MAPPING PROPERTIES FOR CONVOLUTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

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Received 4 March 2002

For $\mu \geq 0$, we consider a linear operator $L_{\mu} : A \to A$ defined by the convolution $f_{\mu} * f$, where $f_{\mu} = (1 - \mu)z_2F_1(a, b, c; z) + \mu z(z_2F_1(a, b, c; z))'$. Let $\varphi^*(A, B)$ denote the class of normalized functions $f$ which are analytic in the open unit disk and satisfy the condition $zf'/f \prec (1 + Az)/(1 + Bz)$, $-1 \leq A < B \leq 1$, and let $R_{\eta}(\beta)$ denote the class of normalized analytic functions $f$ for which there exists a number $\eta \in (-\pi/2, \pi/2)$ such that $\text{Re}(e^{i\eta}(f'(z) - \beta)) > 0$, ($\beta < 1$). The main object of this paper is to establish the connection between $R_{\eta}(\beta)$ and $\varphi^*(A, B)$ involving the operator $L_{\mu}(f)$. Furthermore, we treat the convolution $I = \int_0^z f_{\mu}(t)/t \; dt * f(z)$ for $f \in R_{\eta}(\beta)$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and $S$ denotes the subclass of functions in $A$ which are univalent in $U$. Moreover, let $S^*(\alpha)$ and $K(\alpha)$ be the subclasses of $S$ consisting, respectively, of functions which are starlike of order $\alpha$ and convex of order $\alpha$, where $0 \leq \alpha < 1$ in $U$. Clearly, we have $S^*(\alpha) \subseteq S^*(0) = S^*$, where $S^*$ denotes the class of functions in $A$ which are starlike in $U$ and $K(\alpha) \subseteq K(0) = K$, where $K$ denotes the class of functions in $A$ which are convex in $U$, and we mention the well-known inclusion chain $K \subset S^*(1/2) \subset S^* \subset S$. For the analytic functions $g$ and $h$ on $U$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$ if there exists an analytic function $w$ on $U$ such that $w(0) = 0$, $|w(z)| < 1$, and $g(z) = h(w(z))$ for $z \in U$. We denote this subordinated relation by

$$g < h \quad \text{or} \quad g(z) < h(z) \quad (z \in U).$$

For $-1 \leq A < B \leq 1$, a function $p$, which is analytic in $U$ with $p(0) = 1$, is said to belong to the class $P(A, B)$ if

$$p(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$
The above condition means that $p$ takes the values in the disk with a center $(1-AB)/(1-B^2)$ and a radius $|A-B|/(1-B^2)$. The boundary circle cuts the real axis at the points $(1+A)/(1+B)$ and $(1-A)/(1-B)$. A function $f \in A$ is said to be in $\varphi^*(A,B)$ if $zf'/f \in P(A,B)$, and in $K(A,B)$ if $zf' \in \varphi^*(A,B)$. The class $\varphi^*(A,B)$ was introduced by N. Shukla and P. Shukla [4]. Also, Janowski [2] introduced the class $P(A,B)$. For the fixed natural number $n$, the subclass $P_n(A,B)$ of $P(A,B)$ containing functions $p$ of the form $p(z) = 1 + pnzn + \cdots$, $z \in U$, was defined by Stankiewicz and Waniurski [7]. In addition, Stankiewicz and Trojnar-Spelina [6] investigated a function $p(z) = 1 - pnzn - \cdots$ belongs to the class $R(n,A,B)$, where $A \in R$ and $B \in [0,1]$ if $p(z) \prec (1+Az)/(1-Bz)$. Let $R_\eta(\beta)$ denote the class of functions $f \in A$ for which there exists a number $\eta \in (-\pi/2,\pi/2)$ such that

$$\text{Re}[e^{in}(f'(z) - \beta)] > 0 \quad (z \in U, \beta < 1). \quad (1.4)$$

Clearly, we have $R_\eta(\beta) \subset S$ ($0 \leq \beta < 1$). Furthermore, if a function $f$ of the form (1.1) belongs to the class $R_\eta(\beta)$, then

$$|a_n| \leq \frac{2(1-\beta)\cos\eta}{n} \quad (n \in N \setminus \{1\}). \quad (1.5)$$

The class $R_\eta(\beta)$ was studied by Kanas and Srivastava [3].

The hypergeometric function $\,_{2}F_{1}(a,b,c;z)$ is given as a power series, converging in $U$, in the following way

$$\,_{2}F_{1}(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \quad (1.6)$$

where $a$, $b$, and $c$ are complex numbers with $c \neq 0, -1, -2, \ldots$, and $(\lambda)_n$ denotes the Pochhammer symbol (or the generalized factorial since $(1)_n = n!$) defined, in terms of the Gamma function $\Gamma$, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n \in N = \{1,2,\ldots\}. \end{cases} \quad (1.7)$$

Note that $\,_{2}F_{1}(a,b,c;z)$, for $a = c$ and $b = 1$ (or, alternatively, for $a = 1$ and $b = c$), reduces to the relatively more familiar geometric function. We also
note that $2F_1(a,b,c;1)$ converges for $\text{Re}(c - a - b) > 0$ and is related to the Gamma functions by

$$2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$ (1.8)

The Hadamard product (or convolution) of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$ (1.9)

N. Shukla and P. Shukla [4] studied the mapping properties of a function $f_\mu$ to be as given in

$$f_\mu(z) = (1 - \mu) z_2 F_1(a,b,c;z) + \mu z (z_2 F_1(a,b,c;z))' \quad (\mu \geq 0),$$ (1.10)

and investigated the geometric properties of an integral operator of the form

$$I(z) = \int_0^z f_\mu(t) \frac{t}{t} dt.$$ (1.11)

We now consider a linear operator $L_\mu : A \to A$ defined by

$$L_\mu(f) = f_\mu(z) \ast f(z).$$ (1.12)

For $\mu = 0$ in (1.12), $L_\mu(f) = [I_{a,b,c}(f)](z)$, which was introduced by Hohlov [1]. Also, Kanas and Srivastava [3], and Srivastava and Owa [5] showed that the operator $I_{a,b,c}(f)$ is the natural extensions of the Alexander, Libera, Bernardi, and Carlson-Shaffer operators. In this paper, we find a relation between $R_\eta(\beta)$ and $q_\ast(A,B)$ involving the operator $L_\mu(f)$. Furthermore, we study to obtain some conditions for the starlikeness and convexity of the convolution of $I$ and $f$, which are given by (1.11) and (1.1), respectively, for $f \in R_\eta(\beta)$.

2. Main results. We make use of the following lemma.

**Lemma 2.1 [4].** Sufficient conditions for $f$ of the form (1.1) to be in $q_\ast(A,B)$ and $K(A,B)$ are

$$\sum_{n=2}^{\infty} [(1+B)n - (A+1)] |a_n| \leq B - A,$$

$$\sum_{n=2}^{\infty} n[(1+B)n - (A+1)] |a_n| \leq B - A,$$

respectively.
\textbf{Theorem 2.2.} Let \(a > 1, \ b > 1, \) and \(c > a + b + 1.\) If \(f \in R_\eta(\beta)\) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1 + B) \left( 1 + \frac{\mu ab}{c-a-b-1} \right) - (A + 1) \left( \mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \right] 
\leq (B - A) \left( \frac{1}{2(1 - \beta) \cos \eta} + 1 \right) + \frac{(A + 1)(\mu - 1)(c-1)}{(a-1)(b-1)}
\]

is satisfied, then \(L_\mu(f) \in \varphi^*(A, B).\)

\textbf{Proof.} By Lemma 2.1, it suffices to show that

\[
T_1 := \sum_{n=2}^{\infty} \left[ (1 + B)n - (A + 1) \right] \frac{(1 + (n - 1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq B - A.
\]

Since \(f \in R_\eta(\beta)\) and \(|a_n| \leq 2(1 - \beta) \cos \eta / n.\) Hence,

\[
T_1 \leq \sum_{n=2}^{\infty} \left[ (1 + B)n - (A + 1) \right] \frac{(1 + (n - 1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{2(1 - \beta) \cos \eta}{n}
\]

\[
= 2(1 - \beta) \cos \eta \left\{ (1 + B) \left( \sum_{n=0}^{\infty} \frac{(a)n(b)n}{(c)n(1)n} - 1 \right) 
- \frac{(A + 1)(c-1)}{(a-1)(b-1)} \left( \sum_{n=0}^{\infty} \frac{(a-1)n(b-1)n}{(c-1)n(1)n} - 1 - \frac{(a-1)(b-1)}{c-1} \right) 
+ \frac{(1 + B)\mu ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)n(b+1)n}{(c+1)n(1)n} 
- (A + 1)\mu \left[ \sum_{n=0}^{\infty} \frac{(a)n(b)n}{(c)n(1)n} - 1 
- \frac{c-1}{(a-1)(b-1)} \left( \sum_{n=0}^{\infty} \frac{(a-1)n(b-1)n}{(c)n(1)n} - 1 - \frac{(a-1)(b-1)}{c-1} \right) \right] \right\}
\]

\[
= 2(1 - \beta) \cos \eta \left\{ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1 + B) \left( 1 + \frac{\mu ab}{c-a-b-1} \right) 
+ (A + 1) \left( \mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \right] 
- \left[ 1 + B - (A + 1) \left( 1 - \frac{(\mu-1)(c-1)}{(a-1)(b-1)} \right) \right] \right\}.
\]

(2.4)

Now, this last expression is bounded above by \(B - A\) if (2.2) holds. \(\square\)
If we take $\mu = 0, A = 2\alpha - 1,$ and $B = 1$ in Theorem 2.2, we have the following corollary.

**Corollary 2.3.** Let $a > 1, b > 1,$ and $c > a + b + 1.$ If $f \in R_\eta(\beta)$ and the inequality

$$
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right] 
\leq (1 - \alpha) \left( \frac{1}{2(1-\beta)\cos\eta + 1} \right) - \frac{\alpha(c-1)}{(a-1)(b-1)}
$$

is satisfied, then $z_2F_1(a,b,c;z) \ast f \in S^*(\alpha).$

If we take $\alpha = 0, \beta = 0,$ and $\eta = 0$ in Corollary 2.3, we get the following corollary.

**Corollary 2.4.** Let $a > 1, b > 1,$ and $c > a + b + 1.$ If $f \in S,$ and the inequality

$$
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2}
$$

is satisfied, then $z_2F_1(a,b,c;z) \ast f \in S^*.$

**Theorem 2.5.** Let $a > 0, b > 0,$ and $c > a + b + 2.$ If $f \in R_\eta(\beta),$ and the inequality

$$
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ B - A + ((1+B)(1+2\mu) 
- (A+1)\mu) \frac{ab}{c-a-b-1} + \frac{(1+B)\mu(a_2)(b_2)}{(c-a-b-2)_2} \right] 
\leq (B-A) \left( \frac{1}{2(1-\beta)\cos\eta + 1} \right)
$$

is satisfied, then $L_\mu(f) \in K(A,B).$

**Proof.** The proof follows from Lemma 2.1. Using the method of the proof of Theorem 2.2, we omit the details involved. □

For $\mu = 0, A = 2\alpha - 1,$ and $B = 1,$ Theorem 2.5 yields the following corollary.
Corollary 2.6. Let \( a > 0 \), \( b > 0 \), and \( c > a + b + 2 \). If \( f \in R_\eta(\beta) \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 - \alpha + \frac{ab}{c-a-b-1} \right] \leq (1 - \alpha) \left( \frac{1}{2(1-\beta)\cos\eta} + 1 \right)
\]

(2.8)

is satisfied, then \( z_2 F_1(a,b,c;z) \ast f \in K(\alpha) \).

For \( \alpha = 0 \), \( \beta = 0 \), and \( \eta = 0 \), Corollary 2.6 yields the following corollary.

Corollary 2.7. Let \( a > 0 \), \( b > 0 \), and \( c > a + b + 1 \). If \( f \in S \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 + \frac{ab}{c-a-b-1} \right] \leq \frac{3}{2}
\]

(2.9)

is satisfied, then \( z_2 F_1(a,b,c;z) \ast f \in K \).

In our next theorems, we find the sufficient conditions for \( I \ast f \) to be in \( \varphi^*(A,B) \) and \( K(A,B) \). From the definition of \( I \) given by (1.11), we obtain

\[
I(z) = z + \sum_{n=2}^{\infty} \frac{(1+\mu n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n \quad (\mu \geq 0, \ z \in U).
\]

(2.10)

Theorem 2.8. Let \( a > 1 \), \( b > 1 \), and \( c > a + b \). If \( f \in R_\eta(\beta) \) and the inequality

\[
(1+B - (A+1)\mu)z_2 F_1(a,b,c;1) - (A+1)(1-\mu)z_3 F_3(a,b,1,1,c,2,2;1) \leq (B-A) \left( \frac{1}{2(1-\beta)\cos\eta} + 1 \right)
\]

(2.11)

is satisfied, then \( I \ast f \in \varphi^*(A,B) \).

Proof. By Lemma 2.1, it satisfies to show that

\[
T_2 := \sum_{n=2}^{\infty} (1+B)n - (A+1) \left| \frac{(1-\mu n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} a_n \right| \leq B - A.
\]

(2.12)
Suppose that \( f \in R_\eta(\beta) \). Then by (1.5) we observe that
\[
T_2 \leq \sum_{n=2}^{\infty} \frac{(1 + B)n - (A + 1)}{(1 + B)(1 - \mu) - (A + 1)\mu} \frac{(1 - \mu + n\mu)(a)^{n-1}(b)^{n-1}}{(c)^{n-1}(1)^n} \frac{2(1 - \beta)\cos \eta}{n}
\]
\[
= 2(1 - \beta)\cos \eta \left\{ \left( (1 + B)(1 - \mu) - (A + 1)\mu \right) \sum_{n=2}^{\infty} \frac{(a)^{n-1}(b)^{n-1}}{(c)^{n-1}(1)^n} \right\}
\]
\[
= 2(1 - \beta)\cos \eta \left\{ \left( (1 + B)(1 - \mu) - (A + 1)\mu \right) \left( \frac{c - 1}{(a-1)(b-1)} + {}_2F_1(a, b, c; 1) \right) - (A + 1)(1 - \mu) {}_4F_3(a, b, 1, 1, c, 2, 2; 1) + (1 + B)\mu {}_2F_1(a, b, c; 1) \right\}
\]
\[
\leq B - A
\]
by (2.11). This completes the proof. \( \square \)

Taking \( \mu = 0, A = 2\alpha - 1, \) and \( B = 1 \) in Theorem 2.8, we see the following corollary.

**Corollary 2.9.** Let \( a > 1, b > 1, \) and \( c > a + b. \) If \( f \in R_\eta(\beta) \) and the inequality
\[
_2F_1(a, b, c; 1) - \alpha {}_4F_3(a, b, 1, 1, c, 2, 2; 1) \leq (1 - \alpha) \left( \frac{1}{2(1 - \beta)\cos \eta} + 1 \right)
\]
(2.14)
is satisfied, then \( \int_0^\alpha {}_2F_1(a, b, c; t) dt * f \in S^*(\alpha). \)

Taking \( \alpha = 0, \beta = 0, \) and \( \eta = 0 \) in Corollary 2.9, we get the following corollary.

**Corollary 2.10.** Let \( a > 1, b > 1, \) and \( c > a + b. \) If \( f \in S \) and the inequality
\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \leq \frac{3}{2}
\]
(2.15)
is satisfied, then \( \int_0^\alpha {}_2F_1(a, b, c; t) dt * f \in S^*. \)
**Theorem 2.11.** Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in R_\eta(\beta) \) and the inequality
\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1+B) \left( 1 + \frac{\mu ab}{c-a-b-1} \right) 
+ (A+1) \left( \mu \frac{c-a-b}{(a-1)(b-1)} - \frac{c-a-b}{(a-1)(b-1)} \right) \right]
\leq (B-A) \left( \frac{1}{2(1-\beta)\cos \eta + 1} \right) - \frac{(1-\mu)(A+1)(c-1)}{(a-1)(b-1)}
\]
(2.16)
is satisfied, then \( I^* f \in K(A,B) \).

**Proof.** The proof follows from Lemma 2.1 and by applying similar method as in the proof of Theorem 2.8; we omit the details involved.

If we let \( \mu = 0, \ A = 2\alpha - 1, \) and \( B = 1 \) in Theorem 2.11, we get the following corollary.

**Corollary 2.12.** Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in R_\eta(\beta) \) and the inequality
\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right]
\leq (1-\alpha) \left( \frac{1}{2(1-\beta)\cos \eta + 1} \right) - \frac{\alpha(c-1)}{(a-1)(b-1)}
\]
(2.17)
is satisfied, then \( \int_0^z \! F_1(a,b,c;\ t) \! dt \ast f \in K(\alpha) \).

If we let \( \alpha = 0, \ \beta = 0, \) and \( \eta = 0 \) in Corollary 2.12, we have the following corollary.

**Corollary 2.13.** Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in S \) and the inequality
\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2}
\]
(2.18)
is satisfied, then \( \int_0^z \! F_1(a,b,c;\ t) \! dt \ast f \in K \).

**Acknowledgment.** This work was supported by the Korea Science and Engineering Foundation (KOSEF), project no. 2000-6-101-01-2.

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