A NOTE ON THE REDUCIBILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS

XIAOPING YUAN and ANA NUNES

Received 3 June 2002

The system $\dot{x} = (A + \epsilon Q(t))x$, where A is a constant matrix whose eigenvalues are not necessarily simple and Q is a quasiperiodic analytic matrix, is considered. It is proved that, for most values of the frequencies, the system is reducible.

2000 Mathematics Subject Classification: 37C55, 34A30, 34C20.

1. Introduction and results. Consider the quasiperiodic linear differential equation

$$\dot{x} = (A + \epsilon Q(t))x \quad (1.1)$$

with x an n -dimensional vector, A a constant square matrix of order n , and Q a square matrix of order n , quasiperiodic in time t . We say that a change of variables $x = P(t)y$ is a Lyapunov-Perron (LP) transformation if $P(t)$ is nonsingular and $P(t)$, $P^{-1}(t)$, and $\dot{P}(t)$ are bounded for all $t \in \mathbb{R}$. Moreover, if P , P^{-1} , and $\dot{P}$ are quasiperiodic in time t , we refer to $x = P(t)y$ as a quasiperiodic LP transformation. If there is a quasiperiodic LP transformation $x = P(t)y$ such that y satisfies the equation

$$\dot{y} = By \quad (1.2)$$

with B a constant matrix, then we say that (1.1) is reducible.

The concept of the reducibility was first considered by Lyapunov (see [5]). There are several authors who investigated the reducibility of (1.1) (see, e.g., [1, 2, 6]). The present paper complements the results obtained by Jorba and Simó [2], which we will briefly recall. To this end, we will introduce some notation and definitions that will be used throughout the paper.

We say that a function F is a quasiperiodic function in time t , with the basic frequencies $\omega = (\omega_1, \dots, \omega_r)$, if there exists a function $\mathcal{F}(\theta_1, \dots, \theta_r)$ which is 2π -periodic in all its arguments θ_j , $j = 1, \dots, r$, and such that $F(t) = \mathcal{F}(\omega_1 t, \dots, \omega_r t)$. We call $\mathcal{F}$ the hull of $F(t)$. The function F will be called analytic quasiperiodic in a strip of width δ if, furthermore, $\mathcal{F}$ is analytic in the complex strip $|\operatorname{Im}\theta| < \delta$.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the eigenvalues of A and $\lambda^0(\epsilon) = (\lambda_1^0(\epsilon), \dots, \lambda_n^0(\epsilon))$ the eigenvalues of $\bar{A} := A + \epsilon \bar{Q}$, where $\bar{Q}$ is the average of $Q(t)$,

$$\bar{Q} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Q(t) dt. \quad (1.3)$$

Assume that $Q(t)$ is analytic on the strip of width $\delta_0 > 0$ and that the vector $(\lambda, \sqrt{-1}\omega)$ satisfies the nonresonance conditions

$$|\sqrt{-1}k \cdot \omega + l \cdot \lambda| \geq \frac{c}{|k|^\gamma}, \quad (1.4)$$

where $l \in \mathbb{Z}^n$ with $|l| = 0, 2$ and $0 \neq k \in \mathbb{Z}^r$. It was shown by Jorba and Simó [2] that (1.1) is reducible for ϵ in some Cantorian set $\mathcal{E} \subset (0, \epsilon_0)$, with ϵ_0 sufficiently small, provided that

- (1) the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are different;
- (2) the eigenvalues $\lambda_1^0(\epsilon), \dots, \lambda_n^0(\epsilon)$ of $\bar{A}$ satisfy

$$\left| \frac{d}{d\epsilon} (\lambda_i^0(\epsilon) - \lambda_j^0(\epsilon)) \right|_{\epsilon=0} > 2\rho > 0 \quad (1.5)$$

for some constant ρ and any $1 \leq i < j \leq n$.

In [2], the basic idea is to kill the small perturbation $\epsilon Q(t)$ by KAM iteration. Condition (2) is used to overcome the problem arising from the frequency shift which comes up in this procedure. By a well-known theorem [3, pages 113–115], condition (1) guarantees that the eigenvalues $\lambda_j^0(\epsilon)$ of $\bar{A} = A + \epsilon \bar{Q}$ are differentiable in ϵ , and that therefore condition (2) can be imposed.

A natural question is: what happens when condition (1) or (2) is not satisfied? The main result of the present paper is the following theorem which gives an answer to this question.

THEOREM 1.1. *Let $\Omega_0 \subset \mathbb{R}_+^r$ be a compact set with positive Lebesgue measure and assume that $Q(t)$ is quasiperiodic with frequency $\omega \in \Omega_0$ and analytic in the strip of width $\delta_0 > 0$. Then, for a sufficiently small positive constant γ , there exist a subset $\Omega \subset \Omega_0$ with $\text{Meas}(\Omega_0 \setminus \Omega) = \text{Meas}(\Omega_0)(1 - O(\gamma^{1/n^2}))$ and a sufficiently small constant $\epsilon^* = \epsilon^*(\delta_0, \gamma) > 0$ such that for any $\epsilon \in (0, \epsilon^*)$, system (1.1) is reducible. More exactly, there is an analytic quasiperiodic transformation $x = P(t)\gamma$ such that (1.1) is changed into*

$$\dot{\gamma} = B\gamma, \quad (1.6)$$

where B is a constant matrix with $\|A - B\| = O(\epsilon)$.

The proof is based on the construction of an iterative lemma, Lemma 2.1. In this construction, a finite number of terms in the Fourier expansion of the perturbation are killed in each iteration, and the remainder is included in the higher-order perturbation. The averaged perturbation is included in the time-independent term. To solve the homological equation, avoiding the problem

of small divisors, certain frequencies must be removed from the original frequency set Ω_0 at each iteration step. Showing that the remaining frequencies form a big subset of Ω_0 through the estimates of [Section 3](#) concludes the proof.

REMARK 1.2. When one of λ_j is not simple, the functions $\lambda_j^0(\epsilon)$ are not necessarily differentiable in ϵ . Therefore, in the hypothesis of [Theorem 1.1](#), we have to regard the tangent frequencies ω , instead of ϵ , as the parameters used to overcome the frequency shift in KAM iterative steps. Thus, we cannot find explicitly a tangent frequency vector ω satisfying some Diophantine conditions such that [Theorem 1.1](#) holds true. On the other hand, in [Theorem 1.1](#) it is not necessary to excise a subset of small measure from $(0, \epsilon^*)$. In this sense, [Theorem 1.1](#) complements the results of [2]. Yet another complementary approach is that of [1], where ω is fixed and reducibility is proved for “most” matrices A .

2. Proof of [Theorem 1.1](#). The proof of [Theorem 1.1](#) is based on Newton iteration. Before we state the main iterative lemma, we need to introduce some notation.

In the following, we denote by $C, C_1, C_2, \dots$ positive constants which arise in the estimates, by $\mathfrak{Q}$ the hull of a quasiperiodic function $Q(t)$, and by $\tilde{\mathfrak{Q}}$ the average of $\mathfrak{Q}$ on the r -torus. For a matrix-valued function $Q(t)$, define

$$\|Q\|^D := \sup_{t \in D} \|Q(t)\|, \quad (2.1)$$

where $\|\cdot\|$ is the sup-norm of the matrix.

Denote by m the number of the iterative step, and let

- (1) ϵ_m be the sequence that bounds the size of the perturbation before the m th iteration step with $\epsilon_m = \epsilon^{(1+\rho)^{m-1}}$ and $\rho = 1/3$, for example;
- (2) δ_m be the sequence that measures the size of the analyticity domain in the angular variables after m iteration steps with

$$\delta_m = \delta_0 - \frac{\delta_0}{2}(1 + \dots + m^{-2}) \sum_{j=1}^{\infty} j^{-2} \quad \text{for } m \geq 1; \quad (2.2)$$

- (3) $U_m = U(\delta_m) = \{\theta \in (\mathbb{C}/2\pi\mathbb{Z})^N : |\operatorname{Im} \theta| < \delta_m\}$;
- (4) q_m be the sequence that measures the width of the analyticity domain in the frequency space after m iteration steps with $q_m = \epsilon_{m+1}^{1/4n^2}$, where n is the dimensional number of system (1.1);
- (5) $C(m)$ be a constant of the form $C_1 m^{C_2}$.

Let $\Omega_0 = \Pi_0 \supset \Pi_1 \supset \dots \supset \Pi_{m-1}$ be the closed sets in $\mathbb{R}_+^r$ and let $\Pi_m \subset \Pi_{m-1}$ be as defined inductively in [Section 3](#). Let $\mathbb{O}_l$ be the complex q_l -neighborhood of Π_l for $l = 0, 1, \dots, m$. Assume that, after $m-1$ steps of Newton iteration, we

get a quasiperiodic linear differential equation

$$\dot{x} = (A_{m-1} + \epsilon_m Q_m(t; \omega))x, \quad (2.3)$$

where the following conditions are satisfied:

- (H1)_m $A_{m-1} = A + \epsilon_1 \tilde{\mathcal{Q}}_1(\omega) + \dots + \epsilon_{m-1} \tilde{\mathcal{Q}}_{m-1}(\omega)$, $m \geq 2$, $A_0 = A$ with $\tilde{\mathcal{Q}}_l(\omega)$ analytic in $\mathbb{O}_l$, and $\|\tilde{\mathcal{Q}}_l\|^{\mathbb{O}_l} \leq 1$ for $l = 1, \dots, m-1$;
 (H2)_m the hull $\mathcal{Q}_m$ of $Q_m(t; \omega)$ is analytic in $U_{m-1} \times \mathbb{O}_{m-1}$ and

$$\|\mathcal{Q}_m(\theta, \omega)\|^{U_{m-1} \times \mathbb{O}_{m-1}} \leq 1. \quad (2.4)$$

Let

$$A_m = A_{m-1} + \epsilon_m \tilde{\mathcal{Q}}_m. \quad (2.5)$$

Then (2.3) can be rewritten as

$$\dot{x} = (A_m + \epsilon_m Q_m^*(t))x, \quad (2.6)$$

where $Q_m^*(t) = Q_m(t) - \tilde{\mathcal{Q}}_m$. Following [2], we will find a change of variables

$$x = (E + \epsilon_m P_m(t))y, \quad (2.7)$$

where E is the unit matrix such that (2.3) is changed into

$$\dot{x} = (A_m + O(\epsilon_{m+1}))x \quad (2.8)$$

verifying conditions (H1)_{m+1} and (H2)_{m+1}. This change of variables is given by the following lemma.

LEMMA 2.1 (iterative lemma). *Assume that (H1)_m and (H2)_m are fulfilled. Then there is a quasiperiodic LP transformation*

$$x = (E + P_m(t))y, \quad (2.9)$$

where $P_m(t)$ is quasiperiodic with frequency ω and its hull $\mathcal{P}_m(\theta; \omega)$ is analytic in $U_m \times \mathbb{O}_m$ such that (2.3) is changed into

$$\dot{y} = (A_m + \epsilon_{m+1} Q_{m+1}(t))y, \quad (2.10)$$

where A_m and Q_{m+1} satisfy the conditions (H1)_{m+1} and (H2)_{m+1}.

PROOF. Rewrite (2.3) as

$$\dot{x} = (A_m + \epsilon_m Q_m^*(t))x, \quad (2.11)$$

where $Q_m^*(t) = Q_m(t) - \tilde{\mathcal{Q}}_m$ and $A_m = A_{m-1} + \tilde{\mathcal{Q}}_m$. Hence, we can write

$$\mathcal{Q}_m^*(\theta, \omega) = \sum_{0 \neq k \in \mathbb{Z}^r} \hat{\mathcal{Q}}_m^*(k; \omega) e^{\sqrt{-1}k \cdot \theta}, \quad (2.12)$$

where $\hat{\mathfrak{Q}}_m^*(k; \omega)$ is the k Fourier coefficient of $\mathfrak{Q}_m^*(\theta, \omega)$ in θ . Let $M_m = (1/b_m)|\ln \epsilon_m|$, where $b_m = \delta_{m-1} - \delta_m$, and let

$$\begin{aligned}\mathfrak{Q}_m^{*1}(\theta, \omega) &= \sum_{0 \neq |k| \leq M_m} \hat{\mathfrak{Q}}_m^*(k; \omega) e^{\sqrt{-1}k \cdot \theta}, \\ \mathfrak{Q}_m^{*2}(\theta, \omega) &= \sum_{|k| > M_m} \hat{\mathfrak{Q}}_m^*(k; \omega) e^{\sqrt{-1}k \cdot \theta},\end{aligned}\quad (2.13)$$

so that

$$\mathfrak{Q}_m^*(\theta, \omega) = \mathfrak{Q}_m^{*1}(\theta, \omega) + \mathfrak{Q}_m^{*2}(\theta, \omega). \quad (2.14)$$

We claim that

$$\|\mathfrak{Q}_m^{*2}(\theta, \omega)\|^{U_m \times \mathbb{C}_m} \leq C(m) \epsilon_m, \quad (2.15)$$

where $C(m)$ is a constant of the form $C_1 m^{C_2}$. In fact,

$$\begin{aligned}\|\mathfrak{Q}_m^{*2}(\theta, \omega)\|^{U_m \times \mathbb{C}_m} &\leq \sum_{|k| > M_m} \|\hat{\mathfrak{Q}}_m^*(k; \omega)\|^{\mathbb{C}_m} |e^{\sqrt{-1}k \cdot \theta}|^{U_m} \\ &\leq \sum_{|k| > M_m} \|\hat{\mathfrak{Q}}_m^*(\theta; \omega)\|^{U_{m-1} \times \mathbb{C}_m} e^{-|k|\delta_{m-1}} e^{|k|\delta_m} \\ &\leq \sum_{|k| > M_m} e^{-|k|(\delta_{m-1} - \delta_m)} = \epsilon_m \sum_{|k| > 0} e^{-|k|(\delta_{m-1} - \delta_m)} \\ &\leq C(m) \epsilon_m.\end{aligned}\quad (2.16)$$

Next, we perform the change of variables as in (2.7), where E is the unit matrix in $\mathbb{R}^n$, to transform (2.10) into

$$\dot{\mathcal{Y}} = \left((E + \epsilon_m P_m)^{-1} (A_m + \epsilon_m (A_m P_m - \dot{P}_m + Q_m^{*1}) + \epsilon_{m+1} Q_{m+1}) \right) \mathcal{Y}, \quad (2.17)$$

where

$$\epsilon_{m+1} Q_{m+1} = \epsilon_m Q_m^{*2} + \epsilon_m^2 (E + \epsilon_m P_m)^{-1} Q_m^* P_m. \quad (2.18)$$

We would like to have

$$(E + \epsilon_m P_m)^{-1} (A_m + \epsilon_m (A_m P_m - \dot{P}_m + Q_m^{*1})) = A_m \quad (2.19)$$

and this implies that

$$\dot{P}_m = A_m P_m - P_m A_m + Q_m^{*1}. \quad (2.20)$$

In order to solve this equation, we consider

$$\omega \cdot \frac{\partial \mathcal{P}_m}{\partial \theta} = A_m \mathcal{P}_m - \mathcal{P}_m A_m + \mathfrak{Q}_m^{*1}, \quad (2.21)$$

where $\mathcal{P}$ is the hull of $P(t)$. Write

$$\mathcal{P}_m(\theta, \omega) = \sum_{0 \neq |k| \leq M_m} \hat{\mathcal{P}}_m(k; \omega) e^{\sqrt{-1}k \cdot \theta}. \quad (2.22)$$

Then we get

$$\sqrt{-1}(k \cdot \omega) \hat{\mathcal{P}}_m(k) = A_m \hat{\mathcal{P}}_m(k) - \hat{\mathcal{P}}_m(k) A_m + \hat{\mathcal{Q}}_m^{*1}(k), \quad 0 < |k| \leq M_m, \quad (2.23)$$

where we omit the dependence on ω to simplify the notation. That is,

$$(\sqrt{-1}(k \cdot \omega) E - A_m) \hat{\mathcal{P}}_m(k) + \hat{\mathcal{P}}_m(k) A_m = \hat{\mathcal{Q}}_m^{*1}(k), \quad 0 < |k| \leq M_m. \quad (2.24)$$

By Lemmas A.2 and 3.1, (2.24) is solvable for $\omega \in \mathbb{C}_m$ and

$$\begin{aligned} \|\hat{\mathcal{P}}_m(k; \omega)\|^{e_m} &\leq \left\| ((\sqrt{-1}k \cdot \omega) E_{n^2} - E_n \otimes A_m + A_m^T \otimes E_n)^{-1} \right\|^{e_m} \|\hat{\mathcal{Q}}_m^{*1}(k)\|^{e_m} \\ &\leq \frac{C|k|^\tau}{\gamma_m} \|\mathcal{Q}_m^*\|^{U_{m-1} \times \mathbb{C}_{m-1}} e^{-|k|\delta_{m-1}} \\ &\leq \frac{C|k|^\tau}{\gamma_m} \|\mathcal{Q}_m\|^{U_{m-1} \times \mathbb{C}_{m-1}} e^{-|k|\delta_{m-1}} \\ &\leq \frac{C|k|^\tau}{\gamma_m} e^{-|k|\delta_{m-1}}, \end{aligned} \quad (2.25)$$

where in the last inequality we have used (H2) $_m$. Therefore,

$$\begin{aligned} \|\mathcal{P}_m(\theta, \omega)\|^{U_m \times \mathbb{C}_m} &\leq \sum_{0 < |k| \leq M_m} \|\hat{\mathcal{P}}_m(k; \omega)\|^{e_m} |e^{\sqrt{-1}k \cdot \theta}|^{U_m} \\ &\leq \sum_{k \in \mathbb{Z}^r} \frac{C|k|^\tau}{\gamma_m} e^{-|k|(\delta_{m-1} - \delta_m)} \\ &\leq \frac{C}{\gamma_m} \sum_{k \in \mathbb{Z}^r} |k|^\tau e^{-C_3|k|(\delta_0/m^2)} \leq C(m), \end{aligned} \quad (2.26)$$

where the last inequality follows from Lemma A.1. Then, the function

$$P_m(t) = \mathcal{P}_m(\omega_1 t, \dots, \omega_r t; \omega) \quad (2.27)$$

solves (2.20). By (2.26) and (2.18), it is easy to show that $\|\mathcal{Q}_{m+1}\|^{U_m \times \mathbb{C}_m} \leq 1$. We omit the details. $\square$

PROOF OF THEOREM 1.1. Obviously, (1.1) satisfies the conditions (H) $_m$ with $m = 1$. In fact, condition (H2) $_1$ may always be fulfilled by a suitable rescaling of ϵ . Thus, by Lemma 2.1, there exists a sequence of transformations $x = (E + \epsilon_m P_m(t))y$, $m = 1, 2, \dots$, such that the hulls $\mathcal{P}_m$ of the $P_m(t)$ are analytic in the domains $U_m \times \mathbb{C}_m$, and

$$\|\mathcal{P}_m\|^{U_m \times \mathbb{C}_m} \leq C(m). \quad (2.28)$$

Let

$$U_\infty \times \mathbb{C}_\infty = \bigcap_{m=1}^{\infty} U_m \times \mathbb{C}_m. \quad (2.29)$$

Then, all the $\mathcal{P}_m$, $m = 1, 2, \dots$, are well defined in the domain $U_\infty \times \mathbb{C}_\infty$. Set

$$\begin{aligned} \Phi(\theta, \omega) &= \dots \circ (E + \epsilon_m \mathcal{P}_m(\theta, \omega)) \circ \dots \circ (E + \epsilon_2 \mathcal{P}_2(\theta, \omega)) \circ (E + \epsilon_1 \mathcal{P}_1(\theta, \omega)), \\ \Phi(t; \omega) &= \dots \circ (E + \epsilon_m \mathcal{P}_m(t; \omega)) \circ \dots \circ (E + \epsilon_2 \mathcal{P}_2(t; \omega)) \circ (E + \epsilon_1 \mathcal{P}_1(t; \omega)). \end{aligned} \quad (2.30)$$

Note that $\|E + \epsilon_m \mathcal{P}_m(\theta; \omega)\|^{U_\infty \times \mathbb{C}_\infty} \leq 1 + \epsilon_m C(m) \leq 1 + 2^{-m}$. We see that Φ , and thus Φ , are well defined. Let

$$x = \Phi(t)y. \quad (2.31)$$

Since $\epsilon_m \|\mathcal{Q}_m\|^{U_\infty \times \mathbb{C}_\infty} \leq \epsilon_m \rightarrow 0$ as $m \rightarrow \infty$, the transformation $x = \Phi(t)y$ changes (1.1) into

$$\dot{y} = By, \quad (2.32)$$

where $B = A + \sum_{j=1}^{\infty} \epsilon_m \tilde{\mathcal{Q}}_m$. This, together with Lemma 3.3, completes the proof of Theorem 1.1. $\square$

3. Estimates on the allowed frequencies set. Let Π_l ($0 \leq l \leq m-1$) be a sequence of compact subsets of $\mathbb{R}_+^r$ with $\Omega = \Pi_0 \supset \Pi_1 \supset \dots \supset \Pi_{m-1}$ and denote by $\mathbb{C}_l$ the complex q_l -neighborhood of Π_l , $l = 0, \dots, m-1$. Recall that

$$A_m(\omega) = A + \epsilon_1 \tilde{\mathcal{Q}}_1(\omega) + \dots + \epsilon_m \tilde{\mathcal{Q}}_m(\omega), \quad (3.1)$$

where, for $l = 1, \dots, m-1$, the $\tilde{\mathcal{Q}}_l(\omega)$ are analytic, and real for real arguments in the domain $\mathbb{C}_l$, and $\|\tilde{\mathcal{Q}}_l(\omega)\|^{\mathbb{C}_l} \leq 1$; and $\tilde{\mathcal{Q}}_m(\omega)$ is analytic, and real for real arguments in the domain $\mathbb{C}_{m-1}$, and $\|\tilde{\mathcal{Q}}_m(\omega)\|^{\mathbb{C}_{m-1}} \leq 1$. Denote by $|\cdot|_d$ the determinant of a matrix. Let

$$\mathcal{R}_k(m) := \left\{ \omega \in \Pi_{m-1} : \left| |(\sqrt{-1}k \cdot \omega)E_{n^2} - E_n \otimes A_m + A_m^T \otimes E_n|_d \right| < \frac{\gamma_m}{|k|^{\tau_1}} \right\}, \quad (3.2)$$

where $\gamma_m = \gamma/m^{2n^2}$ and $\tau_1 = (r+1)n^2$,

$$\Pi_m = \Pi_{m-1} \setminus \bigcup_{0 < |k| \leq M_m} \mathcal{R}_k(m), \quad (3.3)$$

where $M_m = |\ln \epsilon_m|/(\delta_{m-1} - \delta_m)$ is the number of Fourier coefficients we must consider at the m th step of the iteration, and denote by $\mathbb{C}_m$ the complex q_m -neighborhood of Π_m .

LEMMA 3.1. *Let $\tau = \tau_1 + n^2 - 1$ and*

$$G(\omega) = (\sqrt{-1}k \cdot \omega)E_{n^2} - E_n \otimes A_m(\omega) + A_m^T(\omega) \otimes E_n. \quad (3.4)$$

Then, for $\omega \in \mathbb{C}_m$ and $0 < |k| \leq M_m$, the inverse of $G(\omega)$ exists and it is analytic in the domain $\mathbb{C}_m$ with

$$\|G^{-1}(\omega)\|^{\mathbb{C}_m} \leq C\gamma_m^{-1}|k|^\tau. \quad (3.5)$$

PROOF. By the definition of Π_m , we get that for $\omega \in \Pi_m$ and $0 < |k| \leq M_m$,

$$|\mathcal{M}_k(\omega)| \geq \frac{\gamma_m}{|k|^{\tau_1}}. \quad (3.6)$$

It is easy to see that

$$\|G(\omega)\|^{\mathbb{C}_m} \leq C_5|k|, \quad k \neq 0, \quad (3.7)$$

where $C_5 = 2(\max\{|\omega| : \omega \in \Pi\} + \|A\| + 1)$. Since $\det G(\omega) = \mathcal{M}_k(\omega)$, $G^{-1}(\omega)$ exists for $\omega \in \Pi_m$ and

$$G^{-1}(\omega) = \frac{\text{adj}G(\omega)}{\mathcal{M}_k(\omega)}, \quad (3.8)$$

where adj is the adjoint of a matrix. Thus, for $0 < |k| \leq M_m$,

$$\|G^{-1}(\omega)\|^{\Pi_m} \leq C_6 \frac{|k|^{n^2-1}}{\gamma_m/|k|^{\tau_1}} = C_6\gamma_m^{-1}|k|^\tau. \quad (3.9)$$

Now, we assume that $\omega \in \mathbb{C}_m$. Then there is an $\omega_0 \in \Pi_m$ such that $|\omega - \omega_0| < q_m$. Thus,

$$\begin{aligned} & \|G^{-1}(\omega_0)\| \|G(\omega) - G(\omega_0)\| \\ & \leq \|G^{-1}(\omega)\|^{\Pi_m} \|\nabla_\omega G(\omega)\|^{\mathbb{C}_m} \cdot |\omega - \omega_0| \\ & \leq C_6\gamma_m^{-1}|k|^\tau \|G(\omega)\|^{\mathbb{C}_{m-1}} \frac{q_m}{q_{m-1} - q_m} \\ & \leq C_6\gamma_m^{-1}|k|^{\tau+1} \frac{q_m}{q_{m-1} - q_m} \\ & \leq C_6M_m^{\tau+1}\gamma_m^{-1} \frac{q_m}{q_{m-1} - q_m} \\ & \leq \frac{C_6m^{(6+2r)n^2} |\ln \epsilon_m|^{\tau+1} \epsilon_{m+1}^{1/4n^2} (\gamma\delta_0)^{-1}}{\epsilon_m^{1/4n^2} - \epsilon_{m+1}^{1/4n^2}} \\ & < \frac{1}{2}. \end{aligned} \quad (3.10)$$

Therefore, $E + G^{-1}(\omega_0)(G(\omega) - G(\omega_0))$ has its inverse which is analytic in $\mathbb{C}_m$ since

$$\begin{aligned} & (E + G^{-1}(\omega_0)(G(\omega) - G(\omega_0)))^{-1} \\ &= \sum_{j=0}^{\infty} (-G^{-1}(\omega_0)(G(\omega) - G(\omega_0)))^j. \end{aligned} \quad (3.11)$$

So, $G(\omega)$ has its inverse for $\omega \in \mathbb{C}_m$ and

$$\begin{aligned} \|G^{-1}(\omega)\| &= \left\| (E + G^{-1}(\omega_0)(G(\omega) - G(\omega_0)))^{-1} \cdot G^{-1}(\omega_0) \right\| \\ &\leq \left\| (E + G^{-1}(\omega_0)(G(\omega) - G(\omega_0)))^{-1} \right\| \cdot \|G^{-1}(\omega_0)\| \\ &\leq C\gamma_m^{-1}|k|^{\tau}. \end{aligned} \quad (3.12) \quad \square$$

LEMMA 3.2. *Let $K = n^2(n+1)^2(\text{diam}\Pi_0)^{r-1}$. Then the Lebesgue measure of $\mathcal{R}_k(m)$ verifies*

$$\text{Meas}\mathcal{R}_k(m) \leq \frac{K\gamma^{1/n^2}}{|k|^{r+1}} \frac{1}{m^2}. \quad (3.13)$$

PROOF. Recall that $q_l = \epsilon_{l+1}^{1/4n^2}$, let $q_l^1 = (5/6)q_l + (1/6)q_{l+1}$, and denote by $\mathbb{C}_l^1$ the complex q_l^1 -neighborhood of Π_l . Obviously, $\mathbb{C}_{l+1} \subset \mathbb{C}_l^1 \subset \mathbb{C}_l$ and $\text{dist}(\partial\mathbb{C}_l^1, \partial\mathbb{C}_l) = (1/6)(q_l - q_{l+1}) > (1/12)q_l$. Noting that $\|\tilde{\mathcal{Q}}_l\|^{\mathbb{C}_l} \leq 1$ and using Cauchy's theorem, we get for $1 \leq s \leq n^2$ and $0 \leq l \leq m-1$,

$$\epsilon_l \|\partial_{\omega}^s \tilde{\mathcal{Q}}_l\|^{\mathbb{C}_l^1} \leq \epsilon_l (12q_l^{-1})^s \|\tilde{\mathcal{Q}}_l\|^{\mathbb{C}_l} \leq \epsilon_l^{1/2}. \quad (3.14)$$

The combination of (3.1) and (3.14) leads to

$$\|\partial_{\omega}^s A_m(\omega)\|^{\mathbb{C}_{m-1}^1} \leq \epsilon^{1/2}. \quad (3.15)$$

Let

$$B(\omega) := -E_n \otimes A_m(\omega) + A_m^T \otimes E_n. \quad (3.16)$$

Then

$$\|\partial_{\omega}^s B(\omega)\|^{\mathbb{C}_{m-1}^1} \leq \epsilon^{1/2}. \quad (3.17)$$

Set

$$\mathcal{M}_k(\omega) = |(\sqrt{-1}k \cdot \omega)E_{n^2} + B(\omega)|_d. \quad (3.18)$$

We are now in a position to estimate $\partial_{\omega}^s \mathcal{M}_k$. To this end, write $B(\omega) = (b_{ij})$. Then

$$\mathcal{M}_k(\omega) = (\sqrt{-1})^{n^2} (k \cdot \omega)^{n^2} + \sum_{1 \leq l \leq n^2-1} \phi_l(\omega) (k \cdot \omega)^{n^2-l}, \quad (3.19)$$

where

$$\phi_l(\omega) = \sum_{1 \leq j_i \leq n^2} \sigma_{j_1 \dots j_l} b_{1j_1} \cdots b_{lj_l} \quad (3.20)$$

and $\sigma_{j_1 \dots j_l} \in \{-1, +1, -\sqrt{-1}, +\sqrt{-1}\}$.

Observe that for $1 \leq l \leq n^2$ and $\omega \in \mathbb{O}_{m-1}^1$,

$$|\partial_{\omega_1}^s b_{ij}| \leq \|\partial_{\omega}^s B(\omega)\|_{m-1}^{\mathbb{O}_{m-1}^1} \leq \epsilon^{1/2}. \quad (3.21)$$

Thus, for $\omega \in \mathbb{O}_{m-1}^1$ and $1 \leq s \leq n^2$,

$$\begin{aligned} \left| \frac{d^s}{d\omega_1^s} (b_{1j_1} \cdots b_{lj_l}) \right| &\leq \left| \sum_{s_1 + \dots + s_l = s} \left(\frac{d^{s_1}}{d\omega_1^{s_1}} b_{1j_1} \right) \cdots \left(\frac{d^{s_l}}{d\omega_1^{s_l}} b_{lj_l} \right) \right| \\ &\leq \epsilon^{(1/2)(s_1 + \dots + s_l)} \sum_{s_1 + \dots + s_l = s} 1 \\ &\leq (2\epsilon^{1/2})^s, \end{aligned} \quad (3.22)$$

and therefore,

$$\left| \frac{d^s}{d\omega_1^s} \phi_l(\omega) \right| \leq \binom{n^2}{l} (2\epsilon^{1/2})^s. \quad (3.23)$$

Without loss of generality, assume that $|k| = |k_1| + \dots + |k_r| \leq r|k_1|$. Hence, for every $\omega \in \mathbb{O}_{m-1}^1$,

$$\begin{aligned} &\left| \frac{d^{n^2}}{d\omega_1^{n^2}} \sum_{1 \leq l \leq n^2-1} \phi_l(\omega) (k \cdot \omega)^{n^2-l} \right| \\ &\leq \sum_{1 \leq l \leq n^2-1} \left| \frac{d^{n^2}}{d\omega_1^{n^2}} (\phi_l(\omega) (k \cdot \omega)^{n^2-l}) \right| \\ &\leq \sum_{1 \leq l \leq n^2-1} \sum_{l \leq s \leq n^2} \binom{n^2}{s} \left| \frac{d^{s^2}}{d\omega_1^{s^2}} \phi_l(\omega) \right| \left| \frac{d^{n^2-s}}{d\omega_1^{n^2-s}} (k \cdot \omega)^{n^2-l} \right| \\ &\leq \sum_{1 \leq l \leq n^2-1} \sum_{l \leq s \leq n^2} \binom{n^2}{l} \binom{n^2}{s} (2\epsilon^{1/2})^s |k_1|^{n^2-s} |k \cdot \omega|^{s-l} n^2! \\ &\leq C_4 \epsilon^{1/2} |k_1|^{n^2-1} n^2!, \end{aligned} \quad (3.24)$$

where C_4 is some constant which depends only on n, r , and on the maximum of $|\omega|$ in Π_0 . Obviously,

$$\frac{d^{n^2}}{d\omega_1^{n^2}} (k \cdot \omega)^{n^2} = n^2! |k_1|^{n^2}. \quad (3.25)$$

Thus, in $\mathbb{C}_{m-1}^1$, we have

$$\left| \frac{d^{n^2}}{d\omega_1^{n^2}} \mathcal{M}_k(\omega) \right| \geq n^2! |k_1|^{n^2} (1 - C_4 \epsilon^{1/2} |k_1|^{-1}) > \frac{1}{2} n^2! |k_1|^{n^2} \quad (3.26)$$

provided that ϵ is small enough so that $C_4 \epsilon^{1/2} < 1/2$. Using (3.26) and Lemma A.3, we get

$$\begin{aligned} \text{Meas} \mathcal{R}_k(m) &\leq n^2 (n^2 + 1) \left(\frac{y_m}{|k|^{\tau_1}} \right)^{1/n^2} (\text{diam} \Pi_0)^{r-1} \\ &\leq \frac{K y^{1/n^2}}{|k|^{r+1}} \cdot \frac{1}{m^2}. \end{aligned} \quad (3.27)$$

This completes the proof. $\square$

By Lemma 2.1, the nested sequence of closed sets

$$\Omega_0 = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_m \supset \cdots \quad (3.28)$$

is defined inductively. The following lemma is a corollary of Lemma 3.2.

LEMMA 3.3. *Let*

$$\Pi_\infty = \bigcap_{m=0}^{\infty} \Pi_m. \quad (3.29)$$

Then $\text{Meas} \Pi_\infty = (\text{Meas} \Pi_0) (1 - O(y^{1/n^2}))$.

Appendix

LEMMA A.1. *For $\delta > 0$ and $\nu > 0$, the following inequality holds true:*

$$\sum_{k \in \mathbb{Z}^N} e^{-2|k|\delta} |k|^\nu \leq \left(\frac{\nu}{e} \right)^\nu \frac{1}{\delta^{\nu+N}} (1+e)^N. \quad (\text{A.1})$$

PROOF. This lemma can be found in [1]. We will find the value of $z \geq 1$ yielding a maximum value for the expression $\nu \ln z - \delta z$. Differentiating it in z and equating the result to zero, we get that $\nu/z = \delta$ and $z = \nu/\delta > 1$. From this it follows that

$$\nu \ln z - \delta z \leq \nu \left(\ln \frac{\nu}{\delta} - 1 \right). \quad (\text{A.2})$$

This expression yields

$$z^\nu \leq \exp(\delta z) \exp \left(\nu \left(\ln \frac{\nu}{\delta} - 1 \right) \right) = \left(\frac{\nu}{e} \right)^\nu \frac{\exp(\delta z)}{\delta^\nu}. \quad (\text{A.3})$$

Thus,

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}^N} e^{-2|k|} |k|^\nu &\leq \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu} \sum_k e^{-|k|\delta} \\
 &= \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu} \left(\frac{1 + \exp(-\delta)}{1 - \exp(-\delta)}\right)^N \\
 &\leq \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^\nu} \left(\frac{1+e}{\delta}\right)^N.
 \end{aligned} \tag{A.4}$$

□

LEMMA A.2. Let A , B , and C be $r \times r$, $s \times s$, and $r \times s$ matrices, respectively; and let X be an $r \times s$ unknown matrix. Then the matrix equation

$$AX + XB = C \tag{A.5}$$

is solvable if and only if the vector equation

$$(E_s \otimes A + B^T \otimes E_r)X' = C' \tag{A.6}$$

is solvable, where $X' = (X_1^T, \dots, X_s^T)^T$, $C' = (C_1^T, \dots, C_s^T)^T$ if we write $X = (X_1, \dots, X_s)$ and $C = (C_1, \dots, C_s)$. Moreover,

$$\|X\| \leq \left\| (E_s \otimes A + B^T \otimes E_r)^{-1} \right\| \|C\| \tag{A.7}$$

if the inverse exists.

PROOF. This lemma can be found in many textbooks on matrix theory; for example, [4, page 256]. □

The following lemma can be found in [7, page 23].

LEMMA A.3. Let $\mathcal{I}$ be an interval in $\mathbb{R}^1$ and $\bar{\mathcal{I}}$ its closure. Suppose that $g: \bar{\mathcal{I}} \rightarrow \mathbb{C}$ is k times continuously differentiable. Let $\mathcal{I}_h = \{x \in \bar{\mathcal{I}} : |g(x)| \leq h\}$, $h > 0$. If, for some constant $d > 0$, $|d^k g(x)/dx^k| \geq d$ for any $x \in \mathcal{I}$, then $\text{Meas } I_h \leq ch^{1/k}$ with $c = 2(2 + 3 + \dots + k + d^{-1})$.

ACKNOWLEDGMENT. This work was carried out during the first author's stay at the Centro de Matemática e Aplicações Fundamentais (CMAF), Universidade de Lisboa with a Postdoctoral Fellowship Grant from the Fundação para a Ciência e Tecnologia.

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Xiaoping Yuan: Department of Mathematics, Fudan University, Shanghai 200433, China

Current address: Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, 1649-003 Lisboa, Portugal

Ana Nunes: Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, 1649-003 Lisboa, Portugal

E-mail address: anunes@lmc.fc.ul.pt