RIESZ BASES AND POSITIVE OPERATORS ON HILBERT SPACE

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It is shown that a normalized Riesz basis for a Hilbert space H (i.e., the isomorphic image of an orthonormal basis in H) induces in a natural way a new, but equivalent, inner product on H in which it is an orthonormal basis, thereby extending the sense in which Riesz bases and orthonormal bases are thought of as being the *same*. A consequence of the method of proof of this result yields a series representation for all positive isomorphisms on a Hilbert space.

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1. Introduction. Let H denote a Hilbert space (assumed real, for notational convenience) with inner product $(\cdot, \cdot)$ and let $\{x_i\}$ be a basis for H having coefficient functionals $\{f_i\}$ denoted by $\{x_i, f_i\}$. We say that $\{x_i, f_i\}$ is a *Riesz basis* for H if it has the property that $\sum a_i x_i$ converges in H if and only if $\{a_i\}$ is in the sequence space l^2 . Equivalently, $\{x_i, f_i\}$ is a Riesz basis for H if and only if there is an isomorphism U on H and some orthonormal basis $\{\phi_i\}$ for H so that $U\phi_i = x_i$ for all i , implying that Riesz bases and orthonormal bases are the “same” in linear-topological terms, but differ in geometrical ones due to the additional orthogonality relations between basis vectors in an orthonormal basis that is lacking in a Riesz basis. The result below ([Theorem 2.1](#)) shows that this is, in a sense, an artificial distinction by showing that every Riesz basis, in fact, is an orthonormal basis for H under a different, but equivalent, inner product.

2. Main results

THEOREM 2.1. *Let $\{x_i, f_i\}$ be a normalized Riesz basis for a Hilbert space H . Then there is an equivalent inner product on H in which $\{x_i\}$ is an orthonormal basis for H under the norm induced by this inner product.*

PROOF. If x and y are any two vectors in H , then the sequences $\{(f_i, x)\}$ and $\{(f_i, y)\}$ are in l^2 , implying that $\sum (f_i, x)(f_i, y)$ converges. Clearly, the bilinear form on $H \times H$, defined by $\langle x, y \rangle = \sum (f_i, x)(f_i, y)$, is then an inner product on H for which $\langle x_i, x_j \rangle = \delta_{ij}$ for all i and j , in which $\{x_i\}$ is an orthonormal set that is also complete, since if $\langle x_n, x \rangle = 0$ for all n , then $0 = \sum (f_i, x_n)(f_i, x) = (f_n, x)$ for all n ; that is, $0 = \sum (f_i, x_n)(f_i, x)$ by definition of the new inner product for all n , implying that $(f_n, x) = 0$ for all n , and hence that $x = 0$.

As usual, the inner product $\langle \cdot, \cdot \rangle$ defines a norm $\| \cdot \|_1$ on H by $\|x\|_1^2 = \langle x, x \rangle = \sum |(f_i, x)|^2$. Since $\{x_i\}$ is a Riesz basis, there is an isomorphism U on H that maps each vector ϕ_i in an orthonormal basis $\{\phi_i\}$ for H to the vector x_i , implying that the isomorphism $V = (U^*)^{-1}U^{-1}$ on H maps x_i to f_i for all i . Since, for any x in H , $\langle x, x \rangle = \sum (f_i, x)(f_i, x) = \sum (f_i, x)(Vx_i, x) = \sum (f_i, x)(Vx_i, x) = (V[\sum (f_i, x)x_i], x) = (Vx, x)$, we see that $(Vx, x) = \sum |(f_i, x)|^2 = \|x\|_1^2$ for all x in H , so V is a positive operator. If we let W denote the positive square root of V , then W is also an isomorphism on H so that, for any x in H , we have $\|x\|_1^2 = (Vx, x) = (Wx, Wx) = \|Wx\|^2 \leq \|W\|^2 \|x\|^2$. In the same way, we see that $\|x\|_1^2 \leq \|W^{-1}\|^2 \|x\|^2$, and it follows that the new norm $\| \cdot \|_1$ is equivalent to the original norm on H . In particular, H is then complete under the new norm, hence a Hilbert space, in which $\{x_i\}$ is then an orthonormal basis, being an orthonormal set, that is complete in the new inner product. $\square$

3. Positive operators. In the proof above we used the fact that if $\{x_i, f_i\}$ is a Riesz basis for a Hilbert space H , then the operator U on H , mapping x_i to f_i , is a positive isomorphism on H . It is interesting to note that, in fact, every positive isomorphism on H is such an operator for some Riesz basis in H , thereby providing a representation for all positive isomorphisms U on a Hilbert space.

THEOREM 3.1. *An operator U on a Hilbert space on H is a positive isomorphism if and only if U is of the form $U = \sum f_i \otimes f_i$ for some Riesz basis $\{x_i, f_i\}$ for H (i.e., $Ux_i = f_i$ for all i).*

PROOF. If $U = \sum f_i \otimes f_i$ for some Riesz basis $\{x_i, f_i\}$ for H , $\{\phi_i\}$ is an orthonormal basis for H , and T is the isomorphism on H mapping ϕ_i to f_i for all i , then $U = \sum T\phi_i \otimes T\phi_i = TT^*$, a positive isomorphism on H .

Conversely, if U is any positive isomorphism on H , then W , the positive square root of U , is also an isomorphism on H . If we set $f_i = W\phi_i$ for some orthonormal basis $\{\phi_i\}$, then $\{f_i\}$ is a Riesz basis for H so that, for any x in H , we have $Ux = W^2x = W[\sum (\phi_i, Wx)\phi_i] = W[\sum (W\phi_i, x)\phi_i] = \sum (f_i, x)W\phi_i = \sum (f_i, x)f_i$. That is, $U = \sum_i f_i \otimes f_i$ and the proof is complete. $\square$

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