ON THE L_w^2 -BOUNDEDNESS OF SOLUTIONS FOR PRODUCTS OF QUASI-INTEGRO DIFFERENTIAL EQUATIONS

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Given a general quasidifferential expressions $\tau_1, \tau_2, \dots, \tau_n$ each of order n with complex coefficients and their formal adjoints are $\tau_1^+, \tau_2^+, \dots, \tau_n^+$ on $[0, b)$, respectively, we show under suitable conditions on the function F that all solutions of the product of the quasi-integrodifferential equation $[\prod_{j=1}^n \tau_j]y = wF(t, y, \int_0^t g(t, s, y, y', \dots, y^{(n^2-1)}(s))ds)$ on $[0, b)$, $0 < b \leq \infty$; $t, s \geq 0$, are bounded and L_w^2 -bounded on $[0, b)$. These results are extensions of those by the author (1994), Wong (1975), Yang (1984), and Zettl (1970, 1975).

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1. Introduction. In [8, 11, 15] Wong and Zettl proved that all solutions of a perturbed linear differential equation belong to $L^2(0, \infty)$ assuming the fact that all solutions of the unperturbed equation possess the same property. In [6] the author extends their results for a general quasidifferential expression τ of arbitrary order n with complex coefficients, and considered the property of boundedness of solutions of a general quasidifferential equation $\tau[y] - \lambda w y = w f(t, y)$, where $\lambda \in \mathbb{C}$, on $[0, b)$, $f(t, s)$ satisfies

$$|f(t, y)| \leq e_1(t) + r_1(t)|y|^\sigma, \quad t \in [0, b) \text{ for some } \sigma \in [0, 1], \quad (1.1)$$

where $e_1(t)$ and $r_1(t)$ are nonnegative continuous functions on $[0, b)$.

Our objective in this paper is to extend the results in [4, 6, 8, 9, 11, 15] to more general class of quasi-integrodifferential equation in the form

$$\left[\prod_{j=1}^n \tau_j \right] y = wF\left(t, y, \int_0^t g\left(t, s, y, y', \dots, y^{(n^2-1)}(s)\right)ds\right) \quad \text{on } [0, b), \quad (1.2)$$

where $0 < b \leq \infty$; $t, s \geq 0$. Also, we prove under suitable condition on the function F that, if all solutions of the equations $(\prod_{j=1}^n \tau_j)y = 0$ and $(\prod_{j=1}^n \tau_j^+)z = 0$ belong to $L_w^2(0, b)$, then all solutions of (1.2) also belong to $L_w^2(0, b)$, where τ_j^+ is the formal adjoint of τ_j , $j = 1, 2, \dots, n$.

We deal throughout this paper with a quasidifferential expression τ_j each of arbitrary order n defined by Shin-Zettl matrices (see [4, 13]) on the interval $I = [0, b)$. The left-hand end point of I is assumed to be regular but the right-hand end point may be regular or singular.

2. Notation and preliminaries. The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively and $N(T)$ will denote its null space. The nullity of T , written $\text{nul}(T)$, is the dimension of $N(T)$ and the deficiency of T , written $\text{def}(T)$, is the codimension of $R(T)$ in H ; thus if T is densely defined and $R(T)$ is closed, then $\text{def}(T) = \text{nul}(T^*)$. The Fredholm domain of T is (in the notation of [2]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H . Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has a closed range and a finite nullity and deficiency.

A closed operator A in a Hilbert space H has property (C), if it has closed range and $\lambda = 0$ is not an eigenvalue, that is, there is some positive number r such that $\|Ax\| \geq r\|x\|$ for all $x \in D(A)$.

Note that, property (C) is equivalent to $\lambda = 0$ being a regular type point of A . This in turn is equivalent to the existence of A^{-1} as a bounded operator on the range of A (which need not be all of H).

Given two operators A and B , both acting in a Hilbert space H , we wish to consider the product operator AB . This is defined as follows

$$D(AB) := \{x \in D(B) \mid Bx \in D(A)\}, \quad (AB)x = A(Bx), \quad \forall x \in D(AB). \quad (2.1)$$

It may happen in general that $D(AB)$ contains only the null element of H . However, in the case of many differential operators, the domains of the product will be dense in H .

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors.

LEMMA 2.1 (cf. [4, Theorem A] and [12]). *Let A and B be closed operators with dense domains in a Hilbert space H . Suppose that $\lambda = 0$ is a regular type point for both operators and $\text{def } A$ and $\text{def } B$ are finite. Then AB is a closed operator with dense domain, has $\lambda = 0$ as a regular type point and*

$$\text{def } AB = \text{def } A + \text{def } B. \quad (2.2)$$

Evidently, Lemma 2.1 extends to the product of any finite number of operators $A_1, A_2, \dots, A_n$.

We now turn to the quasidifferential expressions defined in terms of a Shin-Zettl matrix F on an interval I . The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $n \times n$ matrices $P = \{p_{rs}\}$, $1 \leq r, s \leq n$, whose entries are complex-valued

functions on I which satisfy the following conditions:

$$\begin{aligned} p_{rs} &\in L^1_{\text{loc}}(I) \quad (1 \leq r, s \leq n, n \geq 2), \\ p_{rs} &\neq 0 \quad \text{a.e. on } I \quad (1 \leq r \leq n-1), \\ p_{rs} &= 0 \quad \text{a.e. on } I \quad (2 \leq r+1 < s \leq n). \end{aligned} \quad (2.3)$$

For $P \in Z_n(I)$, the quasiderivatives associated with P are defined by

$$\begin{aligned} \mathcal{Y}^{[0]} &:= \mathcal{Y}, \\ \mathcal{Y}^{[r]} &:= (p_{r,r+1})^{-1} \left\{ (\mathcal{Y}^{[r-1]})' - \sum_{s=1}^r p_{rs} \mathcal{Y}^{[s-1]} \right\} \quad (1 \leq r \leq n-1), \\ \mathcal{Y}^{[n]} &:= (\mathcal{Y}^{[n-1]})' - \sum_{s=1}^n p_{ns} \mathcal{Y}^{[s-1]}, \end{aligned} \quad (2.4)$$

where the prime $'$ denotes differentiation.

The quasidifferential expression τ associated with P is given by

$$\tau[\mathcal{Y}] := i^n \mathcal{Y}^{[n]} \quad (n \geq 2), \quad (2.5)$$

this being defined on the set

$$V(\tau) := \{\mathcal{Y} : \mathcal{Y}^{[r-1]} \in AC_{\text{loc}}(I), r = 1, \dots, n\}, \quad (2.6)$$

where $L^1_{\text{loc}}(I)$ and $AC_{\text{loc}}(I)$ denote, respectively, the spaces of complex-valued Lebesgue measurable functions on I which are locally integrable and locally absolutely continuous on every compact subinterval of I .

The formal adjoint τ^+ of τ defined by the matrix $P^+ \in Z_n(I)$ is given by

$$\begin{aligned} \tau^+[z] &:= i^n z^{[n]} \quad \forall z \in V(\tau^+), \\ V(\tau^+) &:= \{z : z^{[r-1]}_+ \in AC_{\text{loc}}(I), r = 1, \dots, n\}, \end{aligned} \quad (2.7)$$

where $z^{[r-1]}_+$, $r = 1, 2, \dots, n$, are the quasiderivatives associated with the matrix P^+ ,

$$P^+ = \{p^+_{rs}\} = (-1)^{r+s+1} \overline{p}_{n-s+1, n-r+1} \quad \text{for each } r, s; 1 \leq r, s \leq n. \quad (2.8)$$

Note that $(P^+)^+ = P$ and so $(\tau^+)^+ = \tau$. We refer to [2, 3, 6, 7, 13] for a full account of the above and subsequent results on quasidifferential expressions.

Let the interval I have end points a, b ($-\infty \leq a < b \leq \infty$), and let $w : I \rightarrow \mathbb{R}$ be a nonnegative weight function with $w \in L^1_{\text{loc}}(I)$ and $w(x) > 0$ (for almost

all $x \in I$). Then $H = L_w^2(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_I w|f|^2 < \infty$; the inner-product is defined by

$$(f, g) := \int_I w(x) f(x) \overline{g(x)} dx \quad (f, g \in L_w^2(I)). \quad (2.9)$$

The equation

$$\tau[y] - \lambda w y = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I \quad (2.10)$$

is said to be regular at the left end point $a \in \mathbb{R}$, if for all $X \in (a, b)$, $a \in \mathbb{R}$; $w, p_{rs} \in L^1[a, X]$, $(r, s = 1, \dots, n)$. Otherwise (2.10) is said to be singular at a . If (2.10) is regular at both end points, then it is said to be regular; in this case we have,

$$a, b \in \mathbb{R}, \quad w, p_{rs} \in L^1(a, b), \quad (r, s = 1, \dots, n). \quad (2.11)$$

We will be concerned with the case when a is a regular end point of (2.10), the end point b being allowed to be either regular or singular. Note that, in view of (2.8), an end point of I is regular for (2.10), if and only if it is regular for the equation

$$\tau^+[z] - \bar{\lambda} w z = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I. \quad (2.12)$$

Note that, at a regular end point a , $y^{[r-1]}(a)(z_+^{[r-1]}(a))$, $r = 1, \dots, n$, is defined for all $u \in V(\tau)$ ($v \in V(\tau^+)$). Set

$$\begin{aligned} D(\tau) &:= \{y : y \in V(\tau), y, w^{-1}\tau[y] \in L_w^2(a, b)\}, \\ D(\tau^+) &:= \{z : z \in V(\tau^+), z, w^{-1}\tau^+[z] \in L_w^2(a, b)\}. \end{aligned} \quad (2.13)$$

The subspaces $D(\tau)$ and $D(\tau^+)$ of $L_w^2(a, b)$ are domains of the so-called maximal operators $T(\tau)$ and $T(\tau^+)$, respectively, defined by

$$\begin{aligned} T(\tau)y &:= w^{-1}\tau[y] \quad (y \in D(\tau)), \\ T(\tau^+)z &:= w^{-1}\tau^+[z], \quad (z \in D(\tau^+)). \end{aligned} \quad (2.14)$$

For the regular problem, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ are the restrictions of $w^{-1}\tau[y]$ and $w^{-1}\tau^+[z]$ to the subspaces

$$\begin{aligned} D_0(\tau) &:= \{y : y \in D(\tau), y^{[r-1]}(a) = y^{[r-1]}(b) = 0, r = 1, \dots, n\}, \\ D_0(\tau^+) &:= \{z : z \in D(\tau^+), z_+^{[r-1]}(a) = z_+^{[r-1]}(b) = 0, r = 1, \dots, n\}, \end{aligned} \quad (2.15)$$

respectively. The subspaces $D_0(\tau)$ and $D_0(\tau^+)$ are dense in $L_w^2(a, b)$, and $T_0(\tau)$ and $T_0(\tau^+)$ are closed operators (see [2, 3, 6] and [13, Section 3]).

In the singular problem, we first introduce the operators $T'_0(\tau)$ and $T'_0(\tau^+)$; $T'_0(\tau)$ being the restriction of $w^{-1}\tau[\cdot]$ to the subspace

$$D'_0(\tau) := \{y : y \in D(\tau), \text{supp } y \subset (a, b)\} \quad (2.16)$$

and with $T'_0(\tau^+)$ defined similarly. These operators are densely defined and closable in $L_w^2(a, b)$, and we defined the minimal operators $T_0(\tau)$, $T_0(\tau^+)$ to be their respective closures (see [2] and [13, Section 5]). We denote the domains of $T_0(\tau)$ and $T_0(\tau^+)$ by $D_0(\tau)$ and $D_0(\tau^+)$, respectively. It can be shown that

$$\begin{aligned} y \in D_0(\tau) &\Rightarrow y^{[r-1]}(a) = 0 \quad (r = 1, \dots, n), \\ z \in D_0(\tau^+) &\Rightarrow z_+^{[r-1]}(a) = 0 \quad (r = 1, \dots, n), \end{aligned} \quad (2.17)$$

because we are assuming that a is a regular end point. Moreover, in both regular and singular problems, we have

$$T_0^*(\tau) = T(\tau^+), \quad T^*(\tau) = T_0(\tau^+), \quad (2.18)$$

see [13, Section 5] in the case when $\tau = \tau^+$ and compare it with treatment in [2, Section III.10.3] and [3] in general case.

3. Some technical lemmas. The proof of the general theorem is based on the results in this section. We start by listing some properties and results of quasi-differential expressions $\tau_1, \tau_2, \dots, \tau_n$ each of order n . For proofs, the reader is referred to [4, 12, 13, 14].

$$\begin{aligned} (\tau_1 + \tau_2)^+ &= \tau_1^+ + \tau_2^+, \\ (\tau_1 \tau_2)^+ &= \tau_2^+ \tau_1^+, \quad (\lambda \tau)^+ = \bar{\lambda} \tau^+, \quad \text{for } \lambda \text{ is a complex number.} \end{aligned} \quad (3.1)$$

A consequence of properties (3.1) is that if $\tau^+ = \tau$, then $P(\tau)^+ = P(\tau^+)$ for P is any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence the product of regular differential expressions is regular.

LEMMA 3.1 (cf. [4, Theorem 1]). Suppose that τ_j is a regular differential expression on the interval $[0, b]$ such that the minimal operator $T_0(\tau_j)$ has property (C) for $j = 1, 2, \dots, n$. Then

- (i) the product operator $\prod_{j=1}^n [T_0(\tau_j)]$ is closed and has dense domain, property (C), and

$$\text{def} \left[\prod_{j=1}^n T_0(\tau_j) \right] = \sum_{j=1}^n \text{def} [T_0(\tau_j)]; \quad (3.2)$$

- (ii) the operators $T_0(\tau_1 \tau_2 \cdots \tau_n)$ and $\prod_{j=1}^n [T_0(\tau_j)]$ are not equal in general, that is, $[T_0(\tau_1 \tau_2 \cdots \tau_n)] \subseteq \prod_{j=1}^n [T_0(\tau_j)]$.

LEMMA 3.2 (cf. [4, Theorem 2]). Let $\tau_1, \tau_2, \dots, \tau_n$ be regular differential expressions on $[0, b]$. Suppose that $T_0(\tau_j)$ satisfies property (C) for $j = 1, 2, \dots, n$. Then

$$T_0(\tau_1 \tau_2 \cdots \tau_n) = \prod_{j=1}^n T_0(\tau_j) \quad (3.3)$$

if and only if the following partial separation condition is satisfied:

$$\{f \in L_w^2(a, b), f^{[s-1]} \in AC_{\text{loc}}[a, b]\}, \quad (3.4)$$

where s is the order of product expression $(\tau_1 \tau_2 \cdots \tau_n)$ and $(\tau_1 \tau_2 \cdots \tau_n)^+ f \in L_w^2(a, b)$ together imply that $(\prod_{j=1}^k (\tau_j^+))f \in L_w^2(a, b)$, $k = 1, \dots, n-1$.

Furthermore, $T_0(\tau_1 \tau_2 \cdots \tau_n) = \prod_{j=1}^n T_0(\tau_j)$ if and only if

$$\text{def} [T_0(\tau_1 \tau_2 \cdots \tau_n)] = \sum_{j=1}^n \text{def} [T_0(\tau_j)], \quad (3.5)$$

then the product $(\tau_1 \tau_2 \cdots \tau_n)$ is partially separated expressions in $L_w^2(0, b)$ whenever property (3.4) holds.

LEMMA 3.3 (cf. [4, Corollary 1]). Let τ_j be a regular differential expression on $[0, b]$ for $j = 1, \dots, n$. If all solutions of the differential equations $(\tau_j)y = 0$ and $(\tau_j^+)z = 0$ on $[0, b]$ are in $L_w^2(0, b)$ for $j = 1, \dots, n$, then all solutions of $(\tau_1 \tau_2 \cdots \tau_n)y = 0$ and $(\tau_1 \tau_2 \cdots \tau_n)^+z = 0$ are in $L_w^2(0, b)$.

The special case of Lemma 3.3 when $\tau_j = \tau$ for $j = 1, 2, \dots, n$ and τ is symmetric was established in [14]. In this case, it is easy to see that the converse also holds. If all solutions of $\tau^n u = 0$ are in $L_w^2(0, b)$, then all solutions of $\tau y = 0$ must be in $L_w^2(0, b)$. In general, if all solutions of $(\tau_1 \tau_2 \cdots \tau_n)y = 0$

are in $L_w^2(0, b)$, then all solutions of $\tau_n \gamma = 0$ are in $L_w^2(0, b)$ since these also the solutions of $(\tau_1 \tau_2 \cdots \tau_n) \gamma = 0$. If all solutions of the adjoints equation $(\tau_1 \tau_2 \cdots \tau_n)^+ z = 0$ are also in $L_w^2(0, b)$, then it follows similarly that all solutions of $\tau_1^+ z = 0$ are in $L_w^2(0, b)$. So, in particular, for $n = 2$ we have established the following lemma.

LEMMA 3.4. *Suppose that τ_1 , τ_2 , and $\tau_1 \tau_2$ are all regular expressions on $[0, b]$. Then the product is in the maximal deficiency case at b if and only if both τ_1 , τ_2 are in the maximal deficiency case at b , see [4, Corollary 2] for more details.*

Denote by $S(\tau)$ and $S(\tau^+)$ the sets of all solutions of the equations

$$\left(\prod_{j=1}^n \tau_j \right) \gamma = 0, \quad \left(\prod_{j=1}^n \tau_j^+ \right) z = 0, \quad (3.6)$$

respectively. Let $\phi_k(t)$, $k = 1, 2, \dots, n^2$, denote the solutions of the homogeneous equation $(\prod_{j=1}^n \tau_j) \gamma = 0$ determined by the initial conditions

$$\phi_k^{[r]}(t_0) = \delta_{k, r+1} \quad \forall t_0 \in [0, b] \quad (3.7)$$

(where $k = 1, 2, \dots, n^2$; $r = 0, 1, \dots, n^2 - 1$). Let $\phi_k^+(t)$, $k = 1, 2, \dots, n^2$, denote the solutions of the homogeneous equation $(\prod_{j=1}^n \tau_j^+) z = 0$ determined by the initial conditions

$$(\phi_k^+)^{[r]}(t_0) = (-1)^{k+r} \delta_{k, n^2-r} \quad \forall t_0 \in [0, b], \quad (3.8)$$

where $k = 1, 2, \dots, n^2$; $r = 0, 1, \dots, n^2 - 1$.

REMARK 3.5. If all solutions $\phi_k(t)$, $\phi_k^+(t)$, $k = 1, 2, \dots, n^2$, of $(\prod_{j=1}^n \tau_j) \gamma = 0$ and $(\prod_{j=1}^n \tau_j^+) z = 0$, respectively are bounded (L_w^2 -bounded) on $[0, b]$, then $S(\tau)$ and $S(\tau^+)$ are bounded (L^2 -bounded) and hence $S(\tau) \cup S(\tau^+)$ is bounded (L^2 -bounded) on $[0, b]$; see [6] and [7, Lemmas 3.4 and 3.5].

The next lemma is a form of the variation of parameters formula of a general quasidifferential equation, see [6, Section 3] and [7, 13].

LEMMA 3.6. *For f locally integrable, the solution ϕ of the quasidifferential equation*

$$\left(\prod_{j=1}^n \tau_j \right) \gamma = w f \quad \text{on } [0, b] \quad (3.9)$$

satisfying

$$\phi^{[r]}(t_0) = \alpha_{r+1} \quad \forall t_0 \in [0, b), \quad r = 0, 1, \dots, n^2 - 1 \quad (3.10)$$

is given by

$$\phi(t) = \sum_{j=1}^{n^2} \alpha_j \phi_j(t) + \frac{1}{i^{n^2}} \sum_{j,k=1}^{n^2} \zeta^{jk} \phi_j(t) \int_{t_0}^t \overline{\phi_k^+(s)} f(s) w(s) ds \quad (3.11)$$

for some $\alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \mathbb{C}$, where $\phi_j(t)$ and $\phi_k^+(t)$, $j, k = 1, 2, \dots, n^2$, are solutions of the equations in (3.6), respectively, ζ^{jk} is a constant which is independent of t .

In the sequel, we will require the following nonlinear integral inequality which generalizes those integral inequalities used in [1, 5, 9, 10].

LEMMA 3.7 (cf. [5, 10]). Let $u(t)$, $v(t)$, $f(t, s)$, $g_i(t, s)$, and $h_i(t, s)$ ($i = 1, 2, \dots, n^2$) be nonnegative continuous functions defined on the interval I and $I \times I$, respectively, here $I = (0, c)$, $0 < c \leq \infty$, with their ranges in $\mathbb{R}^+$. Let $v(t)$ be nondecreasing on I , and $f(t, s)$, $g_i(t, s)$, and $h_i(t, s)$, ($i = 1, 2, \dots, n^2$) be nondecreasing in t for each $s \in I$ fixed. Suppose that the inequality

$$u(t) \leq v(t) + \int_0^t f(t, s) u(s) ds + \sum_{j=1}^{n^2} \int_0^t g_i(t, s) \left[\int_0^s h_i(t, s) [u(\tau)]^\sigma d\tau \right] ds \quad (3.12)$$

holds for all $t \in I$, where $\sigma \in (0, 1]$ is constant. Then

(i) if $0 < \sigma < 1$,

$$u(t) \leq \left[[v(t)F(t)]^{1-\sigma} + (1-\sigma) \sum_{i=1}^{n^2} G_i(t)F(t) \int_0^t h_i(t, s) ds \right]^{1/(1-\sigma)}, \quad t \in I, \quad (3.13)$$

(ii) if $\sigma = 1$,

$$u(t) \leq v(t) \exp \int_0^t \left[f(t, s) + \sum_{i=1}^{n^2} G_i(t)F(t)h_i(t, s) \right] ds, \quad (3.14)$$

where

$$F(t) = \exp \int_0^t f(t, s) ds, \quad G_i(t) = \int_0^t g_i(s) ds, \quad i = 1, 2, \dots, n^2. \quad (3.15)$$

COROLLARY 3.8 (cf. [9, 10]). Let $u(t)$, $f_1(t)$, $f_2(t)$, $g_1(t, s)$, and $g_2(t, s)$ be nonnegative continuous functions defined on the intervals $I = [0, b)$ and $I \times I$,

respectively. Suppose that the inequality

$$\begin{aligned} u(t) \leq & C + \int_0^t f_1(s)u(s)ds + \int_0^t f_2(s)u^\sigma(s)ds \\ & + \int_0^t \left(\int_0^s g_1(s,x)u(x)dx \right) + \int_0^t (g_2(s,x)u^\sigma(x))ds \end{aligned} \quad (3.16)$$

holds for all $t \in [0, b]$, where $\sigma \in (0, 1]$ and C is constant. Then

$$\begin{aligned} u(t) \leq & \left[C^{(1-p)} + (1-\sigma) \right. \\ & \times \int_0^t \left[f_2(s) + \int_0^s g_2(s,x)dx \right] \\ & \times \exp \left[(1-\sigma) \int_0^s \left[f_1(\tau) + \int_0^\tau g_1(\tau,x)dx \right] d\tau \right] ds \Big]^{1/(1-p)} \\ & \times \exp \left(\int_0^t \left[f_1(s) + \int_0^s g_1(s,x)dx \right] ds \right). \end{aligned} \quad (3.17)$$

4. Boundedness of solutions. In this section, we consider the question of determining conditions under which all solutions of (1.2) are bounded and L_w^2 -bounded.

Suppose there exist nonnegative continuous functions $e_1(t)$, $e_2(t)$, $e_3(t)$, $r_1(t)$, $r_2(t)$, $K_0(t, s)$, and $K_i(t - s)$ on $[0, b]$, $0 < b \leq \infty$; $i = 1, 2, \dots, n^2 - 1$ such that,

$$\begin{aligned} & \left| F\left(t, y, \int_0^t g\left(t, s, y, y', \dots, y^{(n^2-1)}(s)\right)ds\right) \right| \\ & \leq e_1(t) + r_1(t) |y(t)|^\sigma + r_2(t) \left[\int_0^t [e_2(t) + e_3(s) + K_0(t, s) |y(s)|^\sigma] ds \right. \\ & \quad \left. + \left| \int_0^t \sum_{i=1}^{n^2-1} K_i(t-s) y^{(i)}(s) ds \right| \right], \end{aligned} \quad (4.1)$$

for $t, s \geq 0$ and some $\sigma \in [0, 1]$; see [5, 9, 10].

THEOREM 4.1. Suppose that (4.1) is satisfied with $\sigma = 1$, $S(\tau) \cup S(\tau^+)$ is bounded on $[0, b]$, and that

- (a) $k_i^{(\ell)}(0) = 0$ for all $\ell = 0, 1, \dots, i-1$; $i = 1, 2, \dots, n^2 - 1$,
- (b) $e_1(t)$, $r_1(t)$, and $r_2 k_i^{(\ell)}(t) \in L_w^1(0, b)$, $\ell = 0, 1, \dots, i-1$; $i = 1, 2, \dots, n^2 - 1$,
- (c) the following integrals are bounded at $t \rightarrow b$,

- (i) $\int_0^t r_2(s) (\int_0^s [e_2(x) + e_3(x)] dx) w(s) ds,$
- (ii) $\int_0^t r_2(s) (\int_0^s K_0(s, x) dx) w(s) ds,$
- (iii) $\int_0^t r_2(s) (\sum_{i=1}^{n^2-1} \int_0^s |(\partial^i / \partial x^i) K_i(s-x)| dx) w(s) ds.$

Then all solutions of (1.2) are also bounded on $[0, b)$.

PROOF. Note that (4.1) implies that all solutions are defined on $[0, b)$. Let $\{\phi_1(t), \dots, \phi_{n^2}(t)\}$ and $\{\phi_1^+(t), \dots, \phi_{n^2}^+(t)\}$ be two sets of linearly independent solutions of the equations in (3.6), respectively, and let $\phi(t)$ be any solution of (1.2) on $[0, b)$, then by Lemma 3.6, we have

$$\phi(t) = \sum_{j=1}^{n^2} \alpha_j \phi_j(t) + \frac{1}{i^{n^2}} \sum_{j,k=1}^{n^2} \zeta^{jk} \phi_j(t) \int_0^t \overline{\phi_k^+(s)} F(s) w(s) ds. \quad (4.2)$$

Hence,

$$\begin{aligned} |\phi(t)| &\leq \sum_{j=1}^{n^2} |\alpha_j| |\phi_j(t)| \\ &\quad + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\ &\quad \times \int_0^t |\overline{\phi_k^+(s)}| \left[e_1(s) + r_1(s) |\phi(s)| \right. \\ &\quad \left. + r_2(s) \left[\int_0^s [e_2(s) + e_3(x) + K_0(s, x) |\phi(x)|] dx \right. \right. \\ &\quad \left. \left. + \left| \int_0^s \sum_{i=1}^{n^2-1} K_i(s-x) \phi^{(i)}(x) dx \right| \right] \right] w(s) ds. \end{aligned} \quad (4.3)$$

Since $\phi_k^+(t)$ is bounded on $[0, b)$, $k = 1, \dots, n^2$, and $e_1(t) \in L_w^1(0, b)$, then $\phi_k^+(t) e_1(t) \in L_w^1(0, b)$, $k = 1, 2, \dots, n^2$. Setting

$$C_k = \int_0^t |\overline{\phi_k^+(s)}| e_1(s) w(s) ds, \quad (4.4)$$

and integrating the last integral in (4.3) by parts, we have

$$\begin{aligned} &\sum_{i=1}^{n^2-1} \int_0^s K_i(s-x) \phi^{(i)}(x) dx \\ &= \sum_{\ell=0}^{i-1} (-1)^{\ell+1} K_i^{(\ell)}(s) \phi^{(i-1-\ell)}(0) + (-1)^i \int_0^s \frac{\partial^i}{\partial x^i} K_i(s-x) \phi(x) dx, \end{aligned} \quad (4.5)$$

where $K_i^{(\ell)}(0) = 0$ for all $\ell = 0, 1, \dots, i-1$; $i = 1, 2, \dots, n^2-1$. Then (4.3) becomes

$$\begin{aligned}
 |\phi(t)| &\leq \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
 &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 &\times \int_0^t |\overline{\phi_k^+(s)}| \left[r_1(s) |\phi(s)| \right. \\
 &\quad + r_2(s) \left(\int_0^s [e_2(s) + e_3(x) + K_0(s, x) |\phi(x)|] dx \right. \\
 &\quad + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)| |\phi^{(i-1-\ell)}(0)| \\
 &\quad \left. \left. + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| |\phi(x)| dx \right) \right] w(s) ds,
 \end{aligned} \tag{4.6}$$

where $|\phi^{(i-1-\ell)}(0)| \leq \beta$ for all $\ell = 0, 1, \dots, i-1$; $i = 0, \dots, n^2-1$. Let

$$\begin{aligned}
 h(t) &= \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
 &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \int_0^t |\overline{\phi_k^+(s)}| \left[r_2(s) \left[\int_0^s [e_2(s) + e_3(x)] dx \right. \right. \\
 &\quad \left. \left. + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)| \beta \right] \right] w(s) ds.
 \end{aligned} \tag{4.7}$$

Then (4.6) becomes

$$\begin{aligned}
 |\phi(t)| &\leq h(t) \\
 &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 &\times \int_0^t |\overline{\phi_k^+(s)}| \left[r_1(s) |\phi(s)| \right. \\
 &\quad + r_2(s) \left[\int_0^s K_0(s, x) |\phi(x)| dx \right. \\
 &\quad \left. \left. + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| |\phi(x)| dx \right] \right] w(s) ds.
 \end{aligned} \tag{4.8}$$

From our assumptions and conditions (i) and (ii), it follows that $h(t)$ is bounded on $[0, b)$. Applying Lemma 3.7 with $\sigma = 1$, we obtain

$$\begin{aligned}
 |\phi(t)| & \leq h(t) \exp \left\{ \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \right. \\
 & \quad \times \int_0^t |\overline{\phi_k^+(s)}| \left[r_1(s) + r_2(s) \left[\int_0^s \left[K_0(s, x) \right. \right. \right. \\
 & \quad \left. \left. \left. + \sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| \right] dx \right] \right] w(s) ds \Big\}, \quad (4.9)
 \end{aligned}$$

and hence our assumptions and conditions (i), (ii), and (iii) yield that $\phi(t)$ is bounded on $[0, b)$. $\square$

THEOREM 4.2. Suppose that $S(\tau) \cup S(\tau^+) \subset L_w^2(0, b)$ with $\sigma = 1$, and that

- (i) $r_1(t)$ and $r_2(t)$ are bounded on $[0, b)$,
- (ii) $e_1(s)$ and $K_i^{(\ell)}(s) \in L_w^2(0, b)$ for all $\ell = 0, 1, \dots, i-1$; $i = 1, 2, \dots, n^2-1$,
- (iii) $\int_0^t [\int_0^s [e_2(s) + e_3(x)] dx]^2 w(s) ds < \infty$,
- (iv) $\int_0^t [\int_0^s [(1/w)(K_0^2(s, x)) + [\sum_{i=1}^{n^2-1} |(\partial^i / \partial x^i) K_i(s-x)|]^2] dx] w(s) ds < \infty$.

Then all solutions of (1.2) are in $L_w^2(0, b)$.

PROOF. The proof is the same up to (4.5), since $\phi_k^+(s)$, $e_1(s) \in L_w^2(0, b)$ (see Lemma 3.3), then $\phi_k^+(s)e_1(s) \in L_w^1(0, b)$, $k = 1, 2, \dots, n^2$, for all $s \in (0, b)$. By using (4.4) and applying the Cauchy-Schwartz inequality to the integral in (4.6), we have

$$\begin{aligned}
 |\phi(t)| & \leq \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
 & \quad + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 & \quad \times \int_0^t |\overline{\phi_k^+(s)}| \left[\left(\int_0^t |\overline{\phi_k^+(s)}|^2 r_1^2(s) w(s) ds \right)^{1/2} \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^{1/2} \right. \\
 & \quad \left. + \left(\int_0^t |\overline{\phi_k^+(s)}|^2 r_2^2(s) w(s) ds \right)^{1/2} \right. \\
 & \quad \times \left\{ \left(\int_0^t \left[\int_0^s [e_2(s) + e_3(x)] dx \right]^2 w(s) ds \right)^{1/2} \right. \\
 & \quad \left. \left. + \left(\int_0^t \left[\int_0^s K_0(s, x) |\phi(x)| dx \right]^2 w(s) ds \right)^{1/2} \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)\beta| \right]^2 w(s) ds \right)^{1/2} \\
& + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x)\phi(x) \right| dx \right]^2 w(s) ds \right)^{1/2} \Bigg\} \Bigg].
\end{aligned} \tag{4.10}$$

Since $r_1(s), r_2(s)$ are bounded on $[0, b)$ and $\phi_k^+(s) \in L_w^2(0, b)$, then $\phi_k^+(s)r_1(s), \phi_k^+(s)r_2(s) \in L_w^2(0, b)$; $k = 1, 2, \dots, n^2$ for all $s \in [0, b)$ and hence there exist positive constants ζ_1, ζ_2 such that

$$\|\phi_k^+(s)r_i(s)\|_{L_w^2(0, b)} \leq \xi_i \quad \forall k = 1, 2, \dots, n^2; i = 1, 2. \tag{4.11}$$

Therefore (4.10) becomes

$$\begin{aligned}
& |\phi(t)| \\
& \leq \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
& + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \left[\xi_1 \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^{1/2} \right. \\
& + \xi_2 \left\{ \left(\int_0^t \left[\int_0^s [e_2(s) + e_3(x)] dx \right]^2 w(s) ds \right)^{1/2} \right. \\
& + \left(\int_0^t \left[\int_0^s K_0(s, x) |\phi(x)| dx \right]^2 w(s) ds \right)^{1/2} \\
& + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)\beta| \right]^2 w(s) ds \right)^{1/2} \\
& \left. \left. + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x)\phi(x) \right| dx \right]^2 w(s) ds \right)^{1/2} \right\} \right].
\end{aligned} \tag{4.12}$$

Let

$$\begin{aligned}
h(t) &= \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
& + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \left[\xi_2 \left(\int_0^t \left[\int_0^s [e_2(s) + e_3(x)] dx \right]^2 w(s) ds \right)^{1/2} \right. \\
& \left. + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)\beta| \right]^2 w(s) ds \right)^{1/2} \right],
\end{aligned} \tag{4.13}$$

then

$$\begin{aligned}
 |\phi(t)| &\leq h(t) + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 &\times \left[\xi_1 \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^{1/2} \right. \\
 &\quad + \xi_2 \left\{ \left(\int_0^t \left[\int_0^s K_0(s,x) |\phi(x)| dx \right]^2 w(s) ds \right)^{1/2} \right. \\
 &\quad \left. \left. + \left(\int_0^t \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \phi(x) \right| dx \right]^2 w(s) ds \right)^{1/2} \right\} \right].
 \end{aligned} \tag{4.14}$$

Applying the Cauchy-Schwartz inequality and squaring both sides of (4.1), we have

$$\begin{aligned}
 |\phi(t)|^2 &\leq 2h^2(t) \\
 &+ 4 \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)|^2 \\
 &\times \left[\xi_1^2 \left(\int_0^t |\phi(s)|^2 w(s) ds \right) \right. \\
 &\quad + \xi_2^2 \int_0^t \left(\int_0^s \frac{1}{w} \left(K_0^2(s,x) + \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| dx \right]^2 \right) dx \right) \\
 &\quad \left. \times \left(\int_0^t |\phi(s)|^2 w(s) ds \right) \right].
 \end{aligned} \tag{4.15}$$

If $u(t) = \int_0^t |\phi(s)|^2 w(s) ds$, then

$$\begin{aligned}
 u(t) &\leq 2 \int_0^t h^2(s) w(s) ds \\
 &+ 4 \xi_1^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}| \int_0^t |\phi_j(s)|^2 w(s) ds \\
 &+ 4 \xi_2^2 \sum_{j,k=1}^{n^2} \zeta^{jk} \int_0^t |\phi_j(s)|^2 \\
 &\quad \times \left[\int_0^s \left(\int_0^\tau \frac{1}{w} \left(K_0^2(s,x) + \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| \right]^2 \right) dx \right) \right. \\
 &\quad \left. \times u(\tau) w(\tau) d\tau \right] w(s) ds.
 \end{aligned} \tag{4.16}$$

From conditions (ii) and (iii), it follows that the integral $\int_0^t h^2(s)w(s)ds$ will be finite and by using [Lemma 3.7](#), we obtain

$$\begin{aligned}
 u(t) \leq & \left(2 \int_0^t h^2(s)w(s)ds \right) \\
 & \times \exp \left\{ 4\xi_1^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}| \int_0^t |\phi_j(s)|^2 w(s)ds \right. \\
 & + 4\xi_1^2 \sum_{j,k=1}^{n^2} \zeta^{jk} \int_0^t |\phi_j(s)|^2 \\
 & \times \left[\int_0^s \left(\int_0^\tau \frac{1}{w} \left(K_0^2(s,x) + \left[\sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| \right)^2 dx \right) \right. \\
 & \left. \left. \times w(\tau) d\tau \right] w(s) ds \right\}. \tag{4.17}
 \end{aligned}$$

Hence our assumption and condition (iv) yield that $\phi(t) \in L_w^2(0, b)$. $\square$

Next, we consider (4.1) with $0 \leq \sigma < 1$, and we have the following results.

THEOREM 4.3. *Suppose that $S(\tau) \cup S(\tau^+)$ is bounded on $[0, b)$ and that*

- (i) $e_1(s)$ and $r_1(s) \in L_w^2(0, b)$ for all $s \in [0, b)$,
- (ii) $\int_0^t r_2(s) (\int_0^s [e_2(s) + e_3(x)] dx) w(s) ds < \infty$,
- (iii) $\int_0^t r_2(s) (\sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)|) w(s) ds < \infty$,
- (iv) $\int_0^t r_2(s) (\int_0^s K_0(s, x) dx) w(s) ds < \infty$,
- (v) $\int_0^t r_2(s) (\sum_{i=1}^{n^2-1} \int_0^s |(\partial^i / \partial x^i) K_i(s-x)| dx) w(s) ds < \infty$.

Then all solutions of (1.2) are bounded in $[0, b)$.

PROOF. For $0 \leq \sigma < 1$, the proof is the same up to (4.6). In this case (4.6) becomes

$$\begin{aligned}
 |\phi(t)| \leq & \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
 & + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 & \times \int_0^t |\overline{\phi_k^+(s)}| \left[r_1(s) |\phi(s)|^\sigma \right.
 \end{aligned}$$

$$\begin{aligned}
& + r_2(s) \left(\int_0^s [e_2(s) + e_3(x) + K_0(s, x) |\phi(x)|^\sigma] dx \right. \\
& \quad + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)| \beta \\
& \quad \left. + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i^{(i)}(s-x) \right| |\phi(x)| dx \right) w(s) ds.
\end{aligned} \tag{4.18}$$

Let,

$$\begin{aligned}
h(t) = & \sum_{j=1}^{n^2} (C_j + |\alpha_j|) |\phi_j(t)| \\
& + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \int_0^t |\overline{\phi_k^+(s)}| \left[r_2(s) \left[\int_0^s [e_2(s) + e_3(x)] dx \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)| \beta \right] \right] w(s) ds.
\end{aligned} \tag{4.19}$$

Then,

$$\begin{aligned}
& |\phi(t)| \\
& \leq h(t) + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
& \quad \times \int_0^t |\overline{\phi_k^+(s)}| \left[r_1(s) |\phi(s)|^\sigma \right. \\
& \quad \left. + r_2(s) \left[\int_0^s k_0(s, x) |\phi(x)|^\sigma dx \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i^{(i)}(s-x) \right| |\phi(x)| dx \right] \right] w(s) ds.
\end{aligned} \tag{4.20}$$

By hypothesis, there exist positive constants ξ_1 and ξ_2 such that,

$$|\phi_j(t)| \leq \xi_1, \quad |\phi_k^+(t)| \leq \xi_2 \quad \forall j, k = 1, 2, \dots, n^2, \tag{4.21}$$

and from conditions (i), (ii), and (iii), it follows that $h(t)$ is bounded on $[0, b)$, that is, there exists a positive constant ξ_3 such that $h(t) \leq \xi_3$ for all $t \in [0, b)$.

Then,

$$\begin{aligned}
 |\phi(t)| \leq & \xi_3 + n^2 \xi_1 \xi_2 \left[\int_0^t r_1(s) |\phi(s)|^\sigma w(s) ds \right. \\
 & + \int_0^t r_2(s) \left(\int_0^s k_0(s, x) |\phi(x)|^\sigma dx \right) w(s) ds \\
 & \left. + \sum_{i=1}^{n^2-1} \int_0^t \left(r_2(s) \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| |\phi(x)| dx \right) w(s) ds \right].
 \end{aligned}
 \tag{4.22}$$

Applying [Corollary 3.8](#) with $f_1(x) = 0$, we have

$$\begin{aligned}
 |\phi(t)| \leq & \exp \left(\int_0^t r_2(s) \left(\sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| dx \right) w(s) ds \right) \\
 & \times \left\{ \left[\xi_3^{(1-\sigma)} + (1-\sigma) \right. \right. \\
 & \quad \times \int_0^t \left(r_1(s) + \int_0^s r_2(x) k_0(s, x) dx \right) \\
 & \quad \times \exp(1-\sigma) \left(\int_0^t \left(r_2(\tau) \sum_{i=1}^{n^2-1} \int_0^\tau \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| dx \right) \right. \\
 & \quad \left. \left. \times w(\tau) d\tau \right) w(s) ds \right]^{1/(1-\sigma)} \Big\}.
 \end{aligned}
 \tag{4.23}$$

Hence, from conditions (i), (ii), and (iii), it follows that $\phi(s)$ is bounded on $[0, b)$. $\square$

THEOREM 4.4. Suppose that $S(\tau) \cup S(\tau^+) \subset L_w^2(a, b)$, $r_2(s)$ is bounded on $[0, b)$, and the following conditions are satisfied:

- (i) $e_1(s) \in L_w^2(0, b)$ and $r_1(s) \in L_w^{2/(1-\sigma)}(0, b)$ for all $s \in [0, b)$,
- (ii) $\int_0^t (\int_0^s [e_2(s) + e_3(x)] dx)^2 w(s) ds < \infty$,
- (iii) $\int_0^t (\sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)|)^2 w(s) ds < \infty$,
- (iv) $[\int_0^s w^{\sigma/(\sigma-2)} K_0^{2/(2-\sigma)}(s, x) dx]^{(2-\sigma)/2} < \infty$,
- (v) $[\int_0^s w^{-1} |(\partial^i/\partial x^i) K_i(s-x)|^2 dx]^{1/2} w(s) ds < \infty$.

Then all solutions of (1.2) are in $L_w^2(0, b)$.

PROOF. For $0 \leq \sigma < 1$, the proof is the same up to (4.18). Applying the Cauchy-Schwartz inequality to the integrals in (4.18), we have that

$$\begin{aligned}
 & \int_0^t |\overline{\phi_j^+(s)}| r_1(s) |\phi(s)|^\sigma w(s) ds \\
 & \leq \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^{\sigma/2} \left(\int_0^t |\overline{\phi_k^+(s)} r_1(s)|^\mu w(s) ds \right)^{1/\mu}, \\
 & \int_0^s K_0(s \cdot x) |\overline{\phi_k^+(s)}|^\sigma dx \\
 & \leq \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^{\sigma/2} \left(\int_0^s w^{1-\mu} K_0^\mu(s, x) dx \right)^{1/\mu}, \\
 & \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \phi(x) \right| dx \\
 & \leq \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^{1/2} \left(\int_0^s w^{-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right|^2 dx \right)^{1/2},
 \end{aligned} \tag{4.24}$$

where $\mu = 2/(2-\sigma)$. Since $\phi_k^+(s) \in L_w^2(0, b)$ (see Lemma 3.3), $k = 1, \dots, n^2$, and $r_1(s) \in L_w^{2/(1-\sigma)}(0, b)$ by hypothesis, we have $\phi_k^+ r_1 \in L_w^\mu(0, b)$, $k = 1, 2, \dots, n^2$. Using this fact and (4.18) in (4.16), we obtain

$$\begin{aligned}
 |\phi(t)| & \leq h(t) + \sum_{j,k=1}^{n^2} |\zeta^{jk}| |\phi_j(t)| \\
 & \times \left[\xi_0 \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^{\sigma/2} \right. \\
 & + \xi_1 \left(\int_0^t |\overline{\phi_k^+(s)}| r_2(s) \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^{\sigma/2} w(s) ds \right) \\
 & \left. + n^2 \xi_2 \left(\int_0^t |\overline{\phi_k^+(s)}| r_2(s) \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^{1/2} w(s) ds \right) \right],
 \end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
 \xi_0 & = \left(\int_0^t |\overline{\phi_k^+(s)} r_1(s)|^\mu w(s) ds \right)^{1/\mu}, \\
 \xi_1 & = \left(\int_0^t w^{1-\mu} K_0^\mu(s, x) dx \right)^{1/\mu}, \\
 \xi_2 & = \left(\int_0^t w^{-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right|^2 dx \right)^{1/2}, \quad \text{for } i = 1, 2, \dots, n^2.
 \end{aligned} \tag{4.26}$$

Applying the Cauchy-Schwartz inequality again to the integrals in (4.25) and squaring both sides, we have

$$\begin{aligned}
 & |\phi(t)|^2 \\
 & \leq 2h^2(t) + 4 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 |\phi_j(t)|^2 \\
 & \quad \times \left[\xi_0^2 \left(\int_0^t |\phi(s)|^2 w(s) ds \right)^\sigma \right. \\
 & \quad + \xi_1^2 \left(\int_0^t |\overline{\phi_k^+}(s)|^2 w(s) ds \right) \left(\int_0^t r_2^2 \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^\sigma \right) \\
 & \quad + n^4 \xi_0^2 \left(\int_0^t |\overline{\phi_k^+}(s)|^2 w(s) ds \right) \left(\int_0^t r_2^2 \left(\int_0^s |\phi(x)|^2 w(x) dx \right)^\sigma \right. \\
 & \quad \left. \left. \times w(s) ds \right) \right]. \tag{4.27}
 \end{aligned}$$

Let

$$\begin{aligned}
 u(t) &= \int_0^t |\phi(s)|^2 w(s) ds, \\
 \xi_3 &= \left(\int_0^t |\overline{\phi_k^+}(s)|^2 w(s) ds \right)^{1/2}, \quad j = 1, 2, \dots, n^2, \tag{4.28}
 \end{aligned}$$

and integrate (4.27), to obtain

$$\begin{aligned}
 u(t) &\leq 2 \int_0^t h^2(s) w(s) ds \\
 &+ 4 \xi_0^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 \int_0^t |\phi_j(s)|^2 u^\sigma(s) w(s) ds \\
 &+ 4 \xi_1^2 \xi_3^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 \int_0^t |\phi_j(s)|^2 \left(\int_0^s r_2^2(x) u^\sigma(x) w(x) dx \right) w(s) ds \\
 &+ 4 n^4 \xi_2^2 \xi_3^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 \int_0^t |\phi_j(s)|^2 \left(\int_0^s r_2^2(x) u(x) w(x) dx \right) w(s) ds. \tag{4.29}
 \end{aligned}$$

From our assumptions and conditions (ii) and (iii), it follows that the integral $\int_0^t h^2(s) w(s) ds$ is finite, that is, there exists a positive constant ξ_4 such that $\|h(t)\|_{L_w^2(0,b)} \leq \xi_4$ for all $t \in [0, b)$. Applying Corollary 3.8 with $f_1(x) = 0$, we

obtain

$$\begin{aligned}
 u(t) \leq & \exp \left(\int_0^t \left(4n^4 \xi_2^2 \xi_3^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 |\phi_j(s)|^2 \int_0^s r_2^2(x) w(x) dx \right) w(s) ds \right) \\
 & \times \left[\xi_4^{(1-\sigma)} + (1-\sigma) \right. \\
 & \times \int_0^t 4 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 |\phi_j(s)|^2 \left(\xi_0^2 + \xi_1^2 \xi_3^2 \int_0^s r_2^2(x) w(x) dx \right) \\
 & \times \exp \left((1-\sigma) \int_0^s \left[4n^4 \xi_2^2 \xi_3^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}|^2 |\phi_j(x)|^2 \int_0^x r_2^2(\tau) w(\tau) d\tau \right] \right. \\
 & \left. \left. \times w(x) dx \right) w(s) ds \right]^{1/(1-\sigma)}. \quad (4.30)
 \end{aligned}$$

Since $\phi_j(t) \in L_w^2(0, b)$, $j = 1, 2, \dots, n^2$, and $r_2(t)$ is bounded on $[0, b)$, then $\phi(t) \in L_w^2(0, b)$ and hence the result. $\square$

COROLLARY 4.5. Suppose that $S(\tau) \cup S(\tau^+) \subset L_w^2(0, b) \cap L^\infty(0, b)$ and the following conditions are satisfied:

- (i) $e_1(s) \in L_w^2(0, b)$ and $r_1(s) \in L_w^p(0, b)$ for any p , $1 \leq p \leq 2/(1-\sigma)$,
- (ii) $r_2(s)$ and $K_i^{(\ell)}(s) \in L_w^2(0, b) \cap L^\infty(0, b)$ for $\ell = 0, \dots, i-1$; $i = 1, \dots, n^2 - 1$,
- (iii) $\int_0^t (\int_0^s [e_2(s) + e_3(x)] dx)^2 w(s) ds < \infty$,
- (iv) $(\int_0^s w^{\sigma/(\sigma-2)} K_0^{2/(2-\sigma)}(s, x) dx)^{(2-\sigma)/2} < \infty$, $0 \leq \sigma < 1$,
- (v) $[\int_0^s w^{-1} |(\partial^i / \partial x^i) K_i(s-x)|^2 dx]^{1/2} < \infty$, $i = 1, \dots, n^2 - 1$.

Then all solutions of (1.2) belong to $L_w^2(0, b) \cap L^\infty(0, b)$.

PROOF. The proof follows from Theorems 4.3 and 4.4. We refer to [6, 7, 8, 11] for more details. $\square$

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