COMMON PERIODIC POINTS FOR A CLASS OF CONTINUOUS COMMUTING MAPPINGS ON AN INTERVAL

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The existence of common periodic points for a family of continuous commuting self-mappings on an interval is proved and two illustrative examples are given in support of our theorem and definition.

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1. Introduction and preliminaries. All mappings considered here are assumed to be continuous from the interval \( I = [u, v] \) to itself. Let \( F(f) \) and \( P(f) \) be the set of fixed and periodic points of \( f \), respectively, and let \( \overline{P(f)} \) be the closure of \( P(f) \). Denote \( L(x, f) \) by the set of limit points of the sequence \( \{f^n(x)\}_{n=0}^{\infty} \). By Schwartz’s theorem [4], it is easy to show that \( L(x, f) \cap \overline{P(f)} \neq \emptyset \) for each \( x \) in \( I \). Obviously, \( F(f) \) is a closed set and \( \emptyset \neq F(f) \subset P(f) \). Define the classes of mappings

\[
A = \{f : I \to I \mid F(f) = [a_f, b_f], a_f \leq b_f\},
B = \{f : I \to I \mid P(f) = F(f)\},
D = \{f : I \to I \mid P(f) = \overline{P(f)}\}. \tag{1.1}
\]

The following definition was introduced by Cano [2].

DEFINITION 1.1. A class of mappings \( T \) is said to be an \( H \)-class if \( T = T' \cup \{h\} \), where \( T' \) is any subset of \( A \cup B \) composed of commuting mappings and \( h \) is any mapping which commutes with the elements of \( T' \).

Boyce [1] and Huneke [3] showed that if \( f \) and \( g \) are two commuting self-mappings of \( I \), then \( f \) and \( g \) need not have a common fixed point in \( I \). Cano [2] proved the following theorem.

THEOREM 1.2. There is a common fixed point for every \( H \)-class in \( I \).

In this note, we consider a larger class of mappings which has the common periodic point property and properly contains the class \( H \) considered by Cano. Two illustrative examples are given in support of our theorem and definition. We first introduce the following definition.
**Definition 1.3.** A class of mappings $T$ is said to be a $C$-class if $T = T' \cup \{ h \}$ and $T$ is a commuting family of mappings, where $T'$ is any subset of $A \cup D$ and $h$ is any mapping.

Obviously, $B \subset D$. The following example proves that $B$ is a proper subset of $D$.

**Example 1.4.** Let $I = [0,1]$ and $f(x) = 1 - x$. It is easy to show that $F(f) = \{1/2\} \neq [0,1] = P(f) = \overline{P(f)}$, that is, $f \in D$ and $f \not\in B$.

**Remark 1.5.** Clearly, $H$-class is $C$-class, but the converse is not true.

2. **Main results.** Our main result is as follows.

**Theorem 2.1.** There is a common periodic point for every $C$-class in $I$.

**Proof.** Let $T$ be a $C$-class and $T_1$ a finite subset of $T$. We can write $T_1$ as

$$T_1 = \{f_1, f_2, \ldots, f_n\} \cup \{h\} \cup \{g_1, g_2, \ldots, g_m\},$$

(2.1)

where $f_i \in A$, $i = 1, 2, \ldots, n$, and $h$ is a possible arbitrary mapping that commutes with the elements of $T$, $g_j \in D$, $j = 1, 2, \ldots, m$. Suppose that there are different $i, k \in \{1, 2, \ldots, n\}$ such that $F(f_i) \cap F(f_k)$ is not an interval, that is, $F(f_i) \cap F(f_k) = \varnothing$. Let $F(f_i) = [a_i, b_i]$ and $F(f_k) = [a_k, b_k]$. Clearly, $\max\{a_i, a_k\} > \min\{a_i, a_k\}$. Without loss of generality, we can assume $a_k > a_i$. Since $f_i$ and $f_k$ commute and $a_i, b_i \in F(f_i)$, then $f_i(f_k(a_i)) = f_k(f_i(a_i)) = f_k(a_i)$, that is, $f_k(a_i) \in F(f_i)$. Hence, $f_k(a_i) > a_i$. Similarly, we can show that $f_k(b_i) < b_i$. Let $w(x) = f(x) - x$ for $x \in F(f_i)$. Since $w(a_i) > 0$ and $w(b_i) < 0$, there is $c \in (a_i, b_i)$ such that $w(c) = 0$, that is, $f_k(c) = c$. Therefore,

$$c \in (a_i, b_i) \cap F(f_k) \subset F(f_i) \cap F(f_k) \neq \varnothing,$$

(2.2)

a contradiction. Thus, $F(f_i) \cap F(f_k)$ is an interval for any two distinct $i, k \in \{1, 2, \ldots, n\}$. It is easy to show that $\cap_{i=1}^n F(f_i)$ is an interval. Let $\cap_{i=1}^n F(f_i) = [a, b]$. By the commutativity of $h$ with the $f_i$’s, $h$ takes $[a, b]$ into $[a, b]$, and so, it must have a fixed point $z \in [a, b]$. Now, $\{g^n_1(z)\}_{n=0}^\infty$ has a limit point $z_1 \in P(g_1)$ because $P(g_1)$ is a closed set. Clearly, there exists a subsequence $\{g^{n_k}_1(z)\}_{k=1}^\infty$ of $\{g^n_1(z)\}_{n=1}^\infty$ such that

$$\lim_{k \to \infty} g^{n_k}_1(z) = z_1 = g'_1(z_1) \in P(g_1).$$

(2.3)

Since $z \in (\cap_{i=1}^n F(f_i)) \cap F(h)$, by (2.3), we have

$$f_i(g^{n_k}_1(z)) = g^{n_k}_1(f_i(z)) = g^{n_k}_1(z) \to z_1, \quad k \to \infty,$$

$$f_i(g^{n_k}_1(z)) \to f_i(z_1), \quad k \to \infty.$$

(2.4)
From (2.4), we have \( f_i(z_1) \in F(f_i) \). Using the same method, we can show that \( z_1 \in F(h) \). So,

\[
\tag{2.5}
z_1 \in (\bigcap_{i=1}^{n} F(f_i)) \cap F(h) \cap P(g_1).
\]

Similarly, \( \{g_j^n(z_{j-1})\}_{n=0}^{\infty}, j = 2, 3, \ldots, m \), has a limit point

\[
\tag{2.6}
z_j \in (\bigcap_{i=1}^{n} F(f_i)) \cap F(h) \cap (\bigcap_{i=1}^{j} P(g_i)).
\]

Thus,

\[
\tag{2.7}
\emptyset \neq (\bigcap_{i=1}^{n} F(f_i)) \cap F(h) \cap (\bigcap_{j=1}^{m} P(g_j)) \subset \bigcap_{f \in T} P(f)
\]

by the compactness of \( I \). When \( T \) contains no such \( h \), \( T \cap A = \emptyset \), or \( T \cap D = \emptyset \), we have the same result from the above proof. This completes the proof. \( \square \)

We at last give an example in which Theorem 2.1 holds but Theorem 1.2 is not applicable.

**Example 2.2.** Let \( I = [-1, 1] \),

\[
\tag{2.9}
f(x) = \begin{cases} 
1 + x & \text{if } x \in [-1, 0], \\
1 - x & \text{if } x \in (0, 1],
\end{cases}
\]

\[
g(x) = \begin{cases} 
-x & \text{if } x \in [-1, 0], \\
x & \text{if } x \in (0, 1].
\end{cases}
\]

Let \( h \) be a continuous mapping and commute with \( f \) and \( g \). It is easy to see that

\[
\tag{2.10}
F(f) = \left\{ \frac{1}{2} \right\}, \quad P(f) = \overline{P(f)} = [0, 1], \quad F(g) = [0, 1];
\]

that is, \( f \in D, f \not\in B \), and \( g \in A \). Clearly, \( f \) and \( g \) are continuous and

\[
\tag{2.11}
f(g(x)) = g(f(x)) = \begin{cases} 
1 + x & \text{if } x \in [-1, 0], \\
1 - x & \text{if } x \in (0, 1].
\end{cases}
\]

Thus, \( \{f, g, h\} \) is a \( C \)-class but \( \{f, g, h\} \) is not an \( H \)-class. Hence, Theorem 2.1 holds, that is, \( f \), \( g \), and \( h \) have a common periodic point. But Theorem 1.2 is not applicable.

**Remark 2.3.**Example 2.2 and Remark 1.5 prove the greater generality of Theorem 2.1 over Theorem 1.2.
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