

## INVARIANT SUBSPACES FOR POLYNOMIALLY COMPACT ALMOST SUPERDIAGONAL OPERATORS ON $l(p_i)$

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It is shown that almost superdiagonal, polynomially compact operators on the sequence space  $l(p_i)$  have nontrivial, closed invariant subspaces if the *nonlocally convex* linear topology  $\tau(p_i)$  is locally bounded.

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**1. Introduction.** The purpose of this paper is to show that almost superdiagonal, polynomially compact operators on the sequence space  $l(p_i)$  have nontrivial, closed invariant subspaces if the *nonlocally convex* linear topology  $\tau(p_i)$  is locally bounded. The proofs and arguments of this paper are stated within the framework of nonstandard analysis (see [4, Theorem 6.3 and Proposition 5.5]).

**1.1. Preliminaries.** Let  $\{p_i\}_{i=1}^{\infty}$  be a sequence of real numbers such that  $0 < p_i \leq 1$  for each  $i \in \mathbb{N}_+$ , the set of positive integers. Let

$$l(p_i) = \left\{ (\xi_i) \mid \sum_{i=1}^{\infty} |\xi_i|^{p_i} < \infty \right\}, \quad (1.1)$$

where  $\xi_i \in \mathbb{C}$ , the complex numbers. Since  $|\lambda + \beta|^p \leq |\lambda|^p + |\beta|^p$  and  $|\lambda\beta|^p \leq \max(1, |\lambda|)|\beta|^p$  are valid for all  $\lambda, \beta \in \mathbb{C}$  and  $0 < p \leq 1$ , it follows that  $l(p_i)$  is a vector space over  $\mathbb{C}$ . Also,

$$\rho(p_i)(x, y) = \sum_{i=1}^{\infty} |\xi_i - \zeta_i|^{p_i}, \quad (1.2)$$

where  $x = (\xi_i)$  and  $y = (\zeta_i)$ , defines a translation invariant metric on  $l(p_i)$ . Let  $\tau(p_i)$  denote the topology generated on  $l(p_i)$  by  $\rho(p_i)$ . If  $p_i = p \in (0, 1]$  for all  $i \in \mathbb{N}_+$ , then we denote  $l(p_i)$  by  $l^p$  and  $\tau(p_i)$  by  $\tau_p$ .

For  $(l(p_i), \tau(p_i))$ , the following facts are known:

- (1)  $(l(p_i), \tau(p_i))$  is a complete topological vector space;
- (2)  $(l(p_i), \tau(p_i))$  is a locally convex space if and only if  $l(p_i) = l^1$ ;
- (3) the following three conditions on  $\{p_i\}_{i=1}^{\infty} \subset (0, 1]$  are equivalent:

- (a)  $\liminf p_i > 0$ ,
- (b) a subset  $B$  of  $l(p_i)$  is bounded in  $\tau(p_i)$  if and only if it is bounded in the metric  $\rho(p_i)$ ,
- (c)  $(l(p_i), \tau(p_i))$  is locally bounded, that is, there exists a  $\tau(p_i)$ -bounded neighborhood of 0.

(See [5, Lemmas 1 and 2, Theorems 5 and 6].)

Unless stated otherwise, it will be assumed that  $0 < p_i \leq 1$ , for  $i \in \mathbb{N}_+$ , and  $0 < p \leq \liminf p_i$ .

The sequence  $\{e_i\}$  (where  $e_i = (\varepsilon_{ij})$ ,  $\varepsilon_{ii} = 1$ , and  $\varepsilon_{ij} = 0$  for  $i \neq j$ ) will denote the natural Schauder basis for  $(l(p_i), \tau(p_i))$  and  $\{\pi_i \mid i \in \mathbb{N}_+\}$ ,  $\{P_j \mid j \in \mathbb{N}_+\}$ , and  $\{E_j \mid j \in \mathbb{N}_+\}$  will denote the sequences of coordinate functionals, projections, and coordinate spaces, respectively, generated in  $l(p_i)$  by  $\{e_i\}$ . Also, a  $\tau$  or  $\rho$ , when used, will symbolize  $\tau(p_i)$  and  $\rho(p_i)$ , respectively.

Let  $\mathcal{F}[l(p_i)]$  symbolize the collection of all functions mapping  $l(p_i)$  into  $l(p_i)$  and let  $[l(p_i)]$  and  $\mathcal{L}(l(p_i))$  designate the vector spaces of  $\tau(p_i)$ -continuous linear transformation and linear transformations on  $l(p_i)$ , respectively. If  $T, U \in \mathcal{L}(l(p_i))$ , then  $TU$  denotes the composite map of  $T$  and  $U$ . For  $n \in \mathbb{N}$ , the set of natural numbers, and  $T \in \mathcal{L}(l(p_i))$ , define  $T^n$  in the usual manner, that is,  $T^0 = I$ , the identity map,  $T^1 = T$ , and  $T^n = TT^{n-1}$  for  $1 \leq n$ . If  $q(\lambda) = \sum_{k=0}^n c_k \lambda^k$  is a polynomial over  $\mathbb{C}$ , then we define  $q(T) = \sum_{k=0}^n c_k T^k$  for  $T \in \mathcal{L}(l(p_i))$ .

Let  $x \in l(p_i)$ , which implies  $x = \sum_{j=1}^\infty \pi_j(x)e_j$ . If  $T \in [l(p_i)]$ , then  $Tx = \sum_{j=1}^\infty \pi_j(x)Te_j$ . Note that  $\pi_i(Tx) = \pi_i(\sum_{j=1}^\infty \pi_j(x)Te_j) = \sum_{j=1}^\infty \pi_i(Te_j)\pi_j(x)$ , for  $i \in \mathbb{N}_+$ , by the continuity of  $\pi_i$ . Consequently, if  $a_{ij} = \pi_i(Te_j)$ , then  $Tx = \sum_{i=1}^\infty \pi_i(Tx)e_i = \sum_{i=1}^\infty [\sum_{j=1}^\infty a_{ij}\pi_j(x)]e_i$ . Therefore, for  $T \in [l(p_i)]$ , a double sequence  $[a_{ij}] \subset \mathbb{C}$  will be called the *matrix of  $T$  with respect to  $\{e_i\}$*  (or simply the matrix of  $T$ ) if and only if  $a_{ij} = \pi_i(Te_j)$ .

Define  $\mathcal{CF}[l(p_i)] \subset [l(p_i)]$  as follows:  $T \in \mathcal{CF}[l(p_i)]$  if and only if  $T \in [l(p_i)]$  and for  $j \in \mathbb{N}_+$ , there exists  $n \in \mathbb{N}_+$ , depending on  $j$ , such that  $Te_j \in E_n$ . If  $T \in \mathcal{CF}[l(p_i)]$  and  $[a_{ij}]$  is the matrix of  $T$ , then there exists  $n \in \mathbb{N}_+$  such that  $a_{ij} = 0$  for  $n < i$ . Also,  $T, U \in \mathcal{CF}[l(p_i)]$  implies  $TU \in \mathcal{CF}[l(p_i)]$ . For  $T \in \mathcal{CF}[l(p_i)]$ ,  $T$  is said to be *almost superdiagonal (a.sd.)* if and only if  $Te_j \in E_{j+1}$  for each  $j \in \mathbb{N}_+$ .

For  $T \in [l(p_i)]$ ,  $n \in \mathbb{N}$ , and  $[a_{ij}]$ , the matrix of  $T$ , let the matrix of  $T^n$  be denoted by  $[a_{ij}^{(n)}]$ . It can be shown that if  $T$  is almost superdiagonal, then

$$\begin{aligned}
 a_{j+n,j}^{(m)} &= 0 \quad \text{for } m < n, \\
 a_{j+n,j}^{(n)} &= \prod_{i=0}^{n-1} a_{j+i+1,j+i} \quad \text{for } j, m, n \in \mathbb{N}_+
 \end{aligned}
 \tag{1.3}$$

(see [1, Section 3, Theorem 3.6]).

An asterisk appended to the upper-left corner of a symbol indicates the *nonstandard extension* of the object represented by the symbol. The notation

$\mu(0)$  will denote the set of *infinitesimals* of  ${}^*\mathbb{C}$ . An element  $\lambda$  of  ${}^*\mathbb{C}$  is said to be *finite* if and only if  $|\lambda| \leq {}^*\delta$  for some positive  $\delta \in \mathbb{R}$ ; otherwise,  $\lambda$  is said to be *infinite*. It is customary to consider  $\mathbb{C} \subset {}^*\mathbb{C}$ , that is, the elements of  $\mathbb{C}$  are identified with their nonstandard extensions; therefore, the asterisk notation is mostly not used for these elements. However, to bring clarity to some arguments, the asterisk notation for nonstandard extensions of elements of  $\mathbb{C}$  will be used occasionally.

The notation  $\mathcal{N}_\tau(0)$  denotes the  $\tau(p_i)$ -neighborhood filter of zero in  $l(p_i)$  and  $\mu_\tau(0)$  denotes the *monad* of the  $\tau(p_i)$ -neighborhoods of zero in  $l(p_i)$ . An element  $z \in {}^*l(p_i)$  is called *near standard* if and only if there exists a (unique) element in  $l(p_i)$ , denoted by  ${}^\circ z$ , such that  $z - {}^\circ z \in \mu_\tau(0)$ . Also, for  $A \subset {}^*l(p_i)$ , the set  ${}^\circ A \subset l(p_i)$ , called the *standard part* of  $A$ , is defined as follows:  $x \in {}^\circ A$  if and only if there is a  $z \in A$  such that  $z - {}^*x \in \mu_\tau(0)$ . It can be shown that if  $F$  is an *internal* vector subspace of  ${}^*l(p_i)$ , then  ${}^\circ F$  is a  $\tau(p_i)$ -closed linear subspace of  $l(p_i)$  (see [3, Proposition 1.7]). If  $T \in [l(p_i)]$ , then  ${}^*T[\mu_\tau(0)] \subseteq \mu_\tau(0)$ ,  ${}^*T(z)$  is *near standard* if  $z \in {}^*l(p_i)$  is *near standard*, and  ${}^\circ[{}^*T(z)] = T({}^\circ z)$ . We will denote the (*external*) set of all *near standard* points of  ${}^*l(p_i)$  by the notation  $\text{ns}_\tau({}^*l(p_i))$ . The (*external*) set  $\{z \in {}^*l(p_i) \mid \lambda z \in \mu_\tau(0) \text{ for each } \lambda \in \mu(0)\}$  is called the set of *finite points* of  ${}^*l(p_i)$  and is denoted by  $\text{fin}_\tau({}^*l(p_i))$ . Clearly,

$$\text{ns}_\tau({}^*l(p_i)) \subset \text{fin}_\tau({}^*l(p_i)). \tag{1.4}$$

Finally, if  $Y$  is any set belonging to the *superstructure* generated by  $\mathbb{C} \cup l(p_i)$ , then  $\Delta(Y)$  denotes the collection of all finite subsets of the set  $Y$ . Also, the elements of  ${}^*\Delta(Y)$  are called *\*-finite* subsets of  ${}^*Y$ .

**2. Nonstandard properties of  $l(p_i)$ .** The purpose of this section is to state some of the (nonstandard) properties of  $(l(p_i), \tau(p_i))$  that will be used in later arguments. These facts were developed in [2, 3]. The reader is referred to these references for the proofs.

Recall that for  $x \in l(p_i)$ ,

$$P_i(x) = \sum_{j=1}^i \pi_j(x) e_j \in E_i, \tag{2.1}$$

where  $\{e_i\}$  is the natural Schauder basis for  $l(p_i)$ ,  $E_i = \text{sp}(e_1, \dots, e_i)$ , and  $\{\pi_i\}$  is the sequence of scalar projections generated by  $\{e_i\}$ . Let  $\mathcal{FS}(l(p_i))$  be the collection of all finite-dimensional linear subspaces of  $l(p_i)$ . We will let  $d : \mathcal{FS}(l(p_i)) \rightarrow \mathbb{N}$  denote the dimension function. In other words,  $d(F) = n$ , for  $F \in \mathcal{FS}(l(p_i))$ , if and only if  $F = \text{sp}(x_1, \dots, x_n)$  for some linearly independent  $\{x_i\}_{i=1}^n \subset l(p_i)$ . In particular,

$$d(E_i) = i \text{ for each } i \in \mathbb{N}_+. \tag{2.2}$$

**PROPOSITION 2.1** (see [2, Propositions III.1 and III.3]). *If  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ , then the (internal) projection  $P_\alpha : {}^*l(p_i) \rightarrow E_\alpha$  satisfies the following two conditions:*

- (1) *for  $W \in \mathcal{N}_\tau(0)$ , there exists  $V \in \mathcal{N}_\tau(0)$  such that  $P_\alpha[{}^*V] \subset {}^*W$ ;*
- (2) *if  $x \in l(p_i)$ , then  $P_\alpha({}^*x) - {}^*x \in \mu_\tau(0)$  (i.e.,  ${}^\circ[P_\alpha({}^*x)] = x$ ).*

Note that **Proposition 2.1** implies that for  $x \in l(p_i)$  and  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ , we have  $l(p_i) = {}^\circ E_\alpha$  and  $P_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0)$ . It can be shown that

$$z \in \text{ns}_\tau({}^*l(p_i)) \text{ implies } z - P_\alpha(z) \in \mu_\tau(0) \quad (2.3)$$

(see [2, Proposition II.2]). Also, for  $T \in [l(p_i)]$  and  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ , if we define  $T_\alpha = P_\alpha({}^*T)P_\alpha$ , then  $T_\alpha \in {}^*\mathcal{L}(l(p_i))$  (i.e.,  $T_\alpha$  is an *internal* linear transformation on  ${}^*l(p_i)$ ),  $T_\alpha : {}^*l(p_i) \rightarrow E_\alpha$ , and  $T_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0)$ . In addition,  ${}^\circ[T_\alpha({}^*x)] = T(x)$  for  $x \in l(p_i)$  and  ${}^\circ[T_\alpha(z)] = T({}^\circ z)$  for any *near standard*  $z \in {}^*l(p_i)$  (see [2, Propositions II.4 and II.5]). Finally, if  $F \in {}^*\mathcal{F}\mathcal{S}(l(p_i))$  such that  $F \subseteq E_\alpha$  and  $T_\alpha[F] \subseteq F$ , then  $T[{}^\circ F] \subseteq {}^\circ F$  (see [2, Proposition II.6]).

**PROPOSITION 2.2** (see [2, Theorem II.1]). *There exists a function  $\nabla : \mathcal{F}\mathcal{S}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$  that satisfies the following conditions:*

- (1) *if  $F \in \mathcal{F}\mathcal{S}(l(p_i))$ , then  $\nabla(F) : l(p_i) \rightarrow F$ ;*
- (2) *for each  $V \in \mathcal{N}_\tau(0)$  and any nonzero  $x \in l(p_i)$ , there exists a positive  $\lambda \in \mathbb{R}$  such that  $\nabla(F)(\lambda x) \in V$  for all  $F \in \mathcal{F}\mathcal{S}(l(p_i))$ ;*
- (3) *if  $x \in l(p_i)$  such that  $x \in {}^\circ F$  for  $F \in {}^*\mathcal{F}\mathcal{S}(l(p_i))$ , then  ${}^*\nabla(F)({}^*x) - {}^*x \in \mu_\tau(0)$  (i.e.,  ${}^\circ[{}^*\nabla(F)({}^*x)] = x$ ).*

Let  $[\mathcal{F}\mathcal{S}(l(p_i))]$  be the collection of all linear transformations  $Q$  with  $\mathcal{D}(Q)$ ,  $\mathcal{R}(Q) \in \mathcal{F}\mathcal{S}(l(p_i))$ , where  $\mathcal{D}(Q)$  and  $\mathcal{R}(Q)$  are the domain and range of  $Q$ , respectively, (i.e., the domain and range of linear transformation  $Q$  are *finite* dimensional for  $Q \in [\mathcal{F}\mathcal{S}(l(p_i))]$ ). Since the scalar field of  $l(p_i)$  is complex, the following sentence is true.

*If  $E \in \mathcal{F}\mathcal{S}(l(p_i))$  and  $Q \in [\mathcal{F}\mathcal{S}(l(p_i))]$  such that  $Q : E \rightarrow E$ , then for  $n = d(E)$ , there exists  $\{F_j\}_{j=0}^n \in \Delta(\mathcal{F}\mathcal{S}(l(p_i)))$  such that*

- (a)  $F_0 = \{0\}$  and  $F_n = E$ ,
- (b)  $F_{j-1} \subset F_j$  for  $j = 1, \dots, n$ ,
- (c)  $d(F_j) = d(F_{j-1}) + 1$  for  $j = 1, \dots, n$ ,
- (d)  $Q[F_j] \subset F_j$  for  $j = 0, \dots, n$ .

Note that the (logical) constants of the previous statement are  $\mathcal{F}\mathcal{S}(l(p_i))$ ,  $[\mathcal{F}\mathcal{S}(l(p_i))]$ ,  $\Delta(\mathcal{F}\mathcal{S}(l(p_i)))$ , and  $d$ , the dimension function. Therefore, by the *transfer principle*, the following sentence is true.

*If  $E \in {}^*\mathcal{F}\mathcal{S}(l(p_i))$  and  $Q \in {}^*[\mathcal{F}\mathcal{S}(l(p_i))]$  such that  $Q : E \rightarrow E$ , then for  $\alpha = {}^*d(E)$ , there exists  $\{F_\iota\}_{\iota=0}^\alpha \in {}^*\Delta(\mathcal{F}\mathcal{S}(l(p_i)))$  such that*

- (a)  $F_0 = \{0\}$  and  $F_\alpha = E$ ,
- (b)  $F_{\iota-1} \subset F_\iota$  for  $\iota = 1, \dots, \alpha$ ,
- (c)  ${}^*d(F_\iota) = {}^*d(F_{\iota-1}) + 1$  for  $\iota = 1, \dots, \alpha$ ,
- (d)  $Q[F_\iota] \subset F_\iota$  for  $\iota = 0, \dots, \alpha$ .

In [2], this fact was used to obtain the following proposition.

**PROPOSITION 2.3** (see [2, Definition II.2 and Lemma II.8]). *Let  $\nabla : \mathcal{F}\mathcal{S}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$  be the function established by Proposition 2.2 and let  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ . If  $T \in \mathcal{L}(l(p_i))$ , then there exists an internal family  $\{F_\iota\}_{\iota=0}^\alpha \in {}^*\Delta(\mathcal{F}\mathcal{S}(l(p_i)))$  such that the following conditions are fulfilled:*

- (1)  $F_0 = \{0\}$ ,  $F_\alpha = E_\alpha$ , and  $F_{\iota-1} \subset F_\iota$  for  $\iota = 1, \dots, \alpha$ ,
- (2)  ${}^*d(F_\iota) = {}^*d(F_{\iota-1}) + 1$  for  $\iota = 1, \dots, \alpha$ ,
- (3)  $T_\alpha[F_\iota] \subset F_\iota$  for  $\iota = 0, \dots, \alpha$ , where  $T_\alpha = P_\alpha({}^*T)P_\alpha$ ,
- (4)  $\{F_\iota\}_{\iota=0}^\alpha$  and  $\{{}^*\nabla(F_\iota)\}_{\iota=0}^\alpha$  are  ${}^*$ -finite,
- (5)  ${}^*\nabla(F_\iota) : {}^*l(p_i) \rightarrow F_\iota$  such that  $x \in {}^\circ F_\iota$  implies  ${}^*\nabla(F_\iota)({}^*x) - {}^*x \in \mu_\tau(0)$  (i.e.,  ${}^\circ[{}^*\nabla(F_\iota)({}^*x)] = x$ ) for each  $\iota \in \{0, \dots, \alpha\}$ .

Note that  ${}^\circ F_\alpha = {}^\circ E_\alpha = l(p_i)$  and from continuity and Proposition 2.3(3), we infer that  $T[{}^\circ F_\iota] \subset {}^\circ F_\iota$  for  $T \in [l(p_i)]$  and  $\iota \in \{0, \dots, \alpha\}$ . Also, given  $F_{\iota-1}$  and  $F_\iota$ ,  $\iota = 1, \dots, \alpha$ , it can be shown that Proposition 2.3(2) implies that for  $x_1, x_2 \in {}^\circ F_\iota$ , either  $x_1 = \zeta_1 x_2 + y_1$  or  $x_2 = \zeta_2 x_1 + y_2$  for some  $\zeta_1, \zeta_2 \in \mathbb{C}$  and  $y_1, y_2 \in {}^\circ F_{\iota-1}$ . In other words, any two points of  ${}^\circ F_\iota$  are linearly dependent modulo  ${}^\circ F_{\iota-1}$  (see [2, Proposition I.21]).

Observe that for  $T \in [l(p_i)]$ , Proposition 2.3 produces a chain of closed invariant linear subspaces for  $T$ , namely  $\{{}^\circ F_\iota\}_{\iota=0}^\alpha$ . The problem is that we could have  ${}^\circ F_\iota = \{0\}$  for  $\iota \in \{0, \dots, \alpha\} \cap \mathbb{N}$  (i.e., the finite elements of  $\{0, \dots, \alpha\}$ ) and  ${}^\circ F_\iota = l(p_i)$  for  $\iota \in \{0, \dots, \alpha\} \cap ({}^*\mathbb{N} - \mathbb{N})$  (i.e., the infinite elements of  $\{0, \dots, \alpha\}$ ). However, if we could find  $\nu \in \{1, \dots, \alpha\}$  such that  ${}^\circ F_{\nu-1} \neq l(p_i)$  and  ${}^\circ F_\nu \neq \{0\}$ , then either  ${}^\circ F_{\nu-1}$  or  ${}^\circ F_\nu$  is a closed nontrivial linear subspace of  $T$  since  $l(p_i)$  is infinite dimensional and any two points of  ${}^\circ F_\nu$  are linearly dependent modulo  ${}^\circ F_{\nu-1}$ . The next proposition gives sufficient conditions for the existence of such a  $\nu$ .

**PROPOSITION 2.4** (see [2, Definition II.2 and Lemma II.9]). *Let  $T \in \mathcal{L}(l(p_i))$ ,  $\nabla : \mathcal{F}\mathcal{S}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$  be the function established by Proposition 2.2, and let  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ . Let the collection  $[\{F_\iota\}_{\iota=0}^\alpha : \{{}^*\nabla(F_\iota)\}_{\iota=0}^\alpha]$  satisfy the conditions of Proposition 2.3 with respect to  $T$ ,  $\nabla$ , and  $\alpha$ . Let  $U \in {}^*\mathcal{L}(l(p_i))$  such that  $U[\mu_\tau(0)] \subset \mu_\tau(0)$ . If there exists  $x \in l(p_i)$  such that  $U({}^*x) \notin \mu_\tau(0)$  and  $U({}^*\nabla(F_\iota)({}^*x)) \in F_\iota \cap \text{ns}_\tau({}^*l(p_i))$  for each  $\iota \in \{0, \dots, \alpha\}$ , then there exists  $\nu \in \{1, \dots, \alpha\}$  such that  ${}^\circ F_{\nu-1} \neq l(p_i)$  and  ${}^\circ F_\nu \neq \{0\}$ .*

We close this section with a useful characterization of  $\text{fin}_\tau({}^*l(p_i))$ , the finite points of  ${}^*l(p_i)$ .

**PROPOSITION 2.5.** *If  $z \in \text{fin}_\tau({}^*l(p_i))$ , then  $\pi_\iota(z)$  is finite for  $\pi_\iota \in \{*\pi_i \mid i \in \mathbb{N}_+\}$ .*

**PROOF.** Since  $0 < p \leq \liminf p_i$  implies that  $\tau(p_i)$  is locally bounded (see [5, Theorem 6]), there exists a positive  $\delta_0 \in \mathbb{R}$  such that

$$V_0 = S(\rho(p_i); \delta_0) = \{x \in l(p_i) \mid \rho(p_i)(x, 0) \leq \delta_0\} \tag{2.4}$$

is  $\tau(p_i)$ -bounded. Let  $z \in \text{fin}_\tau(*l(p_i))$ . Hence,  $*g_{V_0}(z) \leq *\delta$  for some positive  $\delta \in \mathbb{R}$ , where  $g_{V_0}$  is the gauge of  $V_0$  (see [2, Proposition I.14]). So,  $z \in *(\delta V_0) \subset *S(\rho(p_i); \lambda)$  for  $\lambda = \max(\delta\delta_0, \delta_0, 1)$  since  $V_0$  is closed, balanced and  $\delta S(\rho(p_i); \delta_0) \subset S(\rho(p_i); \lambda)$ . Therefore,  $*\rho(p_i)(z, 0) \leq *\lambda$ , which implies  $|\pi_\iota(z)|^{p_\iota} \leq *\lambda$  for  $\pi_\iota \in *\{\pi_i \mid i \in \mathbb{N}_+\}$ . It suffices to consider the case when  $1 \leq |\pi_\iota(z)|$  for  $\iota \in *\mathbb{N}_+$ . Since  $p \in \mathbb{R}$  and  $0 < p_\iota^{-1} \leq *(p^{-1})$  for  $p_\iota \in *\{p_i\}$ , we have  $|\pi_\iota(z)| \leq (*\lambda)^{p_\iota^{-1}} \leq *( \lambda^{p^{-1}})$  for  $\pi_\iota \in *\{\pi_i \mid i \in \mathbb{N}_+\}$ . We infer that  $\pi_\iota(z)$  is finite for  $\pi_\iota \in *\{\pi_i \mid i \in \mathbb{N}_+\}$ .  $\square$

**3. Polynomially compact almost superdiagonal operators.** We want to show that if  $T \in \mathcal{CF}[l(p_i)]$  is almost superdiagonal and  $q(T)$  is compact for some polynomial  $q(\lambda)$  over  $\mathbb{C}$ , then  $T$  has a nontrivial closed invariant linear subspace. Note that for  $\alpha \in *\mathbb{N}_+ - \mathbb{N}_+$  and  $\nabla : \mathcal{FP}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$ , the function defined by Proposition 2.2, we can use Proposition 2.3 to produce a collection  $[\{F_i\}_{i=0}^\alpha : \{\nabla(F_i)\}_{i=0}^\alpha]$ , for some  $\alpha \in *\mathbb{N}$ , such that  $\{^\circ F_i\}_{i=0}^\alpha$  is a collection of closed invariant linear subspaces of  $T$ . The strategy is to find some  $\alpha \in *\mathbb{N}_+ - \mathbb{N}_+$  such that  $*q(T_\alpha) \in *\mathcal{L}(l(p_i))$  satisfies the hypotheses of Proposition 2.4, where  $T_\alpha = P_\alpha(*T)P_\alpha$ .

First, however, consider a compact operator  $U$  on  $l(p_i)$ .

As stated in the proof of Proposition 2.5, there exists a positive  $\delta_0 \in \mathbb{R}$  such that

$$V_0 = S(\rho(p_i); \delta_0) = \{x \in l(p_i) \mid \rho(p_i)(x, 0) \leq \delta_0\} \quad (3.1)$$

is  $\tau(p_i)$ -bounded since  $0 < p \leq \liminf p_i$  implies that  $\tau(p_i)$  is locally bounded (see the known facts about  $(l(p_i), \tau(p_i))$  in the first section). Therefore,

$$*V_0 \subset \text{fin}_\tau(*l(p_i)) \quad (3.2)$$

(see [2, Corollary I.18]). If  $U \in [l(p_i)]$  such that  $\overline{U[W]}$  is  $\tau(p_i)$ -compact for some  $W \in \mathcal{N}_\tau(0)$ , then  $\overline{U[V_0]}$  is  $\tau(p_i)$ -compact since  $\lambda V_0 \subset W$  for some positive scalar  $\lambda$ . Unless stated otherwise,  $V_0 = S(\rho(p_i); \delta_0)$  will be a fixed,  $\tau(p_i)$ -bounded neighborhood of 0, with  $0 < \delta_0 \leq 1$ . Thus, we have that  $U \in [l(p_i)]$  is compact if and only if  $\overline{U[V_0]}$  is  $\tau(p_i)$ -compact. Also, if  $\overline{U[V_0]}$  is  $\tau(p_i)$ -compact, then  $*U[*V_0] \subset \text{ns}_\tau(*l(p_i))$  (see [2, Proposition I.1]).

**PROPOSITION 3.1.** *If  $U \in [l(p_i)]$  is compact and  $[b_{ij}]$  is the matrix of  $U$ , then  $b_{\iota\kappa} \in \mu(0)$  for  $b_{\iota\kappa} \in *[b_{ij}]$  such that  $\iota, \kappa \in *\mathbb{N}_+ - \mathbb{N}_+$ .*

**PROOF.** Let  $\lambda_\kappa = (*\delta_0)^{p_\kappa^{-1}}$  for  $\kappa \in *\mathbb{N}_+$  and  $p_\kappa \in *\{p_i\}$  and define  $z_\kappa = \lambda_\kappa e_\kappa$  for  $\kappa \in *\mathbb{N}_+$  and  $e_\kappa \in *\{e_i\}$ . We infer that  $*\rho(p_i)(z_\kappa, 0) = |\lambda_\kappa|^{p_\kappa} = *\delta_0$  for  $\kappa \in *\mathbb{N}_+$ , which implies  $z_\kappa \in *V_0$  for  $\kappa \in *\mathbb{N}_+$ . Therefore,  $*Uz_\kappa \in \text{ns}_\tau(*l(p_i))$  for  $\kappa \in *\mathbb{N}_+$  since  $*U[*V_0] \subset \text{ns}_\tau(*l(p_i))$ . So, if  $\iota \in *\mathbb{N}_+ - \mathbb{N}_+$ , then  $*Uz_\kappa - P_{\iota-1}(*Uz_\kappa) \in \mu_\tau(0)$  for  $\kappa \in *\mathbb{N}_+$  by expression (2.3) since  $\iota \in *\mathbb{N}_+ - \mathbb{N}_+$  implies  $\iota - 1 \in *\mathbb{N}_+ - \mathbb{N}_+$ .

Let  $\iota, \kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$  and let  $[b_{ij}]$  be the matrix of  $U$  with respect to  $\{e_i\}$ . Note that  $\pi_\iota({}^*Uz_\kappa) = \lambda_\kappa \pi_\iota({}^*Ue_\kappa) = \lambda_\kappa b_{\iota\kappa}$  for  $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$ ,  $e_i \in {}^*\{e_i\}$ , and  $b_{\iota\kappa} \in {}^*[b_{ij}]$ . Since

$$|\lambda_\kappa b_{\iota\kappa}|^{p_\iota} = {}^*\rho(p_i)(\lambda_\kappa b_{\iota\kappa} e_i, 0) \leq {}^*\rho(p_i)({}^*Uz_\kappa - P_{i-1}({}^*Uz_\kappa), 0), \tag{3.3}$$

we have  $|\lambda_\kappa b_{\iota\kappa}|^{p_\iota} \in \mu(0)$ . Consequently,  $1 \leq p_\iota^{-1} \leq {}^*(p^{-1})$  implies  $\lambda_\kappa |b_{\iota\kappa}| \in \mu(0)$ . Also,  $1 \leq p_\kappa^{-1} \leq {}^*(p^{-1})$  implies  ${}^*(\delta_0^{p_\kappa^{-1}}) \leq \lambda_\kappa \leq {}^*\delta_0$ , which implies  $|b_{\iota\kappa}| \in \mu(0)$ . Therefore,  $b_{\iota\kappa} \in \mu(0)$ . □

**PROPOSITION 3.2.** *If  $U \in [l(p_i)]$  is almost superdiagonal and  $q(U)$  is compact for some complex polynomial  $q(\lambda)$ , then there exists  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$  such that  $a_{\alpha+1, \alpha} \in \mu(0)$  for  $a_{\alpha+1, \alpha} \in {}^*[a_{ij}]$ , where  $[a_{ij}]$  is the matrix of  $U$ .*

**PROOF.** Let  $n$  be the degree of  $q(\lambda) = \sum_{k=0}^n c_k \lambda^k$ , which implies  $c_n \neq 0$ . If  $[b_{ij}]$  is the matrix of  $q(U)$ , with respect to  $\{e_i\}$ , then  $q(U)$  being compact implies  $b_{\iota\kappa} \in \mu(0)$  for  $b_{\iota\kappa} \in {}^*[b_{ij}]$  such that  $\iota, \kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$  by Proposition 3.1. Let  $\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ . Since  $U$  being almost superdiagonal implies  $a_{\kappa+n, \kappa}^{(m)} = 0$  for  $m < n$  and  $a_{\kappa+n, \kappa}^{(n)} = \prod_{i=0}^{n-1} a_{\kappa+i+1, \kappa+i}$  (by expression (1.3) and the transfer principle), we have  $b_{\kappa+n, \kappa} = {}^*c_n \prod_{i=0}^{n-1} a_{\kappa+i+1, \kappa+i}$ . Therefore, if  $\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ , then  ${}^*c_n \notin \mu(0)$  and  $b_{\kappa+n, \kappa} \in \mu(0)$  imply  $a_{\kappa+i_0+1, \kappa+i_0} \in \mu(0)$  for some  $i_0 \in \{0, \dots, n-1\}$ . Let  $\alpha = \kappa + i_0$ . □

**PROPOSITION 3.3.** *If  $U \in {}^*\mathcal{L}(l(p_i))$  such that  $U[\mu_\tau(0)] \subset \mu_\tau(0)$ , then*

$$U[\text{fin}_\tau({}^*l(p_i))] \subset \text{fin}_\tau({}^*l(p_i)). \tag{3.4}$$

**PROOF.** Let  $z \in \text{fin}_\tau({}^*l(p_i))$ . If  $\lambda \in \mu(0)$ , then  $\lambda z \in \mu_\tau(0)$ , which implies  $\lambda Uz = U(\lambda z) \in \mu_\tau(0)$ . Therefore,  $Uz \in \text{fin}_\tau({}^*l(p_i))$ . □

Let  $T \in \mathcal{CF}[l(p_i)]$  be an almost superdiagonal operator such that  $q(T)$  is compact for  $q(\lambda) = \sum_{k=0}^n c_k \lambda^k$ , a polynomial over  $\mathbb{C}$  with  $c_n \neq 0$ . Let  $\nabla : \mathcal{FG}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$  satisfy the conditions of Proposition 2.2 and let  $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$  satisfy the conclusion of Proposition 3.2. Note that the (internal) projection  $P_\alpha : {}^*l(p_i) \rightarrow E_\alpha$  satisfies the conditions of Proposition 2.1. Define  $T_\alpha = P_\alpha({}^*T)P_\alpha$ . Observe that

$$\begin{aligned} {}^*q(T_\alpha) &\in {}^*\mathcal{L}(l(p_i)), & T_\alpha[\mu_\tau(0)] &\subset \mu_\tau(0), \\ {}^*(q(T))[\mu_\tau(0)] &\subset \mu_\tau(0), \end{aligned} \tag{3.5}$$

since  $T$  and  $q(T)$  are continuous.

**PROPOSITION 3.4.** *If  $z \in \mu_\tau(0)$ , then  ${}^*q(T_\alpha)z \in \mu_\tau(0)$ .*

**PROOF.** It suffices to show that  $(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$  for  $m \in \mathbb{N}_+$  (see [2, Proposition I.6]). Note that from (3.5),  $T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0)$ . Assume that

$(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$  for  $m \in \mathbb{N}_+$ . Consequently,

$$(T_\alpha)^{m+1}[\mu_\tau(0)] = T_\alpha[(T_\alpha)^m[\mu_\tau(0)]] \subset T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0). \quad (3.6)$$

Therefore,  $(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$  for each  $m \in \mathbb{N}_+$  by induction.  $\square$

So, one of the conditions of [Proposition 2.4](#) for  $*q(T_\alpha)$  has been satisfied, that is,  $*q(T_\alpha)[\mu_\tau(0)] \subset \mu_\tau(0)$ .

**PROPOSITION 3.5.** *Let  $F \in *F\mathcal{G}(l(p_i))$  such that  $F \subset E_\alpha$ . If  $T_\alpha[F] \subset F$ , then  $(T_\alpha)^m[F] \subset F$  for  $m \in \mathbb{N}$ .*

**PROOF.** If  $(T_\alpha)^m[F] \subset F$  for  $m \in \mathbb{N}$ , then  $(T_\alpha)^{m+1}[F] = T_\alpha[(T_\alpha)^m[F]] \subset T_\alpha[F] \subset F$ . Therefore,  $(T_\alpha)^m[F] \subset F$  for any  $m \in \mathbb{N}$  by induction.  $\square$

Consequently, if  $F \in *F\mathcal{G}(l(p_i))$  such that  $F \subset E_\alpha$  and  $T_\alpha[F] \subset F$ , then

$$*q(T_\alpha)[F] \subset F. \quad (3.7)$$

**PROPOSITION 3.6.** *If  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ , then  $[*q(T_\alpha)z - *(q(T))z] \in \mu_\tau(0)$ .*

**PROOF.** It is sufficient to show that  $*(T^m)z - (T_\alpha)^m z \in \mu_\tau(0)$  for  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$  and  $m \in \mathbb{N}$  (see [2, Proposition I.6]). Let  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ , which implies  $z = \sum_{\kappa=1}^\alpha \pi_\kappa(z)e_\kappa$  for  $\pi_\kappa \in *\{\pi_i \mid i \in \mathbb{N}_+\}$  and  $e_\kappa \in *\{e_i\}$ . Consequently,

$$*Tz = \sum_{\kappa=1}^\alpha \pi_\kappa(z)*Te_\kappa = \sum_{\kappa=1}^\alpha \pi_\kappa(z) \left[ \sum_{\iota=1}^{\alpha+1} a_{\iota\kappa} e_\iota \right] = \sum_{\iota=1}^{\alpha+1} \left[ \sum_{\kappa=1}^\alpha a_{\iota\kappa} \pi_\kappa(z) \right] e_\iota \quad (3.8)$$

since  $T$  is almost superdiagonal. Also,  $a_{\alpha+1,\kappa} = 0$  for  $\kappa < \alpha$ , which implies  $\sum_{\kappa=1}^\alpha a_{\alpha+1,\kappa} \pi_\kappa(z) = a_{\alpha+1,\alpha} \pi_\alpha(z)$ . Therefore,  $*Tz = \sum_{\iota=1}^\alpha [\sum_{\kappa=1}^\alpha a_{\iota\kappa} \pi_\kappa(z)] e_\iota + a_{\alpha+1,\alpha} \pi_\alpha(z) e_{\alpha+1}$ , which implies  $*Tz - P_\alpha(*Tz) = a_{\alpha+1,\alpha} \pi_\alpha(z) e_{\alpha+1}$ . So,

$$*\rho(p_i)(*Tz - P_\alpha(*Tz), 0) = |a_{\alpha+1,\alpha}|^{p_{\alpha+1}} |\pi_\alpha(z)|^{p_{\alpha+1}}. \quad (3.9)$$

Since  $\pi_\alpha(z)$  is finite (by [Proposition 2.5](#)),  $a_{\alpha+1,\alpha} \in \mu(0)$  ([Proposition 3.2](#)), and  $0 < *p \leq p_{\alpha+1} \leq 1$ , we infer that  $|a_{\alpha+1,\alpha}|^{p_{\alpha+1}} |\pi_\alpha(z)|^{p_{\alpha+1}} \in \mu(0)$ , which implies  $*Tz - P_\alpha(*Tz) \in \mu_\tau(0)$ . Therefore,  $*Tz - T_\alpha z \in \mu_\tau(0)$  since  $z \in E_\alpha$  implies  $z = P_\alpha(z)$ .

Now, let  $m \in \mathbb{N}$  such that  $2 \leq m$  and assume that  $*(T^{m-1})z - (T_\alpha)^{m-1}z \in \mu_\tau(0)$  for  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ . If  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ , then

$$*(T^m)z - *T((T_\alpha)^{m-1}z) = *T(*(T^{m-1})z - (T_\alpha)^{m-1}z) \in \mu_\tau(0) \quad (3.10)$$

since  $T \in [l(p_i)]$  implies that  $T$  is linear and  $*T[\mu_\tau(0)] \subseteq \mu_\tau(0)$ . If we set  $y = (T_\alpha)^{m-1}z$ , then  $y \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$  by Propositions 3.3 and 3.5 since  $T_\alpha[E_\alpha] \subset E_\alpha$  and  $(T_\alpha)^{m-1}[\mu_\tau(0)] \subseteq \mu_\tau(0)$  (see the proof of Proposition 3.4). Thus,

$$*T\left((T_\alpha)^{m-1}z\right) - (T_\alpha)^m z = *T(y) - T_\alpha y \in \mu_\tau(0) \tag{3.11}$$

(see the first part of the present proof), which implies

$$\begin{aligned} *(T^m)z - (T_\alpha)^m z &= \left[ *(T^m)z - *T\left((T_\alpha)^{m-1}z\right) \right] \\ &\quad + \left[ *T\left((T_\alpha)^{m-1}z\right) - (T_\alpha)^m z \right] \in \mu_\tau(0). \end{aligned} \tag{3.12}$$

Therefore, by induction, it follows that  $*(T^m)z - (T_\alpha)^m z \in \mu_\tau(0)$  for  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$  and  $m \in \mathbb{N}$ . □

**PROPOSITION 3.7.** *If  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ , then  $*q(T_\alpha)z \in \text{ns}_\tau(*l(p_i))$  (i.e.,  $*q(T_\alpha)z$  is  $\tau(p_i)$ -near standard).*

**PROOF.** Let  $z \in E_\alpha \cap \text{fin}_\tau(*l(p_i))$ . There exists  $n \in \mathbb{N}$  such that  $z \in *(nV_0)$  (see [2, Corollary I.15]). Since  $q(T)$  is compact, it follows that

$$*(q(T))[*(nV_0)] = n*(q(T))[*V_0] \subset n[\text{ns}_\tau(*l(p_i))] \subset \text{ns}_\tau(*l(p_i)) \tag{3.13}$$

(see [2, Proposition I.1 and Corollary I.10]). Therefore,

$$\begin{aligned} *q(T_\alpha)z - \circ[*(q(T))z] &= [*q(T_\alpha)z - *(q(T))z] \\ &\quad + [*(q(T))z - \circ[*(q(T))z]] \in \mu_\tau(0) \end{aligned} \tag{3.14}$$

since  $[*q(T_\alpha)z - *(q(T))z] \in \mu_\tau(0)$  by Proposition 3.6. Therefore,  $*q(T_\alpha)z \in \text{ns}_\tau(*l(p_i))$ . □

We now state and prove the main result.

**THEOREM 3.8.** *Let  $0 < p \leq p_i \leq 1$  and let  $T \in [l(p_i)]$  be almost superdiagonal. If  $q(\lambda)$  is a polynomial over  $\mathbb{C}$  such that  $q(T)$  is compact, then  $T$  has at least one nontrivial  $\tau(p_i)$ -closed invariant linear subspace of  $l(p_i)$ .*

**PROOF.** Let  $[a_{ij}]$  be the matrix of  $T$  with respect to  $\{e_i\}$ . Therefore, there exists  $\alpha \in *\mathbb{N}_+ - \mathbb{N}_+$  such that  $a_{\alpha+1,\alpha} \in \mu(0)$  for  $a_{\alpha+1,\alpha} \in *[a_{ij}]$  by Proposition 3.2. Let  $\nabla : \mathcal{F}\mathcal{S}(l(p_i)) \rightarrow \mathcal{F}[l(p_i)]$  satisfy the conditions of Proposition 2.2 and let the collection  $[\{F_i\}_{i=0}^\alpha : \{*\nabla(F_i)\}_{i=0}^\alpha]$  satisfy the conclusion of Proposition 2.3 with respect to  $T$ ,  $\nabla$ , and  $\alpha$ . From Proposition 2.2(2) (and the transfer principle), we infer the existence of a nonzero  $x_0 \in l(p_i)$  such that  $*\nabla(F)(*x_0) \in *V_0$  for each  $F \in *\mathcal{F}\mathcal{S}(l(p_i))$ , which implies  $*\nabla(F)(*x_0) \in F \cap \text{fin}_\tau(*l(p_i))$  for each

$F \in {}^* \mathcal{F} \mathcal{S}(l(p_i))$  (see expression (3.2)). Consequently,

$${}^*q(T_\alpha)({}^*\nabla(F_l)({}^*x_0)) \in F_l \cap \text{ns}_\tau({}^*l(p_i)) \quad \text{for } l \in \{0, \dots, \alpha\} \quad (3.15)$$

by Proposition 3.7 since, for each  $l \in \{0, \dots, \alpha\}$ ,  ${}^*\nabla(F_l)({}^*x_0) \in F_l \subset E_\alpha$ , by definitions of  $\nabla$ ,  $\{F_l\}_{l=0}^\alpha$  (and the *transfer principle*), and  ${}^*q(T_\alpha)[F_l] \subset F_l$  (see expression (3.7)).

If  $\{x_0, Tx_0, \dots, T^m x_0\}$  is linearly *dependent* for some  $m \in \mathbb{N}_+$ , then the linear space generated by  $\{x_0, Tx_0, \dots, T^{m-1} x_0\}$  is nontrivial, closed, and invariant under  $T$ .

For the remainder of the proof, we will assume that  $\{x_0, Tx_0, \dots, T^m x_0\}$  is linearly *independent* for each  $m \in \mathbb{N}_+$ . Consequently,

$$q(T)(x_0) \neq 0. \quad (3.16)$$

Since the (internal) projection  $P_\alpha$  satisfies Proposition 2.1, we have  ${}^*x_0 - P_\alpha({}^*x_0) \in \mu_\tau(0)$ , which implies  $[{}^*q(T_\alpha)({}^*x_0) - {}^*q(T_\alpha)(P_\alpha({}^*x_0))] \in \mu_\tau(0)$  by Proposition 3.4 and  $[{}^*(q(T))(P_\alpha({}^*x_0)) - {}^*(q(T)(x_0))] \in \mu_\tau(0)$  because  $q(T) \in [l(p_i)]$ . Also,  $P_\alpha({}^*x_0) \in E_\alpha \cap \text{fin}_\tau({}^*l(p_i))$  (see expression (1.4)) implies

$$[{}^*q(T_\alpha)(P_\alpha({}^*x_0)) - {}^*(q(T))(P_\alpha({}^*x_0))] \in \mu_\tau(0) \quad (3.17)$$

by Proposition 3.6. Therefore,

$$\begin{aligned} & {}^*q(T_\alpha)({}^*x_0) - {}^*(q(T)(x_0)) \\ &= [{}^*q(T_\alpha)({}^*x_0) - {}^*q(T_\alpha)(P_\alpha({}^*x_0))] \\ & \quad + [{}^*q(T_\alpha)(P_\alpha({}^*x_0)) - {}^*(q(T))(P_\alpha({}^*x_0))] \\ & \quad + [{}^*(q(T))(P_\alpha({}^*x_0)) - {}^*(q(T)(x_0))], \end{aligned} \quad (3.18)$$

which implies  $[{}^*q(T_\alpha)({}^*x_0) - {}^*(q(T)(x_0))] \in \mu_\tau(0)$ . So,  $q(T)(x_0) \neq 0$  implies

$${}^*q(T_\alpha)({}^*x_0) \notin \mu_\tau(0) \quad (3.19)$$

since  $\tau = \tau(p_i)$  is Hausdorff. Therefore, by Propositions 2.4 and 3.4, expressions (3.15) and (3.19), there exists  $v \in \{1, \dots, \alpha\}$  such that  ${}^\circ F_{v-1} \neq l(p_i)$  and  ${}^\circ F_v \neq \{0\}$ . Since any two points of  ${}^\circ F_v$  are linearly dependent modulo  ${}^\circ F_{v-1}$ , we have that either  ${}^\circ F_{v-1}$  or  ${}^\circ F_v$  is a closed nontrivial linear subspace of  $T$ .  $\square$

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