WE MAKE A STUDY OF POISSON STRUCTURES OF $T^*M$ WHICH ARE GRADED STRUCTURES WHEN
RESTRICTED TO THE FIBERWISE POLYNOMIAL ALGEBRA AND WE GIVE EXAMPLES. A CLASS OF
MORE GENERAL GRADED BIVECTOR FIELDS WHICH INDUCE A GIVEN POISSON STRUCTURE $\omega$ ON THE BASE MANIFOLD $M$ IS CONSTRUCTED. IN PARTICULAR, THE_HORIZONTAL LIFTING OF A
POISSON STRUCTURE FROM $M$ TO $T^*M$ VIA CONNECTIONS GIVES SUCH BIVECTOR FIELDS AND WE DISCUSS THE CONDITIONS FOR THESE LIFTS TO BE POISSON BIVECTOR FIELDS AND THEIR COMPATIBILITY WITH THE CANONICAL POISSON STRUCTURE ON $T^*M$. FINALLY, FOR A 2-FORM $\omega$ ON A RIEMANNIAN MANIFOLD, WE STUDY THE CONDITIONS FOR SOME ASSOCIATED 2-FORMS OF $\omega$ ON $T^*M$ TO DEFINE POISSON STRUCTURES ON COTANGENT BUNDLES.


1. Introduction. In this paper, we present the dual version of the subject discussed in [4] and study graded bivector fields and Poisson structures on the cotangent bundle of a manifold. Although this study is similar to the one in [4], it is motivated by the presence of specific aspects. Indeed, we do not have a natural almost tangent structure and semisprays anymore, but we have the canonical symplectic structure instead. This makes a separate exposition required. Another new aspect that we discuss is that of a base manifold which is a Riemannian space.

2. Graded Poisson structures on cotangent bundles. Let $M$ be an $n$-dimensional differentiable manifold and $\pi : T^*M \to M$ its cotangent bundle. If $(x^i) (i = 1, \ldots, n)$ are local coordinates on $M$, we denote by $(p_i)$ the covector coordinates with respect to the cobasis $(dx^i)$. (We assume that everything is $C^\infty$ in this paper.)

In this section, we discuss graded Poisson structures $W$ on the cotangent bundle $T^*M$ obtained as lifts of Poisson structures $\omega$ on the base manifold $M$, in the sense that the canonical projection $\pi$ is a Poisson mapping (see [4]).

Denote by $S_k(TM)$ the space of $k$-contravariant symmetric tensor fields on $M$ and by $\otimes$ the symmetric tensor product on the algebra $S(TM) = \bigoplus_{k \geq 0} S_k(TM)$. The spaces of fiberwise homogeneous $k$-polynomials

$$\mathcal{H} \mathcal{P}_k(T^*M) := \left\{ \hat{Q} = Q_{i_1 \cdots i_k} p_{i_1} \cdots p_{i_k} \mid Q = Q_{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \in S_k(TM) \right\}$$

(2.1)
are interesting subspaces of the function space $C^\infty(T^*M)$ and play an important role in this paper.

The map
\[
\sim: (S(TM), \odot) \to (\mathcal{P}(T^*M), \cdot), \quad \sim Q := \hat{Q},
\]
(2.2)
where $\mathcal{P}(T^*M) := \oplus_k \mathcal{H} \mathcal{P}_k(T^*M)$ is the polynomial algebra and the dot denotes the usual multiplication, is an isomorphism of algebras.

On $T^*M$ we also have the spaces of (fiberwise) nonhomogeneous polynomials of degree less than or equal to $k$
\[
\mathcal{P}_k(T^*M) := \bigoplus_{h=0}^k \mathcal{H} \mathcal{P}_h.
\]
(2.3)
For $k = 1$, $\mathcal{P}(T^*M) := \mathcal{P}_1(T^*M)$ is the space of affine functions, having the elements of the form
\[
a(x,p) = f(x) + m(X),
\]
(2.4)
where $f \in C^\infty(M), X \in \chi(M)$ (the space of vector fields on $M$), and $m(X) := \sim X$ is the momentum of $X$. (The momentum $m(X)$ is $X$ regarded as a function on $T^*M$.)

The elements of the space $\mathcal{P}_2(T^*M)$ of nonhomogeneous quadratic polynomials are
\[
t(x,p) = f(x) + m(X) + s(Q),
\]
(2.5)
where $Q = Q^{ij}(\partial/\partial x^i) \circ (\partial/\partial x^j)$ is a symmetric contravariant tensor field on $M$ and $s(Q) := \sim Q$.

Hereafter, by a polynomial on $T^*M$, we always mean a fiberwise polynomial. Also, we write $f$ for both $f$ on $M$ and $f \circ \pi$ on $T^*M$.

**Definition 2.1.** A Poisson structure $W$ on $T^*M$ is called polynomially graded if for all $Q,R \in \mathcal{P}(T^*M)$,
\[
Q \in \mathcal{P}_h, \ R \in \mathcal{P}_k \Rightarrow \{Q,R\}_W \in \mathcal{P}_{h+k}.
\]
(2.6)

**Proposition 2.2.** A polynomially graded Poisson structure $W$ on $T^*M$ induces a Poisson structure $w$ on the base manifold $M$ such that the projection $\pi : (T^*M, W) \to (M, w)$ is a Poisson mapping.

**Proof.** Any function $f$ on $M$ is a polynomial $(f \circ \pi) \in \mathcal{P}_0(T^*M)$. By (2.6), for all $f,g \in C^\infty(M), \{f \circ \pi, g \circ \pi\}_W \in C^\infty(M)$ and
\[
\{f,g\}_W := \{f \circ \pi, g \circ \pi\}_W
\]
(2.7)
defines a Poisson structure $w$ on $M$. \qed
Hereafter, the bracket \( \{ \cdot, \cdot \}_W \) will be denoted simply by \( \{ \cdot, \cdot \} \).

If the local coordinate expression of the Poisson structure \( w \) introduced by Proposition 2.2 is

\[
w = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},
\]

(2.8)

Definition 2.1 tells us that \( W \) must have the local coordinate expression

\[
W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + (\phi^i_j(x) + p_a A^a_{ij}(x)) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \frac{1}{2} (\eta_{ij}(x) + p_a B^a_{ij}(x) + p_a p_b C_{ij}^{ab}(x)) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j},
\]

(2.9)

where \( w, \phi, \eta, A, B, \) and \( C \) are local functions on \( M \).

The Poisson structure \( W \) is completely determined by the brackets \( \{ f, g \}, \{ m(X), f \}, \) and \( \{ m(X), m(Y) \} \), where \( f, g \in C^\infty(M) \) and \( X, Y \in \chi(M) \) since the local coordinates \( x^i \) and \( p_i \) are functions of this type \( (p_i = m(\partial/\partial x^i)) \).

By (2.6), the bracket \( \{ m(X), f \} \) is in \( \mathcal{P}_1(T^*M) \), that is,

\[
\{ m(X), f \} = Z_X f + m(\gamma_X f),
\]

(2.10)

where \( Z_X f \in C^\infty(M) \) and \( \gamma_X f \in \chi(M) \).

The map \( \{ m(X), \cdot \} \) is a derivation of \( C^\infty(M) \). Hence, \( Z_X \) is a vector field on \( M \) and the mapping \( \gamma_X : C^\infty(M) \to \chi(M) \) also is a derivation. Therefore, \( \gamma_X f \) depends only on \( df \).

From the Leibniz rule, we get that \( Z_h X = h Z_X \) \( (h \in C^\infty(M)) \) and \( \gamma \) must satisfy

\[
\gamma_h X f = h \gamma_X f + (X_h^w f) X.
\]

(2.11)

The bracket of two affine functions has an expression of the form

\[
\{ m(X), m(Y) \} = \beta(X, Y) + m(V(X, Y)) + s(\Psi(X, Y)),
\]

(2.12)

where \( \beta(X, Y) \in C^\infty(M), V(X, Y) \in \chi(M), \) and \( \Psi(X, Y) \in S_2(TM) \) are skew-symmetric operators. If we replace \( Y \) by \( f Y \) in (2.12), the Leibniz rule gives that \( \beta \) is a 2-form on \( M \) and

\[
V(X, f Y) = f V(X, Y) + (Z_X f) Y,
\]

\[
\Psi(X, f Y) = f \Psi(X, Y) + (\gamma_X f) Y.
\]

(2.13)

**Definition 2.3.** A polynomially graded Poisson structure \( W \) on \( T^*M \) is said to be a graded structure if for all \( Q \in \mathcal{HP}_h \) and for all \( R \in \mathcal{HP}_k \), it follows \( \{ Q, R \}_W \in \mathcal{HP}_{h+k} \).
Remark that a polynomially graded structure on $T^*M$ is graded if and only if $Z_X = 0$, $\beta = 0$, and $V = 0$. In this case, (2.9) reduces to

$$W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + p_a A^i_j(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} + \frac{1}{2} p_a p_b C^{ab}_{ij}(x) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.$$  

(2.14)

As in [4], a bivector field $W$ on $T^*M$ which is locally of the form (2.9) (resp., (2.14)) is called a polynomially graded (resp., graded) bivector field.

**Proposition 2.4.** If $W$ is a graded bivector field on $T^*M$ which is $\pi$-related with a Poisson structure $\omega$ on $M$, there exists a contravariant connection $D$ on the Poisson manifold $(M, \omega)$ such that

$$\{m(X), f\} = -m(D df X), \quad X \in \chi(M), \ f \in C^\infty(M).$$  

(2.15)

Moreover, if $W$ is a graded Poisson structure on $T^*M$, then the connection $D$ is flat.

**Proof.** A contravariant connection on $(M, \omega)$ is a contravariant derivative on $TM$ with respect to the Poisson structure [8].

The required connection is defined by

$$D df X := -\gamma_X f.$$  

(2.16)

That we really get a connection, which is flat in the Poisson case, follows in exactly the same way as in [4].

The relation (2.15) extends to the following proposition.

**Proposition 2.5.** If $Q$ is a symmetric contravariant tensor field on $M$ and $\tilde{Q}$ is its corresponding polynomial, then for any graded Poisson bivector field $W$ on $T^*M$,

$$\{\tilde{Q}, f\}_W = -\tilde{D df} \tilde{Q}.$$  

(2.17)

**Proof.** The contravariant connection $D df$ of (2.17) is extended to $S(TM)$ by

$$(D df Q)(\alpha_1, \ldots, \alpha_k) = X^w_j(Q(\alpha_1, \ldots, \alpha_k))$$

$$- \sum_{i=1}^k Q(\alpha_1, \ldots, D df \alpha_i, \ldots, \alpha_k),$$  

(2.18)

where $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$, and $D df \alpha$ is defined by

$$\langle D df \alpha, X \rangle = X^w_j(\alpha, X) - \langle \alpha, D df X \rangle, \quad X \in \chi(M).$$  

(2.19)
We put
\[ D_{dx^i} \frac{\partial}{\partial x^j} = -\Gamma^k_{ij} \frac{\partial}{\partial x^k}, \] (2.20)
and by a straightforward computation we get for \( \{ \tilde{Q}, f \} \) and \( -\tilde{(D_{df} Q)} \) the same local coordinate expression. (See [4] for the complete proof in the case of a symmetric covariant tensor field on \( M \).)

In order to discuss the next two Jacobi identities, we make some remarks concerning the operator \( \Psi \) of (2.12), which is given in the case of a graded Poisson structure on \( T^*M \) by
\[ \{ m(X), m(Y) \} = s(\Psi(X,Y)), \quad X, Y \in \chi(M). \] (2.21)
With (2.16), the second relation (2.13) becomes
\[ \Psi(X, fY) = f\Psi(X,Y) - \frac{1}{2} (D_{df}X \otimes Y + Y \otimes D_{df}X) \] (2.22)
and this allows us to derive the local coordinate expression of \( \Psi \). If \( X = X^i(\partial/\partial x^i) \) and \( Y = Y^j(\partial/\partial x^j) \), we obtain
\[ \Psi(X,Y) = X^iY^j \Psi \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) + \left( X^h \frac{\partial Y^j}{\partial x^k} \Gamma_{ki}^h - Y^h \frac{\partial X^i}{\partial x^k} \Gamma_{kj}^h \right) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \]
(2.23)
Remark that \( \Psi : TM \times TM \to \bigodot^2 TM \) is a bidifferential operator of the first order.

**Proposition 2.6.** If the operator \( D_{df} \) acts on \( \Psi \) by
\[ (D_{df} \Psi)(X,Y) := D_{df}(\Psi(X,Y)) - \Psi(D_{df}X,Y) - \Psi(X, D_{df}Y), \] (2.24)
the Jacobi identity
\[ \{ \{ m(X), m(Y) \}, f \} + \{ \{ m(Y), f \}, m(X) \} + \{ \{ f, m(X) \}, m(Y) \} = 0 \] (2.25)
has the equivalent form
\[ (D_{df} \Psi)(X,Y) = 0, \quad \forall X, Y \in \chi(M). \] (2.26)

**Proof.** Using (2.15), (2.17), and (2.21) for \( Q = \Psi(X,Y) \), (2.25) becomes (2.26). \( \Box \)
We also find
\[
(D_{df} \Psi)(X, hY) = h(D_{df} \Psi)(X, Y) - [C_D(df, dh) X] \odot Y,
\] (2.27)
and hence we see that (2.26) is invariant by \(X \mapsto fX, Y \mapsto gY\) \((f, g \in C^\infty(M))\) if and only if the curvature \(C_D = 0\).

Concerning the Jacobi identity
\[
\sum_{(X,Y,Z)} \{\{m(X), m(Y)\}, m(Z)\} = 0,
\] (2.28)
(putting indices between parentheses denotes that summation is on cyclic permutations of these indices) remark that one must have an operator \(\Theta\) such that
\[
\{s(G), m(X)\} = \widehat{\Theta(G, X)}, \quad X \in \chi(M), \ G \in S_2(M),
\] (2.29)
and \(\Theta(G, X)\) is a symmetric 3-contravariant tensor field on \(M\).

We get the formula
\[
\Theta(f G, hX) = fh \Theta(G, X) - f(D_{dh} G) \odot X + hG \odot D_{df} X + [f, h]_w G \odot X,
\] (2.30)
and then the local coordinate expression
\[
\Theta(G, X) = G^{ij} X^k \Theta \left( \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right)
\]
\[+ \frac{1}{3} \sum_{(i,j,k)} \left( G^{jh} \frac{\partial X^k}{\partial x^a} \Gamma^a_{ij} + G^{ih} \frac{\partial X^k}{\partial x^a} \Gamma^a_{kj} - \frac{\partial G^{ij}}{\partial x^a} \chi^h \Gamma^k_{ij} 
\]
\[+ w^{ab} \frac{\partial G^{ij}}{\partial x^a} \frac{\partial X^k}{\partial x^b} \right) \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j} \odot \frac{\partial}{\partial x^k}.
\] (2.31)

Using the operator \(\Theta\), the Jacobi identity (2.28) becomes
\[
\sum_{(X,Y,Z)} \Theta(\Psi(X,Y), Z) = 0,
\] (2.32)
and we may summarize our analysis concerning the graded Poisson structures on \(T^*M\) in the following proposition.

**Proposition 2.7.** A graded Poisson structure \(W\) on \(T^*M\) with the bracket \(\{\cdot, \cdot\}\) is defined by
\(\text{(a)}\) a Poisson structure \(w\) on the base manifold \(M\) such that
\[
\{f, g\}_w = \{f, g\}_w, \quad f, g \in C^\infty(M);
\] (2.33)
(b) a flat contravariant connection $D$ on $(M, w)$ such that
\[
\{m(X), f\} = -m(D_{df}X), \quad X \in C^\infty(M); \tag{2.34}
\]

(c) an operator $\Psi : TM \times TM \to \mathcal{O}^2 TM$ such that
\[
\{m(X), m(Y)\} = s(\Psi(X, Y)), \quad X, Y \in \chi(M), \tag{2.35}
\]

and formula (2.26) holds;

(d) an operator $\Theta$ defined by (2.29), satisfying (2.32).

To give examples, we consider the following situation similar to [4].
Let $(M, w)$ be an $n$-dimensional Poisson manifold and suppose that its symplectic foliation $S$ is contained in a regular foliation $\mathcal{F}$ on $M$ such that $T\mathcal{F}$ is a foliated bundle, that is, there are local bases $\{Y_u\} (u = 1, \ldots, p, \ p = \text{rank} \mathcal{F})$ of $T\mathcal{F}$ with transition functions constant along the leaves of $\mathcal{F}$. Consider a decomposition
\[
TM = T\mathcal{F} \oplus \nu \mathcal{F}, \tag{2.36}
\]
where $\nu \mathcal{F}$ is a complementary subbundle of $T\mathcal{F}$, and $\mathcal{F}$-adapted local coordinates $(x^a, y^u) (a = 1, \ldots, n - p)$ on $M$ [7].

The Poisson bivector $w$ has the form
\[
w = \frac{1}{2} w^{uv}(x, y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \quad (w^{vu} = -w^{uv}) \tag{2.37}
\]
since $S \subseteq \mathcal{F}$.

If $\{\beta^u\}, \{\tilde{\beta}^v\} (u, v = 1, \ldots, p)$ are the dual cobases of $\{Y_u\}, \{\tilde{Y}_v\}$ ($\beta^u(Y_v) = \delta^u_v$), then their transition functions are constant along the leaves of $\mathcal{F}$.

Now, for all $\alpha \in T^*M$, $\alpha = \zeta_a dx^a + \varepsilon_u \beta^u$ and we may consider $(x^a, y^u, \zeta_a, \varepsilon_u)$ as distinguished local coordinates on $T^*M$. The transition functions are
\[
\tilde{x}^a = \tilde{x}^a(x), \quad \tilde{y}^u = \tilde{y}^u(x, y), \quad \tilde{\zeta}_u = \frac{\partial x^a}{\partial \tilde{x}^u} \zeta_a, \quad \tilde{\varepsilon}_u = a^v_u(x) \varepsilon_v. \tag{2.38}
\]

**PROPOSITION 2.8.** Under the previous hypotheses, $W$ given with respect to the distinguished local coordinates by
\[
W = \frac{1}{2} w^{uv}(x, y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \tag{2.39}
\]
defines a graded Poisson bivector on $T^*M$.

**PROOF.** From (2.38) it follows that $W$ of (2.39) is a global tensor field on $T^*M$. The Schouten-Nijenhuis bracket $[W, W]$ has the same expression as $[w, w]$ on $M$, and thus the Poisson condition $[W, W] = 0$ holds.
To prove that $W$ is graded, we also consider natural coordinates and show that the expression of $W$ with respect to these coordinates becomes of the form (2.14) (see [4]).

There are some interesting particular cases of Proposition 2.8.

(a) The Poisson structure $w$ is regular, and the bundle $TS$ is a foliated bundle; in this case we may take $\mathcal{F} = S$.

(b) The symplectic foliation $S$ is contained in a regular foliation $\mathcal{F}$ which admits adapted local coordinates $(x^u, y^u)$ with local transition functions

$$\bar{y}^v = p^v_u(x) y^u + q^v(x).$$

(The foliation $\mathcal{F}$ is a leaf-wise, locally affine and regular.) In this case, $(\partial/\partial y^u) = \sum_v a^v_u(x) (\partial/\partial \bar{y}^v)$ and we may use the local vector fields $Y_u = \partial/\partial y^u$.

(c) There exists a flat linear connection $\nabla$ (possibly with torsion) on the Poisson manifold $(M, w)$. In this case, we may consider as leaves of $\mathcal{F}$ the connected components of $M$, and the local $\nabla$-parallel vector fields have constant transition functions along these leaves. Therefore, we may take them as $Y_i$ ($i = 1, \ldots, n$).

In particular, we have the result of (c) for a locally affine manifold $M$ (where $\nabla$ has no torsion), using as $Y_i$ local $\nabla$-parallel vector fields, and also for a parallelizable manifold $M$ (where we have global vector fields $Y_i$).

As a consequence, Proposition 2.8 holds for the Lie-Poisson structure [8] of any dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, the graded Poisson structure being defined on $T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$.

3. Graded bivector fields on cotangent bundles. In this section, we discuss graded bivector fields on a cotangent bundle $T^*M$, which may be seen as lifts of a given Poisson structure $w$ on $M$, that satisfy less restrictive existence conditions than in the case of graded Poisson structures.

Recall the following definition from [4]. Let $\mathcal{F}$ be an arbitrary regular foliation, with $p$-dimensional leaves, on an $n$-dimensional manifold $N$. We denote by $C^\infty_{\text{fol}}(N)$ the space of foliated functions (the functions on $N$ which are constant along the leaves of $\mathcal{F}$). A transversal Poisson structure of $(N, \mathcal{F})$ is a bivector field $w$ on $N$ such that

$$\{f, g\} := w(df, dg), \quad f, g \in C^\infty_{\text{fol}}(N)$$

is a Lie algebra bracket on $C^\infty_{\text{fol}}(N)$. A bivector field $w$ on $N$ defines a transversal Poisson structure of $(N, \mathcal{F})$ if and only if [4]

$$(\mathcal{L}_Y w)|_{\text{Ann} T\mathcal{F}} = 0, \quad [w, w]|_{\text{Ann} T\mathcal{F}} = 0,$$

for all $Y \in \Gamma(T\mathcal{F})$ (the space of global cross sections of $T\mathcal{F}$), where $\text{Ann} T\mathcal{F} \subseteq \Omega^1(N)$ is the annihilator space of $T\mathcal{F}$. ($\Omega^1(N)$ denotes the space of Pfaff forms on $N$.)
The cotangent bundle $T^*M$ of any manifold $M$ has the vertical foliation $\mathcal{F}$ by fibers with the tangent distribution $V := T\mathcal{F}$.

Obviously, the set of foliated functions on $T^*M$ may be identified with $C^\infty(M)$.

**Proposition 3.1.** Any polynomially graded bivector field $W$ on $T^*M$, which is $\pi$-related with a Poisson structure of $M$, is a transversal Poisson structure of $(T^*M,V)$.

**Proof.** The local coordinate expression of $W$ is of the form (2.9), and $W$ is $\pi$-related with the bivector field $\varpi$ defined on $M$ by the first term of (2.9). Then, (3.2) holds because $\varpi$ is a Poisson bivector on $M$. \qed

**Definition 3.2.** A transversal Poisson structure of the vertical foliation of $T^*M$ will be called a semi-Poisson structure on $T^*M$.

**Remark 3.3.** The structures $W$ of Proposition 3.1 are polynomially graded semi-Poisson structures on $T^*M$.

In what follows, we discuss some interesting classes of graded semi-Poisson structures of $T^*M$. Then, we give a method to construct all the graded semi-Poisson bivector fields on $T^*M$, which induce the same Poisson structure $\varpi$ on the base manifold $M$.

Let $D$ be a contravariant derivative on a Poisson manifold $(M,\varpi)$. First, for all $Q \in S_k(TM)$, define $^sDQ \in S_{k+1}(TM)$ by

\[
(^sDQ)(\alpha_1,\ldots,\alpha_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (D_{\alpha_i}Q)(\alpha_1,\ldots,\hat{\alpha_i},\ldots,\alpha_{k+1}),
\]

where $\alpha_1,\ldots,\alpha_{k+1} \in \Omega^1(M)$ and the hat denotes the absence of the corresponding factor.

If $X = X^i(\partial/\partial x^i) \in \chi(M)$, then $DX$, defined by $(DX)(\alpha_1,\alpha_2) = (D_{\alpha_1}X)\alpha_2$, is a 2-contravariant tensor field on $M$, and

\[
DX = D^jX^i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},
\]

where $D^jX^i = (D_{dx^i}X)dx^j = D_{dx^i}X^j - X(D_{dx^i}dx^j)$. According to (2.20), we must have

\[
D_{dx^i}dx^j = \Gamma_{ij}^k dx^k
\]

and obtain

\[
D^jX^i = (dx^i)^jX^j - \Gamma_{ij}^k dx^k = \{x^i, X^j\} - \Gamma_{ij}^k X^k.
\]
Then

\[ sDX = \frac{1}{2} (D^j X^j + D^i X^i) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \]  \hspace{1cm} (3.7)

and we get

\[ sDX = \frac{1}{2} \left( \{x^i, x^j\}_w + \{x^j, x^i\}_w - \Gamma^j_k X^k - \Gamma^i_k X^k \right) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \]  \hspace{1cm} (3.8)

**Proposition 3.4.** Let \((M, w)\) be a Poisson manifold and \(D\) a contravariant derivative of \((M, w)\). The bivector field \(W_1\) on \(T^*M\), of bracket \(\{\cdot, \cdot\}_W\) defined by the conditions

\[ \{f, g\}_{W_1} := \{f, g\}_w, \]  \hspace{1cm} (3.9)

\[ \{m(X), f\}_{W_1} := -m(Df X), \]  \hspace{1cm} (3.10)

\[ \{m(X), m(Y)\}_{W_1} = \frac{1}{2} s[\{sD(X, Y) - \{sDX, Y\} - \{X, sDY\}], \]  \hspace{1cm} (3.11)

where \(f, g \in C^\infty(M), X, Y \in \chi(M), \) and \(\langle \cdot, \cdot \rangle\) is the Schouten-Nijenhuis bracket of symmetric tensor fields (defined by the natural Lie algebroid of \(M\)) [1, 4], defines a graded semi-Poisson structure on \(T^*M\) which is \(\pi\)-related with \(w\).

**Proof.** If the local coordinate expression of \(w\) is (2.8), using (3.8) and the properties of \(\langle \cdot, \cdot \rangle\) [1, 4], we get

\[ W_1 = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} - p_a \Gamma^a_j i \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial p_i} \]  \hspace{1cm} (3.12)

\[ - \frac{1}{4} p_a p_b \left[ \frac{\partial}{\partial x^j} (\Gamma^a_i + \Gamma^a_j) - \frac{\partial}{\partial x^i} (\Gamma^a_i + \Gamma^a_j) \right] \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}. \]

**Remark 3.5.** The relation (3.11) provides us with the expression of the operator \(\Psi_{W_1}\) associated to \(W_1\) (see (2.21)):
where \( \#_w : T^* M \to TM \) is defined by \( \beta(\alpha^\sharp) = w(\alpha, \beta) \) for all \( \beta \in \Omega^1(M) \), and the upper index \( H \) denotes the horizontal lift with respect to \( \nabla \) (see [2, 9]). In local coordinates, we get

\[
K = p_a w^{ai} \frac{\partial}{\partial x^i} + \frac{1}{2} p_a p_b (w^{ak} \Gamma^b_{ki} + w^{bk} \Gamma^a_{ki}) \frac{\partial}{\partial p_i}.
\]  

(3.15)

On \( T^* M \), we have the canonical symplectic form \( \omega = d\lambda = dp_i \wedge dx^i \), where \( \lambda = p_i dx^i \) is the Liouville form, and the vector bundle isomorphism

\[
\#_\omega : T^* M \to TM, \quad i_X \omega \in T^* M \to X \in TM
\]  

(3.16)

leads to the canonical Poisson bivector \( W_0 := \#_\omega \omega \) on \( T^* M \). It follows that

\[
W_0(dF, dG) = \omega(\#(dF), \#(dG)), \quad F, G \in C^\infty(T^* M),
\]  

(3.17)

and, locally, one has

\[
W_0 = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i}.
\]  

(3.18)

**PROPOSITION 3.6.** If \((M, w)\) is a Poisson manifold, then the bivector field

\[
W_2 = \frac{1}{2} \mathcal{L}_K W_0
\]  

defines a graded semi-Poisson structure on \( T^* M \) which is \( \pi \)-related with \( w \).

**PROOF.** We get

\[
W_2 = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{2} p_a (\nabla_j w^{ai} + 2 w^{ik} \Gamma^a_{kj}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j}
\]

\[
+ \frac{1}{4} p_a p_b \left[ \frac{\partial}{\partial x^j} (w^{ak} \Gamma^b_{ki} + w^{bk} \Gamma^a_{ki}) - \frac{\partial}{\partial x^i} (w^{ak} \Gamma^b_{kj} + w^{bk} \Gamma^a_{kj}) \right] \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j},
\]  

(3.20)

where \( \nabla_j w^{ai} \) are the components of the \((2,1)\)-tensor field on \( M \) defined by \( X \to \nabla_X w, X \in \mathfrak{X}(M) \).

We will say that \( W_2 \) of (3.19) is the graded \( \nabla \)-lift of the Poisson structure \( w \) of \( M \).

Using local coordinates and the notation of (2.2), we get

\[
\mathcal{L}_K \tilde{Q} = \tilde{D} \tilde{Q},
\]  

(3.21)

where \( D \) is the contravariant derivative induced by the linear connection \( \nabla \), defined by \( D_{df} = \nabla_{(df)^\sharp} \) (see [8]).
From (3.19) we have
\[
\{F_1, F_2\}_{w_2} := W_2(dF_1, dF_2) = \frac{1}{2} \bigl( \mathcal{L}_K \left( \{F_1, F_2\}_{w_0} \right) - \{\mathcal{L}_K F_1, F_2\}_{w_0} - \{F_1, \mathcal{L}_K F_2\}_{w_0} \bigr),
\]
(3.22)

where \( F_1, F_2 \in C^\infty( T^*M) \).

If \( Q_1, Q_2 \in S(TM) \), using (3.21) and the relation
\[
\{\tilde{Q}, \tilde{H}\}_{w_0} := \tilde{i}(Q, H), \quad Q, H \in S(TM)
\]
(3.23)
(see [1, 4]), we get the explicit formula
\[
\{\tilde{Q}_1, \tilde{Q}_2\}_{w_2} = \frac{1}{2} \cdot \frac{1}{2} \tilde{\iota}(\tilde{\nabla}f, \tilde{g}) - \tilde{i}(\tilde{f}, \tilde{g}), \quad \forall f, g \in C^\infty(M);
\]
(3.24)

**Proposition 3.7.** The graded \( \nabla \)-lift \( W_2 \) of \( w \) is characterized by the following:

(i) the Poisson structure induced on \( M \) by \( W_2 \) is \( w \), that is,
\[
\{f, g\}_{w_2} = \{f, g\}_w, \quad \forall f, g \in C^\infty(M);
\]
(3.25)

(ii) for every \( f \in C^\infty(M) \) and \( X \in \chi(M) \),
\[
\{m(X), f\}_{w_2} = -m(\tilde{D}f, X),
\]
(3.26)

where \( \tilde{D} \) is the contravariant derivative of \((M, w)\) defined by
\[
\tilde{D}_{\alpha\beta} = D_{\alpha\beta} + \frac{1}{2} \nabla.w(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(M),
\]
(3.27)

where the contravariant derivative \( D \) is induced by \( \nabla \) and \( (\nabla.w)(\alpha, \beta) \) is the 1-form \( X \rightarrow (\nabla.X.w)(\alpha, \beta) \);

(iii) for any vector fields \( X \) and \( Y \) of \( M \),
\[
\{m(X), m(Y)\}_{w_2} = \frac{1}{2} (\tilde{\iota}D^w(X, Y) - \tilde{\iota}D^w(Y, X) - \{X, D^w(Y)\}).
\]
(3.28)

**Proof.** (i) If \( f \in C^\infty(M) \), then \( Df = -X^w_f \) and from (3.22), (3.23), and the formula
\[
\langle Q, f \rangle = \tilde{i}(d(f)Q), \quad f \in C^\infty(M), \quad Q \in S_p(TM),
\]
(3.29)

we get
\[
\{f, g\}_{w_2} = -\frac{1}{2} (\langle Df, g \rangle + \langle f, Dg \rangle) = \frac{1}{2} (X^w_f g - X^w_g f) = \{f, g\}_w.
\]
(3.30)
(ii) As $W_2$ is graded, the bracket $\{m(X), f\}_{W_2}$ must be of the form (3.26).

Denoting
$$\tilde{D}_{dx^i}dx^j = \Gamma^i_k dx^k,$$
(3.31) gives us
$$\dot{\Gamma}^i_k = \Gamma^i_k + \frac{1}{2} \nabla_k w^{ij},$$
(3.32)
where
$$\Gamma^i_k = -w^{ih}\Gamma^j_h,$$
(3.33)
where the coefficients of the linear connection $\nabla$ and hence (3.27).

(iii) Equation (3.28) is a direct consequence of (3.24).\qed

Notice from (3.28) that the operator $\Psi_{W_2}$ associated to $W_2$ has the same expression as $\Psi_{W_1}$ of (3.13), but in the case of $W_1$, the contravariant derivative $D$ is induced by a linear connection $\nabla$ on $M$.

**Proposition 3.8.** If the graded semi-Poisson structure $W_1$ is defined by a linear connection on $(M, w)$, then it coincides with $W_2$ if and only if $w$ is $\nabla$-parallel.

**Proof.** Compare the characteristic conditions of Propositions 3.4 and 3.7 (or the coefficients of $(\partial/\partial x^i) \wedge (\partial/\partial p_j)$ of (3.12) and of (3.20), using (3.33)). \qed

We prove now the following proposition.

**Proposition 3.9.** Let $(M, w)$ be a Poisson manifold and $\pi : T^*M \to M$ its cotangent bundle. The graded semi-Poisson structures $W$ on $T^*M$ which are $\pi$-related with $w$ are defined by the relations
$$\{f, g\}_W = \{f, g\}_w, \quad \{m(X), f\}_W = -m(D_dX),$$
$$\{m(X), m(Y)\}_W = s(\Psi(X, Y)), \quad f, g \in C^\infty(M), \quad X, Y \in \chi(M),$$
(3.34)
where $D$ is an arbitrary contravariant connection of $(M, w)$ and the operator $\Psi$ is given by
$$\Psi = \Psi_0 + A + T,$$
(3.35)
where $\Psi_0$ is the operator $\Psi$ of a fixed graded semi-Poisson structure and $A : TM \times TM \to \odot^2TM$ is a skew-symmetric, first-order, bidifferential operator such that
$$A(X, fY) = fA(X, Y) - \tau(df, X) \odot Y,$$
(3.36)
where $\tau$ is a $(2,1)$-tensor field on $M$ and $T$ is a $(2,2)$-tensor field on $M$ with the properties $T(Y,X) = -T(X,Y)$ and $T(X,Y) \in S_2(TM)$ for all $X,Y \in \chi(M)$.

**Proof.** If two graded semi-Poisson bivector fields, $\pi$-related with $\omega$, have associated the same contravariant connection $D$, it follows from (2.22) that the difference $\Psi' - \Psi$ is a tensor field $T$, as in Proposition 3.8. To change $D$ means to pass to a contravariant connection $D' = D + \tau$, where $\tau$ is a $(2,1)$-tensor field on $M$ and from (2.22) again, it follows that $A = \Psi' - \Psi$ becomes a bidifferential operator with the property (3.35). $\Box$

4. Horizontal lifts of Poisson structures. In this section, we define and study an interesting class of semi-Poisson structures on $T^*M$ which are produced by a process of horizontal lifting of Poisson structures from $M$ to $T^*M$ via connections.

On $T^*M$, we distinguish the vertical distribution $V$, tangent to the fibers of the projection $\pi$ and, by complementing $V$ by a distribution $H$, called horizontal, we define a nonlinear connection on $T^*M$ [5, 6].

We have (adapted) bases of the form

$$V = \text{span}\left\{\frac{\partial}{\partial p_i}\right\}, \quad H = \text{span}\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{ij} \frac{\partial}{\partial p_j}\right\}, \quad (4.1)$$

and $N_{ij}$ are the coefficients of the connection defined by $H$.

Equivalently, a nonlinear connection may be seen as an almost product structure $\Gamma$ on $T^*M$ such that the eigendistribution corresponding to the eigenvalue $-1$ is the vertical distribution $V$ [6].

We assume that the nonlinear connection above is symmetric, that is, $N_{ji} = N_{ij}$. This condition is independent [6] of the local coordinates.

The complete integrability of $H$, in the sense of the Frobenius theorem, is equivalent to the vanishing of the curvature tensor field

$$R = R_{kij} dx^i \wedge dx^j \otimes \frac{\partial}{\partial p_k}, \quad R_{kij} = \frac{\delta N_{kj}}{\delta x^i} - \frac{\delta N_{ki}}{\delta x^j}. \quad (4.2)$$

For a later utilization, we also notice the formulas [5, 6]

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R_{kij} \frac{\partial}{\partial p_k}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j}\right] = -\Phi^i_{jk} \frac{\partial}{\partial p_k}, \quad \Phi^i_{jk} = -\frac{\partial N_{ik}}{\partial x^j}. \quad (4.3)$$

Let $\omega$ be a bivector on $M$ with the local coordinate expression (2.8).

**Definition 4.1.** The horizontal lift of $\omega$ to the cotangent bundle $T^*M$ is the (global) bivector field $\omega^H$ defined by

$$\omega^H = \frac{1}{2} \omega^{ij}(x) \frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j}. \quad (4.4)$$
**Proposition 4.2.** Let \((M, w)\) be a Poisson manifold. If the connection \(\Gamma\) on \(T^* M\) is defined by a linear connection \(\nabla\) on \(M\), the bivector \(w^H\) defines a graded semi-Poisson structure on \(T^* M\).

**Proof.** In this case, the coefficients of \(\Gamma\) are
\[
N_{ij} = -p_k \Gamma^k_{ij},
\]
where \(\Gamma^k_{ij}\) are the coefficients of \(\nabla\) and, with respect to the bases \(\{\partial/\partial x^i, \partial/\partial p_j\}\), the local expression of \(w^H\) becomes
\[
W = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + w^{ik} \Gamma^a_{kj} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_a} + \frac{1}{2} w^{jh} \Gamma^a_{ki} \Gamma^b_{jp} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.
\]

**Proposition 4.3.** The horizontal lift \(w^H\) is a Poisson bivector on the cotangent bundle \(T^* M\) if and only if \(w\) is a Poisson bivector on the base manifold \(M\) and
\[
R(X^H_f, X^H_g) = 0, \quad \forall f, g \in \mathcal{C}^\infty(M),
\]
where \(X^H_f\) denotes the usual horizontal lift \([2, 9]\), from \(M\) to \(T^* M\), of the \(w\)-Hamiltonian vector field \(X_f\) on \(M\).

**Proof.** We compute the bracket \([w^H, w^H]\) with respect to the bases (4.1) and get that the Poisson condition \([w^H, w^H] = 0\) is equivalent with the pair of conditions
\[
\sum_{(i, j, k)} w^{hk} \frac{\partial w^{ij}}{\partial x^h} = 0, \quad w^{il} w^{jh} R_{klh} = 0.
\]
(Putting indices between parentheses denotes that summation is on cyclic permutations of these indices.)

The first condition in (4.8) is equivalent to \([w, w] = 0\) and the second is the local coordinate expression of (4.7).

**Remark 4.4.** If \(w\) is defined by a symplectic form on \(M\), condition (4.8) becomes \(R = 0\).
COROLLARY 4.5. If \((M, w)\) is a Poisson manifold and the connection \(\Gamma\) on \(T^*M\) is defined by a linear connection \(\nabla\) on \(M\), the bivector \(w^H\) defines a Poisson structure on \(T^*M\) if and only if the curvature \(C_D\) of the contravariant connection induced by \(\nabla\) on \(TM\) vanishes. In this case, \(w^H\) is a graded Poisson structure on \(T^*M\).

PROOF. If \(R^h_{kij}\) are the components of the curvature \(R_\nabla\), then

\[
R_{kij} = -p_h R^h_{kij}
\]

and (4.9) becomes

\[
R_\nabla (\sharp \alpha, \sharp \beta) Z = 0, \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall Z \in \chi(M),
\]

or, equivalently,

\[
R_\nabla (X_f, X_g) Z = 0, \quad \forall f, g \in C^\infty(M), \quad \forall Z \in \chi(M).
\]

This is equivalent to \(C_D = 0\).

In the case where \(w^H\) is a Poisson bivector, it is interesting to study its compatibility with the canonical Poisson structure \(W_0\) of (3.17).

PROPOSITION 4.6. If \(w^H\) is a Poisson bivector, then it is compatible with \(W_0\) if and only if

\[
\frac{\partial w^{ij}}{\partial x^k} + w^{ih} \Phi^j_{hk} - w^{jh} \Phi^i_{hk} = 0, \quad w^{ih} R_{hjk} = 0.
\]

PROOF. By a straightforward computation, we get that the compatibility condition \([w^H, W] = 0\) is equivalent to (4.13).

The Bianchi identity [6]

\[
R_{kij} + R_{ijk} + R_{jki} = 0
\]

shows that the second relation in (4.13) implies (4.7). Then we have the following corollary.

COROLLARY 4.7. If \((M, w)\) is a Poisson manifold and the cotangent bundle \(T^*M\) is endowed with a symmetric nonlinear connection, then \(w^H\) is a Poisson bivector on \(T^*M\) compatible with \(W_0\) if and only if conditions (4.13) hold.

REMARK 4.8. Considering the isomorphism

\[
\Psi : V_u \rightarrow H^*_u, \quad \Psi \left( X_k \frac{\partial}{\partial p_k} \right) = X_k dq^k,
\]
where \( u \in T^* M \) and \( H_u^* \) is the dual space of \( H_u \), the second condition in (4.13) may be written in the equivalent form

\[
[\Psi(R(X,Y))](\sharp_w \alpha)^H = 0, \quad \forall X,Y \in \mathcal{X}(T^* M), \, \forall \alpha \in \Omega^1(M).
\] (4.16)

We recall that a symmetric linear connection \( \nabla \) on a Poisson manifold \((M, w)\) is called a Poisson connection if \( \nabla w = 0 \). Such connections exist if and only if \( w \) is regular, that is, rank \( w = \text{const} \) (see [8]).

**Proposition 4.9.** Let \((M, w)\) be a regular Poisson manifold with a Poisson connection \( \nabla \). Then the bivector \( w^H \), defined with respect to \( \nabla \), is a Poisson structure on \( T^* M \) compatible with the canonical Poisson structure \( W_0 \) if and only if the 2-form

\[
(X,Y) \mapsto R_{\nabla}(X,Y) (\sharp_w \alpha), \quad X,Y \in \mathcal{X}(M)
\] (4.17)

vanishes for every Pfaff form \( \alpha \) on \( M \).

**Proof.** With (4.5), the first condition in (4.13) becomes \( \nabla w = 0 \), which we took as a hypothesis. The second condition in (4.13) becomes

\[
w^{ih} R^i_{hjk} = 0,
\] (4.18)

and we get the required conditions. \( \square \)

**Remark 4.10.** If \( w \) is defined by a symplectic structure of \( M \), then (4.17) means \( R_{\nabla} = 0 \).

5. Poisson structures derived from differential forms. If \( \omega \) is a 2-form on a Riemannian manifold \((M, g)\), we associate with it a 2-form \( \Theta(\omega) \) on the cotangent bundle \( \pi : T^* M \to M \), and considering (pseudo-)Riemannian metrics on \( T^* M \) related to \( g \), we study the conditions for \( \Theta(\omega) \) to produce a Poisson structure on this bundle.

Let \((M, g)\) be an \( n \)-dimensional manifold and \( \nabla \) its Levi-Civita connection. If \( \Gamma^k_{ij} \) are the local coefficients of \( \nabla \), a connection \( \Gamma \) with the coefficients (4.5) is obtained on \( T^* M \).

The system of local 1-forms \((dx^i, \delta p_i) \) \((i = 1, \ldots, n)\), where

\[
\delta p_i := dp_i + N_{ij} dx^j,
\] (5.1)

defines the dual bases of the bases \( \{dx^i, \partial/\partial p_i\} \).

The components of the curvature form are given by (4.2). Since the connection is symmetric, the Bianchi identity (4.14) holds. The elements \( \Phi^k_{ij} \) of (4.3) are

\[
\Phi^k_{ij} = \Gamma^k_{ij}.
\] (5.2)
The Riemannian metric $g$ provides the “musical” isomorphism $\sharp_g : T^*M \rightarrow TM$ and the codifferential

$$\delta_g : \Omega^k(M) \rightarrow \Omega^{k-1}(M), \quad (\delta_g \alpha)_{i_1 \cdots i_{k-1}} = -g^{st} \nabla_t \alpha_{s_{i_1} \cdots i_{k-1}}, \quad (5.3)$$

where $k \geq 1$,

$$\alpha = \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(M), \quad (5.4)$$

and $(g^{st})$ are the entries of the inverse of the matrix $(g_{ij})$ [8].

Let

$$\omega = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j, \quad \omega_{ji} = -\omega_{ij}, \quad (5.5)$$

be a 2-form on $M$.

**Definition 5.1.** The 2-form $\Theta(\omega)$ on $T^*M$ given by

$$\Theta(\omega) = \pi^* \omega - d\lambda, \quad (5.6)$$

where $\lambda$ is the Liouville form, is said to be the associated 2-form of $\omega$.

With respect to the cobases $(dx^i, \delta p_i)$, we get

$$\Theta(\omega) = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j + dx^i \wedge \delta p_i. \quad (5.7)$$

Now, we consider two (pseudo-)Riemannian metrics $G_1$ and $G_2$ on $T^*M$ and study the conditions for the bivectors $W_i = \sharp_{G_i} \Theta(\omega)$ ($i = 1, 2$) to define Poisson structures on $T^*M$. The Poisson condition $[W_i, W_i] = 0, \ i = 1, 2$, is equivalent to [8]

$$\delta_{G_i} (\Theta(\omega) \wedge \Theta(\omega)) = 2 \Theta(\omega) \wedge \delta_{G_i} \Theta(\omega), \quad i = 1, 2. \quad (5.8)$$

First, consider [5, 6] the pseudo-Riemannian metric $G_1$ of signature $(n, n)$

$$G_1 = 2 \delta p_i \odot dx^i. \quad (5.9)$$

To find the condition which ensures that (5.8) holds, we need the local expression of the codifferential $\delta_{G_1}$ of $G_1$. Denote by $\nabla$ the Levi-Civita connection of $G_1$, and for simplicity we put

$$\nabla_i = \nabla_{\delta/\delta x^i}, \quad \nabla^i = \nabla_{\delta/\delta p_i}. \quad (5.10)$$
The connection $\tilde{\nabla}$ is defined by [6]

$$
\begin{align*}
\tilde{\nabla}^i \frac{\partial}{\partial p_j} &= 0, & \tilde{\nabla}^i \frac{\partial}{\partial p_j} &= -\Gamma^j_{ik} \frac{\partial}{\partial p_k}, \\
\tilde{\nabla}^i \frac{\delta}{\delta q^j} &= 0, & \tilde{\nabla}^i \frac{\delta}{\delta q^j} &= \Gamma^j_{ik} \frac{\delta}{\delta q^k} - p_h R^h_{ijk} \frac{\partial}{\partial p_k}.
\end{align*}
$$

(5.11)

**Proposition 5.2.** The bivector $\sharp G_1 \Theta(\omega)$ defines a Poisson structure on the cotangent bundle $T^* M$ if and only if $\omega$ is a closed 2-form on $M$ and $\Gamma^a_{ai} = 0$, for all $i = 1, \ldots, n$. In this case, $\Theta(\omega)$ is a symplectic form.

**Proof.** The proof is by a long computation in local coordinates. After computing the exterior product $\Theta(\omega) \wedge \Theta(\omega)$, we get

$$
\delta_{G_1} (\Theta(\omega) \wedge \Theta(\omega)) = \frac{2}{3!} \sum_{(i,j,k)} \nabla_i \omega_{jk} dx^i \wedge dx^j \wedge dx^k.
$$

(5.12)

Then we compute $\delta_{G_1} \Theta(\omega)$ and obtain

$$
\Theta(\omega) \wedge \delta_{G_1} \Theta(\omega) = \frac{2}{3!} \sum_{(i,j,k)} \omega_{ij} \Gamma^a_{ak} dx^i \wedge dx^j \wedge dx^k
+ (\delta^k_i \Gamma^a_{aj} - \delta^k_j \Gamma^a_{ai}) dx^i \wedge dx^j \wedge \delta_p_k.
$$

(5.13)

Equation (5.8) implies

$$
\delta^k_j \Gamma^a_{ai} - \delta^k_i \Gamma^a_{aj} = 0, \quad \forall i, j, k = 1, \ldots, n.
$$

(5.14)

Making the contraction $k = j$, it follows that $\Gamma^a_{ai} = 0$. Conversely, if $\Gamma^a_{ai} = 0$, then (5.14) holds. Also, since $\nabla$ is symmetric, we get

$$
\sum_{(i,j,k)} \frac{\partial \omega_{jk}}{\partial x^l} = \sum_{(i,j,k)} \nabla_i \omega_{jk}.
$$

(5.15)

Therefore, the condition $\sum_{(i,j,k)} \nabla_i \omega_{jk} = 0$ is equivalent to $d\omega = 0$. 

We consider now the Riemannian metric of Sasaki type

$$
G_2 = g_{i,j} dx^i \odot dx^j + g^{ij} \delta p_i \odot \delta p_j
$$

(5.16)

(see [3] for the Sasaki metric).

**Lemma 5.3.** The local coordinate expression of the Levi-Civita connection $\hat{\nabla}$ of $G_2$ is

$$
\begin{align*}
\hat{\nabla}^i \frac{\partial}{\partial p_j} &= 0, & \hat{\nabla}^i \frac{\partial}{\partial p_j} &= -\frac{1}{2} R^j_{ik} \frac{\delta}{\delta q^k} - \Gamma^j_{ik} \frac{\partial}{\partial p_k}, \\
\hat{\nabla}^i \frac{\delta}{\delta q^j} &= 0, & \hat{\nabla}^i \frac{\delta}{\delta q^j} &= \Gamma^j_{ik} \frac{\delta}{\delta q^k} - \frac{1}{2} R_{kij} \frac{\partial}{\partial p_k}.
\end{align*}
$$

(5.17)
where the notations of (5.10) are used again and $R^{jk}_i$ (also $R^i_{jk}$) are obtained from $R_{kij}$ by the operation of lifting the indices, that is,

$$R^{jk}_i = g^{ia} g^{kb} R_{abi}, \quad R^i_{jk} = g^{ia} g^{kb} R_{abj}. \quad (5.18)$$

**Proof.** The result is proved by a straightforward computation. \[ \square \]

**Proposition 5.4.** The bivector $\delta G_2 \Theta(\omega)$ defines a Poisson structure on the cotangent bundle $T^*M$ if and only if

$$\nabla \omega = 0, \quad g^{ab} R^k_{abi} = 0, \quad \omega^{ab} R^k_{iab} = 0, \quad (5.19)$$

where $\omega^{ab} = g^{ai} g^{bj} \omega_{ij}$ are the components of the bivector $w = \#_g \omega$ on $M$.

**Proof.** By a new long computation again, we get

$$\frac{1}{2} \delta G_2 (\Theta(\omega) \wedge \Theta(\omega)) = \frac{1}{3!} g^{ab} \nabla_a \left( \sum_{(i,j,k)} \omega_{ij} \omega_{kb} \right) dx^i \wedge dx^j \wedge dx^k$$

$$- g^{ab} \sum_{(i,j,k)} (\nabla_a \omega_{ij} \delta^i_k) dx^i \wedge dx^j \wedge \delta p_k$$

$$+ \frac{1}{2} \omega_{ab} (R^{kab} \delta^i_j - R^{jab} \delta^i_k) dx^i \wedge \delta p_j \wedge \delta p_k,$$

where

$$\Theta(\omega) \wedge \delta G_2 \Theta(\omega) = \frac{1}{3!} \sum_{(i,j,k)} (\delta G_2 \Theta(\omega))_k dx^i \wedge dx^j \wedge dx^k$$

$$+ \frac{1}{2} \left[ \delta^k_i (\delta G_2 \Theta(\omega))_j - \delta^k_j (\delta G_2 \Theta(\omega))_i \right] dx^i \wedge dx^j \wedge \delta p_k, \quad (5.20)$$

where

$$\delta G_2 \Theta(\omega) = (\delta G_2 \Theta(\omega))_k dx^k = g^{ab} \left( \nabla_a \omega_{kb} - \frac{1}{2} R_{abk} \right) dx^k. \quad (5.21)$$

Identifying the coefficients, the Poisson condition (5.8) for $W_2$ becomes

$$g^{ab} \sum_{(i,j,k)} \omega_{ij} R^h_{abk} = 0, \quad g^{ab} \sum_{(i,j,k)} (\nabla_a \omega_{ij}) \omega_{kb} = 0, \quad (5.22)$$

$$\nabla \omega = 0, \quad g^{ab} R^k_{abi} = 0, \quad (5.23)$$

$$\omega^{ab} R^k_{iab} = 0. \quad (5.24)$$

We remark that the conditions (5.23) imply (5.22) because if $\nabla \omega = 0$, then $\nabla_a \omega_{ij} = 0$, and $g^{ab} R^k_{abi} = 0$ implies $g^{ab} \omega_{ij} R^h_{abk} = 0$. \[ \square \]
**Remark 5.5.** If the bivector $\sharp G_G \Theta(\omega)$ defines a Poisson structure on $T^* M$, then $w = \sharp g \omega$ defines a Poisson structure on $M$, as the second condition in (5.22) is equivalent to the Poisson condition [8] $\sum_{(i,j,k)} w^i a \nabla_a w^j k = 0$. (5.25)

(The local coordinate expression of $w$ is (2.8).)

**Corollary 5.6.** If $\sharp G_G \Theta(\omega)$ is a Poisson bivector on $T^* M$, then the scalar curvature $r$ of $(M, g)$ vanishes.

**Proof.** The expression of $r$ is $r = g^{ab} R_{ab}$, where $R_{ba} = R^k_{akb} = R_{ab}$ are the components of the Ricci tensor, and if we make the contraction $k = i$ in the second relation in (5.19), we get $g^{ab} R^i_{akb} = 0$, and whence $r = 0$. □

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**References**


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