpha(A) = \text{Int}_\alpha(A) \cup \text{Fr}_\alpha(A)$;
- (3) $\text{Int}_\alpha(A) \cap \text{Fr}_\alpha(A) = \emptyset$;
- (4) $b_\alpha(A) \subset \text{Fr}_\alpha(A)$;
- (5) $\text{Fr}_\alpha(A) = b_\alpha(A) \cup D_\alpha(A)$;
- (6) A is an α -open set if and only if $\text{Fr}_\alpha(A) = D_\alpha(A)$;
- (7) $\text{Fr}_\alpha(A) = \text{Cl}_\alpha(A) \cap \text{Cl}_\alpha(X \setminus A)$;
- (8) $\text{Fr}_\alpha(A) = \text{Fr}_\alpha(X \setminus A)$;
- (9) $\text{Fr}_\alpha(A)$ is α -closed;
- (10) $\text{Fr}_\alpha(\text{Fr}_\alpha(A)) \subset \text{Fr}_\alpha(A)$;
- (11) $\text{Fr}_\alpha(\text{Int}_\alpha(A)) \subset \text{Fr}_\alpha(A)$;
- (12) $\text{Fr}_\alpha(\text{Cl}_\alpha(A)) \subset \text{Fr}_\alpha(A)$;
- (13) $\text{Int}_\alpha(A) = A \setminus \text{Fr}_\alpha(A)$.

PROOF. (2) $\text{Int}_\alpha(A) \cup \text{Fr}_\alpha(A) = \text{Int}_\alpha(A) \cup (\text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A)) = \text{Cl}_\alpha(A)$.

(3) $\text{Int}_\alpha(A) \cap \text{Fr}_\alpha(A) = \text{Int}_\alpha(A) \cap (\text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A)) = \emptyset$.

(5) Since $\text{Int}_\alpha(A) \cup \text{Fr}_\alpha(A) = \text{Int}_\alpha(A) \cup b_\alpha(A) \cup D_\alpha(A)$, $\text{Fr}_\alpha(A) = b_\alpha(A) \cup D_\alpha(A)$.

(7) $\text{Fr}_\alpha(A) = \text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A) = \text{Cl}_\alpha(A) \cap \text{Cl}_\alpha(X \setminus A)$.

(9) $\text{Cl}_\alpha(\text{Fr}_\alpha(A)) = \text{Cl}_\alpha(\text{Cl}_\alpha(A) \cap \text{Cl}_\alpha(X \setminus A)) \subset \text{Cl}_\alpha(\text{Cl}_\alpha(A)) \cap \text{Cl}_\alpha(\text{Cl}_\alpha(X \setminus A)) = \text{Fr}_\alpha(A)$.

Hence, $\text{Fr}_\alpha(A)$ is α -closed.

(10) $\text{Fr}_\alpha(\text{Fr}_\alpha(A)) = \text{Cl}_\alpha(\text{Fr}_\alpha(A)) \cap \text{Cl}_\alpha(X \setminus \text{Fr}_\alpha(A)) \subset \text{Cl}_\alpha(\text{Fr}_\alpha(A)) = \text{Fr}_\alpha(A)$.

(12) $\text{Fr}_\alpha(\text{Cl}_\alpha(A)) = \text{Cl}_\alpha(\text{Cl}_\alpha(A)) \setminus \text{Int}_\alpha(\text{Cl}_\alpha(A)) = \text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(\text{Cl}_\alpha(A)) = \text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A) = \text{Fr}_\alpha(A)$.

(13) $A \setminus \text{Fr}_\alpha(A) = A \setminus (\text{Cl}_\alpha(A) \setminus \text{Int}_\alpha(A)) = \text{Int}_\alpha(A)$. $\square$

The converses of (1) and (4) of [Theorem 2.12](#) are not true in general, as shown by [Example 2.13](#).

EXAMPLE 2.13. Consider the topological space (X, τ) given in [Example 2.3](#). If $A = \{c\}$, then $\text{Fr}(A) = \{b, c\} \not\subseteq \{c\} = \text{Fr}_\alpha(A)$, and if $B = \{a, b\}$, then $\text{Fr}_\alpha(B) = \{c\} \not\subseteq b_\alpha(B)$.

DEFINITION 2.14. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be α -continuous [1] if $f^{-1}(V) \in \tau^\alpha$ for every $V \in \sigma$ and, equivalently, if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \tau^\alpha$ with $x \in U$ such that $f(U) \subset V$.

In the following theorem, $\# \alpha$ -c. denotes the set of points x of X for which a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is not α -continuous.

THEOREM 2.15. $\# \alpha$ -c. is identical with the union of the α -frontiers of the inverse images of α -open sets containing $f(x)$.

PROOF. Suppose that f is not α -continuous at a point x of X . Then, there exists an open set $V \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \tau^\alpha$ containing x . Hence, we have $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \tau^\alpha$ containing x . It follows that $x \in \text{Cl}_\alpha(X \setminus f^{-1}(V))$. We also have $x \in f^{-1}(V) \subset \text{Cl}_\alpha(f^{-1}(V))$. This means that $x \in \text{Fr}_\alpha(f^{-1}(V))$.

Now, let f be α -continuous at $x \in X$ and $V \subset Y$ any open set containing $f(x)$. Then, $x \in f^{-1}(V)$ is an α -open set of X . Thus, $x \in \text{Int}_\alpha(f^{-1}(V))$ and therefore $x \notin \text{Fr}_\alpha(f^{-1}(V))$ for every open set V containing $f(x)$. $\square$

DEFINITION 2.16. $\text{Ext}_\alpha(A) = \text{Int}_\alpha(X \setminus A)$ is said to be an α -exterior of A .

THEOREM 2.17. For a subset A of a space X , the following statements hold:

- (1) $\text{Ext}(A) \subset \text{Ext}_\alpha(A)$ where $\text{Ext}(A)$ denotes the exterior of A ;
- (2) $\text{Ext}_\alpha(A)$ is α -open;
- (3) $\text{Ext}_\alpha(A) = \text{Int}_\alpha(X \setminus A) = X \setminus \text{Cl}_\alpha(A)$;
- (4) $\text{Ext}_\alpha(\text{Ext}_\alpha(A)) = \text{Int}_\alpha(\text{Cl}_\alpha(A))$;

- (5) If $A \subset B$, then $\text{Ext}_\alpha(A) \supset \text{Ext}_\alpha(B)$;
- (6) $\text{Ext}_\alpha(A \cup B) \subset \text{Ext}_\alpha(A) \cup \text{Ext}_\alpha(B)$;
- (7) $\text{Ext}_\alpha(A \cap B) \supset \text{Ext}_\alpha(A) \cap \text{Ext}_\alpha(B)$;
- (8) $\text{Ext}_\alpha(X) = \emptyset$;
- (9) $\text{Ext}_\alpha(\emptyset) = X$;
- (10) $\text{Ext}_\alpha(A) = \text{Ext}_\alpha(X \setminus \text{Ext}_\alpha(A))$;
- (11) $\text{Int}_\alpha(A) \subset \text{Ext}_\alpha(\text{Ext}_\alpha(A))$;
- (12) $X = \text{Int}_\alpha(A) \cup \text{Ext}_\alpha(A) \cup \text{Fr}_\alpha(A)$.

PROOF. (4) $\text{Ext}_\alpha(\text{Ext}_\alpha(A)) = \text{Ext}_\alpha(X \setminus \text{Cl}_\alpha(A)) = \text{Int}_\alpha(X \setminus (X \setminus \text{Cl}_\alpha(A))) = \text{Int}_\alpha(\text{Cl}_\alpha(A))$.

(10) $\text{Ext}_\alpha(X \setminus \text{Ext}_\alpha(A)) = \text{Ext}_\alpha(X \setminus \text{Int}_\alpha(X \setminus A)) = \text{Int}_\alpha(X \setminus (X \setminus \text{Int}_\alpha(X \setminus A))) = \text{Int}_\alpha(\text{Int}_\alpha(X \setminus A)) = \text{Int}_\alpha(X \setminus A) = \text{Ext}_\alpha(A)$.

(11) $\text{Int}_\alpha(A) \subset \text{Int}_\alpha(\text{Cl}_\alpha(A)) = \text{Int}_\alpha(X \setminus \text{Int}_\alpha(X \setminus A)) = \text{Int}_\alpha(X \setminus \text{Ext}_\alpha(A)) = \text{Ext}_\alpha(\text{Ext}_\alpha(A))$. $\square$

EXAMPLE 2.18. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{c, d\}, X\}$. Hence, $\tau^\alpha = \{\emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. If $A = \{a\}$ and $B = \{b\}$. Then, $\text{Ext}_\alpha(A) \not\subset \text{Ext}(A)$, $\text{Ext}_\alpha(A \cap B) \neq \text{Ext}_\alpha(A) \cap \text{Ext}_\alpha(B)$, and $\text{Ext}_\alpha(A \cup B) \neq \text{Ext}_\alpha(A) \cup \text{Ext}_\alpha(B)$.

3. A new decomposition of α -continuity

DEFINITION 3.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly α -continuous [3] if, for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \tau^\alpha$ containing x such that $f(U) \subset \text{Cl}(V)$.

THEOREM 3.2 (Noiri [3]). A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly α -continuous if and only if, for every open set V of Y , $f^{-1}(V) \subset \text{Int}_\alpha(f^{-1}(\text{Cl}(V)))$.

The following notion is motivated by the above theorem.

DEFINITION 3.3. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is relatively weakly α -continuous, if for each $x \in X$ and each open set V in Y containing $f(x)$, the set $f^{-1}(V)$ is α -open in the subspace $f^{-1}(\text{Cl}(V))$.

THEOREM 3.4. An α -continuous function is relatively weakly α -continuous.

PROOF. Straightforward. $\square$

The following example shows that the converse of Theorem 3.4 is not true.

EXAMPLE 3.5. Let X be the set of all real numbers, τ the indiscrete topology for X , and σ the discrete topology for X . Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then, f is relatively weakly α -continuous but it is not weakly α -continuous (hence it is not α -continuous) because $\text{Int}_\alpha(f^{-1}(\text{Cl}(V))) = \emptyset$ for any subset V of (X, σ) .

EXAMPLE 3.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$, and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then, f is weakly α -continuous but not relatively weakly α -continuous.

Examples 3.5 and 3.6 show that weakly α -continuous and relatively weakly α -continuous are independent.

The significance of relatively weakly α -continuous is that it yields a decomposition of α -continuous with weakly α -continuous as the other factor.

THEOREM 3.7. *A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is α -continuous if and only if it is weakly α -continuous and relatively weakly α -continuous.*

PROOF. The necessity is given by Theorem 3.4 and by the fact that every α -continuous function is weakly α -continuous.

SUFFICIENCY. Let V be an open set in Y . Since f is relatively weakly α -continuous, we have $f^{-1}(V) = f^{-1}(\text{Cl}(V)) \cap W$, where W is an α -open set of X . Suppose that $x \in f^{-1}(V)$. This means that $f(x) \in V$ and also $x \in W$. By the fact that f is weakly α -continuous, there exists $U \in \tau^\alpha$ containing x such that $f(U) \subset \text{Cl}(V)$. Therefore, $U \subset f^{-1}(\text{Cl}(V))$. We can take U to be a subset of W . It follows that $x \in U \subset f^{-1}(\text{Cl}(V)) \cap W = f^{-1}(V)$ and thus the claim follows. $\square$

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