ON A CLASS OF HOLOMORPHIC FUNCTIONS DEFINED BY THE RUSCHEWEYH DERIVATIVE

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Received 23 September 2002

By using the Ruscheweyh operator $D_m f(z)$, $z \in U$, we will introduce a class of holomorphic functions, denoted by $M_m^m(\alpha)$, and obtain some inclusion relations.

2000 Mathematics Subject Classification: 30C45.

1. Introduction and preliminaries. Denote by $U$ the unit disc of the complex plane

$$U = \{z \in \mathbb{C}; |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in $U$. We let

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \cdots, z_1 \in U\}$$

with $A_1 = A$.

We let $\mathcal{H}[a,n]$ denote the class of analytic functions in $U$ of the form

$$f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots, z \in U.$$

If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function $w$ analytic in $U$, with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $K = \{f \in A: \text{Re}(zf''(z)/f'(z)) + 1 > 0, z \in U\}$ denote the class of normalized convex functions in $U$. We use the following subordination results.

**Lemma 1.1** (Miller and Mocanu [2, page 71]). Let $h$ be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex with $\text{Re} \gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z),$$

then

$$f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots, z \in U.$$
then \( p(z) < g(z) < h(z) \), where

\[
g(z) = \frac{y}{nz^{y/n}} \int_0^z \frac{h(t) \cdot t^{(y/n)-1}}{t} \, dt.
\]

(1.5)

The function \( g \) is convex and is the best \((a, n)\) dominant.

**Lemma 1.2** (Miller and Mocanu [1]). Let \( g \) be a convex function in \( U \) and let

\[
h(z) = g(z) + n\alpha z g'(z),
\]

(1.6)

where \( \alpha > 0 \) and \( n \) is a positive integer. If \( p(z) = g(0) + p_n z^n + \cdots \) is holomorphic in \( U \) and

\[
p(z) + \alpha z p'(z) < h(z),
\]

(1.7)

then

\[
p(z) < g(z)
\]

(1.8)

and this result is sharp.

**Definition 1.3** [4]. For \( f \in A \) and \( m \geq 0 \), the operator \( D^m f \) is defined by

\[
D^m f(z) = f(z) \ast \frac{z}{(1-z)^{m+1}} = \frac{z}{m!} [z^{m-1} f(z)]^{(m)}, \quad z \in U,
\]

(1.9)

where \( \ast \) stands for convolution.

**Remark 1.4.** We have

\[
D^0 f(z) = f(z), \quad z \in U,
\]

\[
D^1 f(z) = zf'(z), \quad z \in U,
\]

\[
2D^2 f(z) = z \cdot [D^1 f(z)]' + D^1 f(z),
\]

\[
(m+1)D^{m+1} f(z) = z[D^m f(z)]' + mD^m f(z).
\]

(1.10)

2. Main results

**Definition 2.1.** If \( \alpha < 1 \) and \( m, n \in \mathbb{N} \), let \( M^m_n (\alpha) \) denote the class of functions \( f \in A_n \) which satisfy the inequality

\[
\text{Re} (D^m f)'(z) > \alpha.
\]

(2.1)

**Theorem 2.2.** If \( \alpha < 1 \) and \( m, n \in \mathbb{N} \), then

\[
M^{m+1}_n (\alpha) \subset M^m_n (\delta),
\]

(2.2)
where
\[
\delta = \delta(\alpha, n, m) = 2\alpha - 1 + 2 \cdot (1 - \alpha) \cdot \frac{m+1}{n} \beta \left( \frac{m+1}{n} \right),
\]
\[
\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt.
\] (2.3)

**Proof.** Let \( f \in M^{m+1}_n(\alpha) \). By using the properties of the operator \( D^m f(z) \), we have
\[
(m + 1)D^{m+1} f(z) = z \cdot (D^m f)'(z) + mD^m f(z), \quad z \in U.
\] (2.4)

Differentiating (2.4), we obtain
\[
(m + 1)[D^{m+1} f(z)]' = z \cdot (D^m f)''(z) + (D^m f)'(z) + m(D^m f)'(z)
\]
\[
= z(D^m f)''(z) + (m + 1)(D^m f)'(z).
\] (2.5)

If we let \( p(z) = (D^m f)'(z) \), then \( p'(z) = (D^m f)''(z) \) and (2.4) becomes
\[
[D^{m+1} f(z)]' = p(z) + \frac{1}{m+1} z \cdot p'(z).
\] (2.6)

Since \( f \in M^{m+1}_n(\alpha) \), by using Definition 2.1, we have
\[
\Re \left[ p(z) + \frac{1}{m+1} z p'(z) \right] > \alpha
\] (2.7)

which is equivalent to
\[
p(z) + \frac{1}{m+1} z p'(z) < \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z).
\] (2.8)

By using Lemma 1.1, we have
\[
p(z) < g(z) < h(z),
\] (2.9)

where
\[
g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z \frac{1 + (2\alpha - 1)t}{1+t} \cdot t^{(m+1)/n-1} dt.
\] (2.10)

The function \( g \) is convex and is the best dominant.

From \( p(z) < g(z) \), it results that
\[
\Re p(z) > \delta = g(1) = \delta(\alpha, n, m),
\] (2.11)
where
\[
g(1) = \frac{m+1}{n} \int_0^1 t^{(m+1)/n-1} \cdot \frac{1+(2\alpha-1)t}{1+t} \, dt
\]
\[
= 2\alpha - 1 + 2 \cdot \frac{m+1}{n} \cdot (1-\alpha) \beta \left( \frac{m+1}{n} \right),
\]

from which we deduce that $M_n^{m+1}(\alpha) \subset M_n^m(\delta)$. 

For $n = 1$, this result was obtained in [3].

**Theorem 2.3.** Let $g$ be a convex function, $g(0) = 1$, and let $h$ be a function such that
\[
h(z) = g(z) + \frac{1}{m+1} z g'(z).
\]

If $f \in A_n$ and verifies the differential subordination
\[
(D^{m+1}f)'(z) \prec h(z),
\]
then
\[
(D^mf)'(z) \prec g(z).
\]

**Proof.** From
\[
(m+1)D^{m+1}f(z) = z \cdot (D^mf)'(z) + mD^mf(z),
\]
we obtain
\[
(m+1)[D^{m+1}f(z)]' = (D^mf)'(z) + z(D^mf)''(z) + m(D^mf)'(z)
\]
\[
= z(D^mf)''(z) + (m+1)(D^mf)'(z).
\]

If we let $p(z) = (D^mf)'(z)$, then we obtain
\[
[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1} z p'(z)
\]
and (2.14) becomes
\[
p(z) + \frac{1}{m+1} z p'(z) \prec g(z) + \frac{1}{m+1} z g'(z) \equiv h(z).
\]

By using Lemma 1.2, we have
\[
p(z) \prec g(z), \text{ i.e., } (D^mf)'(z) \prec g(z).
\]

For $n = 1$, this result was obtained in [3].
Theorem 2.4. Let \( h \in \mathcal{H}[U] \), with \( h(0) = 1 \), \( h'(0) \neq 0 \), which verifies the inequality

\[
\text{Re} \left[ 1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2(m+1)}, \quad m \geq 0.
\]

If \( f \in A_n \) and verifies the differential subordination

\[
[D^{m+1}f(z)]' < h(z), \quad z \in U,
\]

then

\[
[D^mf(z)]' < g(z),
\]

where

\[
g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1} dt.
\]

The function \( g \) is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1, 2] shows that the function \( g \) is convex. From

\[
(m+1)D^{m+1}f(z) = z[D^mf(z)]' + mD^mf(z),
\]

we obtain

\[
(m+1)[D^{m+1}f(z)]' = z[D^mf(z)]'' + (m+1)[D^mf(z)]'.
\]

If we let \( p(z) = [D^mf(z)]' \), then we obtain

\[
[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1}zp'(z)
\]

and (2.22) becomes

\[
p(z) + \frac{1}{m+1}zp'(z) < h(z).
\]

By using Lemma 1.1, we have

\[
p(z) < g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1} dt.
\]

Theorem 2.5. Let \( g \) be a convex function, \( g(0) = 1 \), and

\[
h(z) = g(z) + nzg'(z).
\]

If \( f \in A_n \) and verifies the differential subordination

\[
[D^mf(z)]' < h(z), \quad z \in U,
\]
then

\[
\frac{D^m f(z)}{z} < g(z).
\]  

(2.32)

**Proof.** We let \( p(z) = D^m f(z)/z, \ z \in U, \) and we obtain

\[
D^m f(z) = zp(z).
\]  

(2.33)

By differentiating, we obtain

\[
[D^m f(z)]' = p(z) + zp'(z), \quad z \in U.
\]  

(2.34)

Then (2.31) becomes

\[
p(z) + zp'(z) < h(z) = g(z) + zg'(z).
\]  

(2.35)

By using Lemma 1.2, we have (1.8).

\[\square\]

**References**


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