EXTENDED BLOCKER, DELETION, AND CONTRACTION MAPS ON ANTICHAINS

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Families of maps on the lattice of all antichains of a finite bounded poset that extend the blocker, deletion, and contraction maps on clutters are considered. Influence of the parameters of the maps is investigated. Order-theoretic extensions of some principal relations for the set-theoretic blocker, deletion, and contraction maps on clutters are presented.

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1. Introduction and preliminary. Let P be a finite bounded poset of cardinality greater than one. We can define some maps on the lattice of all antichains $\mathcal{A}(P)$ of the poset P that naturally extend the set-theoretic blocker, deletion, and contraction maps on clutters; such maps were considered in [4, 5].

A set H is called a *blocking set* for a nonempty family $\mathcal{G} = \{G_1, \dots, G_m\}$ of nonempty subsets of a finite set if, for each $k \in \{1, \dots, m\}$, it holds $|H \cap G_k| > 0$. The family of all inclusionwise minimal blocking sets for $\mathcal{G}$ is called the *blocker* of $\mathcal{G}$. We denote the blocker of $\mathcal{G}$ by $\mathcal{B}(\mathcal{G})$.

A family of subsets of a finite *ground set* S is called a *clutter* or a *Sperner family* if no set from that family contains another. The empty clutter $\emptyset$ containing no subsets of S and the clutter $\{\hat{0}\}$ whose unique set is the empty subset $\hat{0}$ of S are called the *trivial clutters* on S . The set-theoretic *blocker map* reflects a nontrivial clutter to its blocker, and that map reflects a trivial clutter to the other trivial clutter: $\mathcal{B}(\emptyset) = \{\hat{0}\}$ and $\mathcal{B}(\{\hat{0}\}) = \emptyset$.

Let $X \subseteq S$ and $|X| > 0$. The set-theoretic *deletion* $(\backslash X)$ and *contraction* $(/X)$ maps are defined in the following way: if $\mathcal{G}$ is a nontrivial clutter on S , then the *deletion* $\mathcal{G} \backslash X$ is the family $\{G \in \mathcal{G} : |G \cap X| = 0\}$ and the *contraction* $\mathcal{G}/X$ is the family of all inclusionwise minimal sets from the family $\{G - X : G \in \mathcal{G}\}$. The *deletion* and *contraction* for the trivial clutters coincide with the clutters $\emptyset \backslash X = \emptyset/X = \emptyset$ and $\{\hat{0}\} \backslash X = \{\hat{0}\}/X = \{\hat{0}\}$. The maps $(\backslash \hat{0})$ and $(/\hat{0})$ are the identity map on clutters; for any clutter $\mathcal{G}$, we by definition have $\mathcal{G} \backslash \hat{0} = \mathcal{G}/\hat{0} = \mathcal{G}$.

Let $\mathcal{G}$ be a clutter on the ground set S . Given a subset $X \subseteq S$, we have

$$\mathcal{B}(\mathcal{B}(\mathcal{G})) = \mathcal{G}, \tag{1.1}$$

$$\mathcal{B}(\mathcal{G}) \backslash X = \mathcal{B}(\mathcal{G}/X), \quad \mathcal{B}(\mathcal{G})/X = \mathcal{B}(\mathcal{G} \backslash X). \tag{1.2}$$

Recall that the atoms of the poset P are the elements covering its least element. Let X be a subset of the atom set P^a of P . (We denote the empty subset of P^a by $\emptyset^a$.) We use the denotation $\mathfrak{b} : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$ for the order-theoretic blocker map from [4], and we use the denotations $(\backslash X), (/X) : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$ for the order-theoretic operators of deletion and contraction from [5], respectively. We do not recall those concepts here because the map $\mathfrak{b}$ is the $(\emptyset^a, 0)$ -blocker map from Definition 2.1 of the present paper and the maps $(\backslash X)$ and $(/X)$ are the $(X, 0)$ -deletion and $(X, 0)$ -contraction maps from Definition 3.1 of the present paper, respectively.

For any antichain A of P , the following relations hold in $\mathfrak{A}(P)$:

$$\mathfrak{b}(\mathfrak{b}(\mathfrak{b}(A))) = \mathfrak{b}(A), \quad (1.3)$$

$$\mathfrak{b}(A) \backslash X \leq \mathfrak{b}(A/X) \leq \mathfrak{b}(A) \leq \mathfrak{b}(A)/X \leq \mathfrak{b}(A \backslash X). \quad (1.4)$$

Equality (1.3) from [4] goes back to (1.1) from [2, 3]. Comparison (1.4) from [5] goes back to (1.2) from [6].

In the present paper, we consider families of the so-called (X, k) -blocker, (X, k) -deletion, and (X, k) -contraction maps on $\mathfrak{A}(P)$ parametrized by subsets $X \subseteq P^a$ and numbers $k \in \mathbb{N}$, $k < |P^a|$. We show that for all pairs of the above-mentioned parameters X and k , the essential properties of the maps remain similar to those of the $(\emptyset^a, 0)$ -blocker, $(X, 0)$ -deletion, and $(X, 0)$ -contraction maps on $\mathfrak{A}(P)$ that were investigated in [4, 5]. In particular, we present analogues of relations (1.3) and (1.4) in Proposition 2.6(ii) and Theorem 3.7.

We refer the reader to [7, Chapter 3] for basic information and terminology in the theory of posets.

We use $\min Q$ to denote the set of all minimal elements of a poset Q . If Q has a least element, then it is denoted $\hat{0}_Q$; if Q has a greatest element, then it is denoted $\hat{1}_Q$.

Throughout the paper, P stands for a finite bounded poset of cardinality greater than one, that is, P by definition has the least and greatest elements that are distinct. We denote by $\mathfrak{I}(A)$ and $\mathfrak{f}(A)$ the order ideal and filter of P generated by an antichain A , respectively.

All antichains of P compose a distributive lattice denoted $\mathfrak{A}(P)$; in the present paper, antichains are by definition partially ordered in the following way; if $A', A'' \in \mathfrak{A}(P)$, then we set

$$A' \leq A'' \quad \text{iff} \quad \mathfrak{f}(A') \subseteq \mathfrak{f}(A''). \quad (1.5)$$

We call the least and greatest elements $\hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}$ of $\mathfrak{A}(P)$ the *trivial antichains* of P because, in the context of the present paper, they are counterparts of the trivial clutters. Here, $\hat{0}_{\mathfrak{A}(P)}$ is the empty antichain of P and $\hat{1}_{\mathfrak{A}(P)}$ the one-element antichain $\{\hat{0}_P\}$. We denote by $\vee$ and $\wedge$ the operations of join and meet

in the lattice $\mathfrak{A}(P)$; if $A', A'' \in \mathfrak{A}(P)$, then

$$\begin{aligned} A' \vee A'' &= \min(A' \cup A''), \\ A' \wedge A'' &= \min(\mathfrak{f}(A') \cap \mathfrak{f}(A'')). \end{aligned} \quad (1.6)$$

2. (X, k) -blocker map. In this section, we consider a family of maps on antichains of a finite bounded poset that extend the set-theoretic blocker map on clutters. From now on, X is always a subset of P^a and k is a nonnegative integer less than $|P^a|$.

DEFINITION 2.1. The (X, k) -blocker map on $\mathfrak{A}(P)$ is the map $\mathfrak{b}_k^X : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$,

$$A \mapsto \min\{b \in P : |\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap (P^a - X)| > k \ \forall a \in A\} \quad (2.1)$$

if A is nontrivial, and

$$\hat{0}_{\mathfrak{A}(P)} \mapsto \hat{1}_{\mathfrak{A}(P)}, \quad \hat{1}_{\mathfrak{A}(P)} \mapsto \hat{0}_{\mathfrak{A}(P)}. \quad (2.2)$$

Given an antichain $A \in \mathfrak{A}(P)$, the antichain $\mathfrak{b}_k^X(A)$ is the (X, k) -blocker of A in P .

We use the denotations $\mathfrak{b}_k$ and $\mathfrak{b}^X$ instead of the denotations $\mathfrak{b}_k^{\emptyset^a}$ and $\mathfrak{b}_0^X$, respectively. The $(\emptyset^a, 0)$ -blocker map is the blocker map $\mathfrak{b}$ on $\mathfrak{A}(P)$ considered in [4]. Given $A \in \mathfrak{A}(P)$, the antichain $\mathfrak{b}(A)$ is called the blocker of A in P .

If $\{a\}$ is a one-element antichain of P , then we write $\mathfrak{b}_k^X(a)$ instead of $\mathfrak{b}_k^X(\{a\})$. Let $a \neq \hat{0}_P$. Since the blocker map on $\mathfrak{A}(P)$ is antitone, for every $E \subseteq \mathfrak{b}(a) - X$, we have $\{a\} \leq \mathfrak{b}(\mathfrak{b}(a)) \leq \mathfrak{b}(\mathfrak{b}(a) - X) \leq \mathfrak{b}(E) \leq \mathfrak{b}(a)$.

The following statement immediately follows from Definition 2.1.

LEMMA 2.2. Let A be a nontrivial antichain of P . If $\mathfrak{b}_k^X(A) \neq \hat{0}_{\mathfrak{A}(P)}$, then, for each $a \in A$ and for all $b \in \mathfrak{b}_k^X(A)$, it holds that

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^a - X)| > k. \quad (2.3)$$

Let $a \in P$, $a \neq \hat{0}_P$. From now on, $\mathcal{T}_a$ denotes the family of subsets of the atom set P^a defined as follows:

$$\mathcal{T}_a = \{E \subseteq \mathfrak{b}(a) - X : |E| = k + 1\}. \quad (2.4)$$

Let $\mathcal{L}(P^a)$ denote the Boolean lattice of all subsets of the atom set P^a , and let $\mathcal{L}(P^a)^{(k+1)}$ denote the subset of all elements of rank $k + 1$ of $\mathcal{L}(P^a)$. Given a $(k + 1)$ -subset $E \subseteq P^a$, we denote by $\varepsilon(E)$ the least upper bound for E in $\mathcal{L}(P^a)$; conversely, given an element $\mathfrak{e} \in \mathcal{L}(P^a)^{(k+1)}$, we denote by $\varepsilon^{-1}(\mathfrak{e})$ the $(k + 1)$ -subset of all atoms of $\mathcal{L}(P^a)$ that are comparable with $\mathfrak{e}$.

Let A be a nontrivial antichain of P . If $|\mathfrak{b}(a) - X| \leq k$ for some $a \in A$, then [Definition 2.1](#) implies $\mathfrak{b}_k^X(A) = \hat{0}_{\mathfrak{A}(P)}$. In the case $|\mathfrak{b}(a) - X| > k$ for all $a \in A$, [Proposition 2.3](#) describes two alternative ways of elementwise finding the (X, k) -blocker of A ; it involves the set-theoretic blocker $\mathfrak{B}(\cdot)$ of a set family.

PROPOSITION 2.3. *Let A be a nontrivial antichain of P . If $|\mathfrak{b}(a) - X| > k$, for all $a \in A$, then*

$$\mathfrak{b}_k^X(A) = \bigwedge_{a \in A} \bigvee_{E \in \mathcal{T}_a} \mathfrak{b}(E) = \bigvee_{\mathfrak{e} \in \mathfrak{B}(\{\{\varepsilon(E): E \in \mathcal{T}_a\}: a \in A\})} \bigwedge \mathfrak{b}(\varepsilon^{-1}(\mathfrak{e})). \quad (2.5)$$

PROOF. We have

$$\mathfrak{b}_k^X(A) = \bigwedge_{a \in A} \mathfrak{b}_k^X(a), \quad (2.6)$$

and an order-theoretic argument shows that, for every $a \in A$, it holds that

$$\mathfrak{b}_k^X(a) = \bigvee_{E \in \mathcal{T}_a} \mathfrak{b}(E), \quad (2.7)$$

where $\mathfrak{b}(E) = \bigwedge_{e \in E} \{e\}$.

The inclusion $\mathfrak{b}_k^X(A) \supseteq \bigvee_{\mathfrak{e} \in \mathfrak{B}(\{\{\varepsilon(E): E \in \mathcal{T}_a\}: a \in A\})} \bigwedge_{\mathfrak{e} \in \mathfrak{e}} \mathfrak{b}(\varepsilon^{-1}(\mathfrak{e}))$ follows from [Definition 2.1](#). To prove the inclusion

$$\mathfrak{b}_k^X(A) \subseteq \bigvee_{\mathfrak{e} \in \mathfrak{B}(\{\{\varepsilon(E): E \in \mathcal{T}_a\}: a \in A\})} \bigwedge_{\mathfrak{e} \in \mathfrak{e}} \mathfrak{b}(\varepsilon^{-1}(\mathfrak{e})), \quad (2.8)$$

assume that it does not hold. Consider an element $b \in \mathfrak{b}_k^X(A)$ such that it does not belong to the right-hand side of (2.8). In this case, there is an element $a \in A$ such that $|\mathcal{I}(b) \cap \mathcal{I}(a) \cap (P^a - X)| \leq k$. It means that the left-hand side of (2.8) is not an (X, k) -blocker of A , a contradiction. $\square$

The following lemma clarifies how the parameters of the (X, k) -blocker map influence the image of $\mathfrak{A}(P)$; additionally, the lemma states that $\mathfrak{b}_k^X$ is antitone.

LEMMA 2.4. (i) *Let $Y \subseteq P^a$, $Y \supseteq X$, and let j be a nonnegative integer, $j \leq k$. If $A \in \mathfrak{A}(P)$, then*

$$\mathfrak{b}_j^X(A) \geq \mathfrak{b}_k^X(A) \geq \mathfrak{b}_k^Y(A). \quad (2.9)$$

(ii) *For all $A', A'' \in \mathfrak{A}(P)$ such that $A' \leq A''$, it holds that*

$$\mathfrak{b}_k^X(A') \geq \mathfrak{b}_k^X(A''). \quad (2.10)$$

PROOF. (i) There is nothing to prove if A is trivial. Suppose that A is a nontrivial antichain of P . For each element $a \in A$, we by (2.7) have

$$\mathfrak{b}_k^X(a) = \bigvee_{E \in \mathcal{T}_a} \mathfrak{b}(E) \geq \bigvee_{\substack{E \subseteq \mathfrak{b}(a) - Y: \\ |E| = k+1}} \mathfrak{b}(E) = \mathfrak{b}_k^Y(a). \quad (2.11)$$

With respect to (2.6), this yields

$$\mathfrak{b}_k^X(A) = \bigwedge_{a \in A} \mathfrak{b}_k^X(a) \geq \bigwedge_{a \in A} \mathfrak{b}_k^Y(a) = \mathfrak{b}_k^Y(A). \quad (2.12)$$

The relation $\mathfrak{b}_j^X(A) \geq \mathfrak{b}_k^X(A)$ is proved in a similar way.

(ii) If A' is a trivial antichain, then the assertion immediately follows from Definition 2.1. Suppose that A' is nontrivial. For every $a' \in A'$, there is $a'' \in A''$ such that $\{a'\} \leq \{a''\}$ and, as a consequence, it holds the inclusion $\mathfrak{b}(a') \supseteq \mathfrak{b}(a'')$, (2.7) implies $\mathfrak{b}_k^X(a') \geq \mathfrak{b}_k^X(a'')$, and the proof is completed by applying (2.6). $\square$

In addition to Lemma 2.4(ii), we need the following statement to describe the structure of the image of $\mathfrak{A}(P)$ under the (X, k) -blocker map.

LEMMA 2.5. *For any $A \in \mathfrak{A}(P)$, it holds that*

$$\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) \geq A. \quad (2.13)$$

PROOF. If A is a trivial antichain of P , then the lemma follows from Definition 2.1 because, in this case, we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = A$. Suppose that A is nontrivial. If $\mathfrak{b}_k^X(A) = \hat{0}_{\mathfrak{A}(P)}$, then we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = \hat{1}_{\mathfrak{A}(P)} \geq A$ and we are done. Finally, suppose that $\mathfrak{b}_k^X(A)$ is a nontrivial antichain. On the one hand, according to Lemma 2.2, for each $a \in A$ and for all $b \in \mathfrak{b}_k^X(A)$, it holds that

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^a - X)| > k. \quad (2.14)$$

On the other hand, we, by Definition 2.1, have

$$\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = \min \{g \in P : |\mathfrak{I}(g) \cap \mathfrak{I}(b) \cap (P^a - X)| > k \ \forall b \in \mathfrak{b}_k^X(A)\}. \quad (2.15)$$

Hence, we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) \geq A$. $\square$

We complete this section by applying a standard technique of the theory of posets to the lattice $\mathfrak{A}(P)$ and the (X, k) -blocker map on it. See, for instance, [1, Chapter IV] on (co)closure operators.

PROPOSITION 2.6. (i) *The composite map $\mathfrak{b}_k^X \circ \mathfrak{b}_k^X$ is a closure operator on $\mathfrak{A}(P)$.*

(ii) *The poset $\mathfrak{B}_k^X(P) = \{\mathfrak{b}_k^X(A) : A \in \mathfrak{A}(P)\}$ is a self-dual lattice; the restriction map $\mathfrak{b}_k^X|_{\mathfrak{B}_k^X(P)}$ is an anti-automorphism of $\mathfrak{B}_k^X(P)$. The lattice $\mathfrak{B}_k^X(P)$ is a meet-subsemilattice of the lattice $\mathfrak{A}(P)$.*

(iii) *For every $B \in \mathfrak{B}_k^X(P)$, its preimage $(\mathfrak{b}_k^X)^{-1}(B)$ under the (X, k) -blocker map is a convex join-subsemilattice of the lattice $\mathfrak{A}(P)$. The greatest element of $(\mathfrak{b}_k^X)^{-1}(B)$ is $\mathfrak{b}_k^X(B)$.*

PROOF. In view of Lemmas 2.4(ii) and 2.5, assertions (i) and (ii) are a corollary of [1, Propositions 4.36 and 4.26]. To prove (iii), choose arbitrary elements $A', A'' \in (\mathfrak{b}_k^X)^{-1}(B)$, where $B = \mathfrak{b}_k^X(A)$ for some $A \in \mathfrak{A}(P)$, and note that $\mathfrak{b}_k^X(A' \vee A'') = \mathfrak{b}_k^X(A') \wedge \mathfrak{b}_k^X(A'') = B$. If $B = \hat{0}_{\mathfrak{A}(P)}$, then $\mathfrak{b}_k^X(B) = \hat{1}_{\mathfrak{A}(P)}$ is the greatest element of $(\mathfrak{b}_k^X)^{-1}(B)$. If $B = \hat{1}_{\mathfrak{A}(P)}$, then $(\mathfrak{b}_k^X)^{-1}(B)$ is the one-element subposet $\{\hat{0}_{\mathfrak{A}(P)}\}$ of $\mathfrak{A}(P)$. Finally, if B is a nontrivial antichain of P , then the element $\mathfrak{b}_k^X(B) = \mathfrak{b}_k^X(\mathfrak{b}_k^X(A))$ is by (2.15) the greatest element of $(\mathfrak{b}_k^X)^{-1}(B)$. Since the (X, k) -blocker map is antitone, we can see that the subposet $(\mathfrak{b}_k^X)^{-1}(B)$ of $\mathfrak{A}(P)$ is convex. $\square$

We call the poset $\mathfrak{B}_k^X(P)$ from Proposition 2.6(ii) the *lattice of (X, k) -blockers* in P . The poset $\mathfrak{B}(P) = \mathfrak{B}_0^{\varnothing^a}(P)$ is called in [4] the *lattice of blockers* in P .

3. (X, k) -deletion and (X, k) -contraction maps. In this section, we consider order-theoretic extensions of the set-theoretic deletion and contraction maps on clutters.

DEFINITION 3.1. (i) If $\{a\}$ is a nontrivial one-element antichain of P , then the (X, k) -deletion $\{a\} \setminus_k X$ and (X, k) -contraction $\{a\} /_k X$ of $\{a\}$ in P are the antichains

$$\{a\} \setminus_k X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| \leq k, \\ \hat{0}_{\mathfrak{A}(P)}, & \text{if } |\mathfrak{b}(a) \cap X| > k, \end{cases} \quad (3.1)$$

$$\{a\} /_k X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| \leq k, \\ \mathfrak{b}_k^X(\mathfrak{b}_k^X(a)), & \text{if } |\mathfrak{b}(a) \cap X| > k, \mathfrak{b}(a) \not\subseteq X, \\ \hat{1}_{\mathfrak{A}(P)}, & \text{if } |\mathfrak{b}(a) \cap X| > k, \mathfrak{b}(a) \subseteq X. \end{cases} \quad (3.2)$$

(ii) If A is a nontrivial antichain of P , then the (X, k) -deletion $A \setminus_k X$ and (X, k) -contraction $A /_k X$ of A in P are the antichains

$$A \setminus_k X = \bigvee_{a \in A} (\{a\} \setminus_k X), \quad A /_k X = \bigvee_{a \in A} (\{a\} /_k X). \quad (3.3)$$

(iii) The (X, k) -deletion and (X, k) -contraction of the trivial antichains of P are

$$\begin{aligned}\hat{0}_{\mathfrak{A}(P)} \setminus_k X &= \hat{0}_{\mathfrak{A}(P)} /_k X = \hat{0}_{\mathfrak{A}(P)}, \\ \hat{1}_{\mathfrak{A}(P)} \setminus_k X &= \hat{1}_{\mathfrak{A}(P)} /_k X = \hat{1}_{\mathfrak{A}(P)}.\end{aligned}\tag{3.4}$$

(iv) The map

$$(\setminus_k X) : \mathfrak{A}(P) \longrightarrow \mathfrak{A}(P), \quad A \longmapsto A \setminus_k X,\tag{3.5}$$

is the *operator of (X, k) -deletion* on $\mathfrak{A}(P)$.

The map

$$(/_k X) : \mathfrak{A}(P) \longrightarrow \mathfrak{A}(P), \quad A \longmapsto A /_k X,\tag{3.6}$$

is the *operator of (X, k) -contraction* on $\mathfrak{A}(P)$.

Given an antichain $A \in \mathfrak{A}(P)$, we use the denotations $A \setminus X$ and A / X instead of the denotations $A \setminus_0 X$ and $A /_0 X$, respectively. The $(X, 0)$ -deletion map $(\setminus X) : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$ and the $(X, 0)$ -contraction map $(/ X) : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$ are the *operators of deletion* and *contraction* on $\mathfrak{A}(P)$, respectively, considered in [5].

The following observation is an immediate consequence of [Definition 3.1](#). If $a', a'' \in P$ and $\{a'\} \leq \{a''\}$ in $\mathfrak{A}(P)$, then

$$\{a'\} \setminus_k X \leq \{a''\} \setminus_k X, \quad \{a'\} /_k X \leq \{a''\} /_k X;\tag{3.7}$$

hence, in view of (3.3) and (3.4), we can formulate the following lemma.

LEMMA 3.2. *If $A', A'' \in \mathfrak{A}(P)$ and $A' \leq A''$, then*

$$A' \setminus_k X \leq A'' \setminus_k X, \quad A' /_k X \leq A'' /_k X.\tag{3.8}$$

Moreover, if $\{a\}$ is a one-element antichain of P , then we have

$$\{a\} \setminus_k X \leq \{a\} \leq \{a\} /_k X,\tag{3.9}$$

and a more general statement is true.

LEMMA 3.3. *If $A \in \mathfrak{A}(P)$, then*

$$A \setminus_k X \leq A \leq A /_k X.\tag{3.10}$$

Another consequence of [Definition 3.1](#) is that, for a one-element antichain $\{a\}$ of P , it holds that

$$\mathfrak{b}_k^X(a) \setminus_k X \leq \mathfrak{b}_k^X(\{a\} /_k X) \leq \mathfrak{b}_k^X(a) \leq \mathfrak{b}_k^X(a) /_k X \leq \mathfrak{b}_k^X(\{a\} \setminus_k X).\tag{3.11}$$

Let $\{a\}$ be a nontrivial one-element antichain of P . We obviously have $(\{a\} \setminus_k X) \setminus_k X = \{a\} \setminus_k X$. We show that $(\{a\} /_k X) /_k X = \{a\} /_k X$. If $|\hat{b}(a) \cap X| \leq k$, then [Definition 3.1](#) implies $(\{a\} /_k X) /_k X = \{a\} /_k X$; further, if $|\hat{b}(a) \cap X| > k$ and $\hat{b}(a) \subseteq X$, then [Definition 3.1](#) implies $(\{a\} /_k X) /_k X = \{a\} /_k X = \hat{1}_{\mathfrak{A}(P)}$. Suppose that $|\hat{b}(a) \cap X| > k$ and $\hat{b}(a) \not\subseteq X$. In this case, on the one hand, we have $(\{a\} /_k X) /_k X \geq \{a\} /_k X$ by [Lemma 3.3](#), on the other hand, for every element $b \in \{a\} /_k X = \hat{b}_k^X(\hat{b}_k^X(a))$, we have $\hat{b}_k^X(b) \geq \hat{b}_k^X(a)$, and, as a consequence, we have $(\{a\} /_k X) /_k X = \bigvee_{b \in \{a\} /_k X} (\{b\} /_k X) \leq \hat{b}_k^X(\hat{b}_k^X(a)) = \{a\} /_k X$. We arrive at the conclusion that $(\{a\} /_k X) /_k X = \{a\} /_k X$. With respect to (3.3), we can formulate the following lemma.

LEMMA 3.4. *If $A \in \mathfrak{A}(P)$, then*

$$(A \setminus_k X) \setminus_k X = A \setminus_k X, \quad (A /_k X) /_k X = A /_k X. \quad (3.12)$$

Lemmas 3.2, 3.3, and 3.4 lead to a characterization of the (X, k) -deletion and (X, k) -contraction maps in terms of (co)closure operators.

PROPOSITION 3.5. *The map $(\setminus_k X)$ is a coclosure operator on $\mathfrak{A}(P)$. The map $(/_k X)$ is a closure operator on $\mathfrak{A}(P)$.*

The following proposition is a counterpart of [Lemma 2.4\(i\)](#).

PROPOSITION 3.6. *Let $Y \subseteq P^a$, $Y \supseteq X$, and let m be an integer, $k \leq m < |P^a|$. If $A \in \mathfrak{A}(P)$, then*

$$\begin{aligned} A \setminus_m X &\geq A \setminus_k X \geq A \setminus_k Y, \\ A /_k X &\leq A /_k Y \leq A /_m Y. \end{aligned} \quad (3.13)$$

PROOF. If A is a trivial antichain, then the proposition follows from (3.4). Suppose that A is nontrivial. For each $a \in A$, (3.1) implies $\{a\} \setminus_k X \geq \{a\} \setminus_k Y$, (3.2) implies $\{a\} /_k X \leq \{a\} /_k Y$, and (3.3) yields

$$\begin{aligned} A \setminus_k X &= \bigvee_{a \in A} (\{a\} \setminus_k X) \geq \bigvee_{a \in A} (\{a\} \setminus_k Y) = A \setminus_k Y, \\ A /_k X &= \bigvee_{a \in A} (\{a\} /_k X) \leq \bigvee_{a \in A} (\{a\} /_k Y) = A /_k Y. \end{aligned} \quad (3.14)$$

Other relations are proved in a similar way. $\square$

We denote the images $(\setminus_k X)(\mathfrak{A}(P)) = \{A \setminus_k X : A \in \mathfrak{A}(P)\}$ and $(/_k X)(\mathfrak{A}(P)) = \{A /_k X : A \in \mathfrak{A}(P)\}$ by $\mathfrak{A}(P) \setminus_k X$ and $\mathfrak{A}(P) /_k X$, respectively. We can interpret well-known properties of (semi)lattice maps and (co)closure operators on lattices in the case of the (X, k) -deletion and (X, k) -contraction maps.

Definition 3.1 implies that the maps $(\backslash_k X), (/_k X) : \mathfrak{A}(P) \rightarrow \mathfrak{A}(P)$ are upper $\{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$ -homomorphisms, that is, for all $A', A'' \in \mathfrak{A}(P)$, we have $(A' \vee A'') \backslash_k X = (A' \backslash_k X) \vee (A'' \backslash_k X)$ and $(A' \vee A'') /_k X = (A' /_k X) \vee (A'' /_k X)$, and, moreover, we have $\hat{0}_{\mathfrak{A}(P)} \backslash_k X = \hat{0}_{\mathfrak{A}(P)} /_k X = \hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)} \backslash_k X = \hat{1}_{\mathfrak{A}(P)} /_k X = \hat{1}_{\mathfrak{A}(P)}$.

The posets $\mathfrak{A}(P) \backslash_k X$ and $\mathfrak{A}(P) /_k X$, with the partial orders induced by the partial order on $\mathfrak{A}(P)$, are lattices.

We call the poset $\mathfrak{A}(P) \backslash_k X$ the *lattice of (X, k) -deletions* in P , and we call the poset $\mathfrak{A}(P) /_k X$ the *lattice of (X, k) -contractions* in P .

The lattice $\mathfrak{A}(P) \backslash_k X$ is a join-subsemilattice of $\mathfrak{A}(P)$. Denote by $\wedge_{\mathfrak{A}(P) \backslash_k X}$ the operation of meet in $\mathfrak{A}(P) \backslash_k X$. If $D', D'' \in \mathfrak{A}(P) \backslash_k X$, then we have $D' \wedge_{\mathfrak{A}(P) \backslash_k X} D'' = (D' \wedge D'') \backslash_k X$.

The lattice $\mathfrak{A}(P) /_k X$ is a sublattice of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P) \backslash_k X$, then the preimage $(\backslash_k X)^{-1}(D)$ of D under the (X, k) -deletion map is the closed interval $[D, D \vee \bigvee_{E \subseteq X: |E|=k+1} \hat{6}(E)]$ of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P) /_k X$, then the preimage $(/_k X)^{-1}(D)$ of D under the (X, k) -contraction map is a convex join-subsemilattice of the lattice $\mathfrak{A}(P)$, with the greatest element D .

Relations (1.2) and (1.4) have the following analogue.

THEOREM 3.7. *If $A \in \mathfrak{A}(P)$, then*

$$\hat{6}_k^X(A) \backslash_k X \leq \hat{6}_k^X(A /_k X) \leq \hat{6}_k^X(A) \leq \hat{6}_k^X(A) /_k X \leq \hat{6}_k^X(A \backslash_k X). \quad (3.15)$$

PROOF. There is nothing to prove if A is a trivial antichain. Suppose that A is nontrivial. The relations

$$\hat{6}_k^X(A) \backslash_k X \leq \hat{6}_k^X(A) \leq \hat{6}_k^X(A) /_k X, \quad \hat{6}_k^X(A /_k X) \leq \hat{6}_k^X(A) \leq \hat{6}_k^X(A \backslash_k X) \quad (3.16)$$

follow from Lemmas 3.3 and 2.4(ii).

We need the following auxiliary relations. If A' and A'' are arbitrary antichains of P , then

$$(A' \wedge A'') \backslash_k X \leq (A' \backslash_k X) \wedge (A'' \backslash_k X), \quad (3.17)$$

$$(A' \wedge A'') /_k X \leq (A' /_k X) \wedge (A'' /_k X). \quad (3.18)$$

To prove $\hat{6}_k^X(A) \backslash_k X \leq \hat{6}_k^X(A /_k X)$, we use (3.17) and (3.11), and we see that

$$\begin{aligned} \hat{6}_k^X(A) \backslash_k X &= \left(\bigwedge_{a \in A} \hat{6}_k^X(a) \right) \backslash_k X \leq \bigwedge_{a \in A} (\hat{6}_k^X(a) \backslash_k X) \leq \bigwedge_{a \in A} \hat{6}_k^X(\{a\} /_k X) \\ &= \hat{6}_k^X \left(\bigvee_{a \in A} (\{a\} /_k X) \right) = \hat{6}_k^X(A /_k X). \end{aligned} \quad (3.19)$$

To prove $\mathfrak{b}_k^X(A)/_kX \leq \mathfrak{b}_k^X(A \setminus_k X)$, we use (3.18) and (3.11), and we see that

$$\begin{aligned} \mathfrak{b}_k^X(A)/_kX &= \left(\bigwedge_{a \in A} \mathfrak{b}_k^X(a) \right) /_kX \leq \bigwedge_{a \in A} (\mathfrak{b}_k^X(a) /_kX) \leq \bigwedge_{a \in A} \mathfrak{b}_k^X(\{a\} \setminus_k X) \\ &= \mathfrak{b}_k^X \left(\bigvee_{a \in A} (\{a\} \setminus_k X) \right) = \mathfrak{b}_k^X(A \setminus_k X). \end{aligned} \tag{3.20}$$

□

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