A general procedural lemma for fixed-point theorems for three and four maps in a $D$-metric space is proved, and it is further applied for proving the common fixed-point theorems of three and four maps in a $D$-metric space satisfying certain contractive conditions.

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1. Introduction. The concept of a $D$-metric space is introduced by the first author in [2]. A nonempty set $X$, together with a function $\rho : X \times X \times X \to [0, \infty)$, is called a $D$-metric space with $D$-metric $\rho$ if $\rho$ satisfies the following properties:

(i) $\rho(x, y, z) = 0 \iff x = y = z$ (coincidence),
(ii) $\rho(x, y, z) = \rho(p\{x, z, y\})$ (symmetry), where $p$ is a permutation function,
(iii) $\rho(x, y, z) \leq \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

A few details along with some specific examples of a $D$-metric space appear in [3]. A sequence $\{x_n\} \subset X$ is said to be convergent to a point $x \in X$ if

$$\lim_{m,n \to \infty} \rho(x_m, x_n, x) = 0. \quad (1.1)$$

Similarly, a sequence $\{x_n\} \subset X$ is called $D$-Cauchy if

$$\lim_{m,n,p \to \infty} \rho(x_m, x_n, x_p) = 0. \quad (1.2)$$

A complete $D$-metric space is one in which every $D$-Cauchy sequence converges to a point. Further, a subset $S$ of a $D$-metric space $X$ is called bounded if there exists a constant $k > 0$ such that $\rho(x, y, z) \leq k$ for all $x, y, z \in S$, and the constant $k$ is called a $D$-bound of $S$. The infimum of all such $D$-bounds $k$ is called the diameter of $S$, and it is denoted by $\delta(S)$. Finally, it is known that a
mapping \( f : X \to X \) is continuous if and only if, for any sequence \( \{x_n\} \subset X \), \( x_n \to x^* \) implies \( fx_n \to fx^* \).

It has been shown in [5] that the D-metric \( \rho \) is continuous on \( X^3 \) in the topology of D-metric convergence which is Hausdorff. For details of a D-metric space, the reader is referred to Dhage [5].

The general existence principles for the fixed-point theorems for a single and a pair of maps in D-metric space have been established in Dhage and Rhoades [6] and Dhage [1], respectively. However, the extension of these existence principles to three or four maps is not possible.

In this paper, we just formulate the general procedure for common fixed-point theorems for more than two selfmaps of a D-metric space and discuss some of its applications.

We need the following auxiliary results in the sequence.

**Proposition 1.1.** Let \( \{x_n\} \) be a sequence in a D-metric space \( X \) satisfying

\[
\rho(x_n, x_{n+1}, z) \leq \lambda \rho(x_{n-1}, x_n, z)
\]

for all \( n \in \mathbb{N} \) and \( z \in \{x_n\} \), where \( 0 \leq \lambda < 1 \). Then,

\[
\rho(x_n, x_{n+1}, x_m) \leq \lambda^n k
\]

for all \( m > n \), where \( k = \left(\frac{2}{1-\lambda}\right) \max\{\rho(x_0, x_0, x_1), \rho(x_0, x_1, x_1)\} \).

**Proof.** From (1.3),

\[
\rho(x_n, x_{n+1}, x_m) \leq \lambda \rho(x_{n-1}, x_n, x_m)
\]

for each \( m > n \in \mathbb{N} \). By induction,

\[
\rho(x_n, x_{n+1}, x_m) \leq \lambda^n \rho(x_0, x_1, x_m).
\]

Let \( q = \max\{\rho(x_0, x_0, x_1), \rho(x_0, x_1, x_1)\} \). Using the tetrahedral inequality,

\[
\begin{align*}
\rho(x_0, x_1, x_m) & \leq \rho(x_0, x_1, x_{m-1}) + \rho(x_{m-1}, x_0, x_m) + \rho(x_{m-1}, x_m, x_1) \\
& = \rho(x_0, x_1, x_{m-1}) + \rho(x_{m-1}, x_m, x_0) + \rho(x_{m-1}, x_m, x_1) \\
& \leq \rho(x_0, x_1, x_{m-1}) + \lambda^{m-1} \rho(x_0, x_1, x_0) + \lambda^{m-1} \rho(x_0, x_1, x_1) \\
& \leq \rho(x_0, x_1, x_{m-1}) + 2\lambda^{m-1} q \\
& \leq \rho(x_0, x_1, x_{m-2}) + 2\lambda^{m-1} q + 2\lambda^{m-2} q \\
& \vdots \\
& \leq \rho(x_0, x_1, x_1) + 2(\lambda + \lambda^2 + \cdots + \lambda^{m-1}) q \\
& \leq 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{m-1}) q < \frac{2}{1-\lambda} q = k.
\end{align*}
\]

Substituting (1.7) into (1.6) yields the desired inequality (1.4). \( \square \)
**Proposition 1.2.** Every sequence \( \{x_n\} \subset X \) satisfying (1.3) is bounded with a \( D \)-bound \( k = (2/(1 - \lambda)) \max \{\rho(x_0,x_0,x_1),\rho(x_0,x_1,x_1)\} \).

**Proof.** Let \( q = \max \{\rho(x_0,x_0,x_1),\rho(x_0,x_1,x_1)\} \). Then, for any integers \( r \geq s \geq n \), there exists positive integer \( p \) and \( t \) such that
\[
\rho(x_n,x_r,x_s) = \rho(x_n,x_{n+p},x_{n+t}) \\
\leq \rho(x_n,x_{n+1},x_{n+t}) + \rho(x_n,x_{n+1},x_{n+p}) + \rho(x_{n+1},x_{n+p},x_{n+t}) \\
\leq 2\lambda^n q + \rho(x_{n+1},x_{n+p},x_{n+q}) \\
\leq 2\lambda^n q + 2\lambda^{n+1} q + \rho(x_{n+2},x_{n+p},x_{n+t})
\]
\[
\vdots
\]
\[
\leq 2 \left( \sum_{j=1}^{n-p-2} \lambda^j \right) q + \rho(x_{n+1},x_{n+p},x_{n+r}) \\
\leq 2 \left( \sum_{j=1}^{n+p-1} \lambda^j \right) q < 2 \left( \sum_{j=1}^{\infty} \lambda^j \right) q < \frac{2}{1-\lambda} q = k.
\]
(1.8)

Then, \( \{x_n\} \) is bounded and the proof is complete. \( \square \)

2. **Main results.** Before going to the main results of this paper, we state a lemma proved in Dhage [4].

**Lemma 2.1** \((D\text{-Cauchy principle})\). Let \( \{y_n\} \) be a bounded sequence in \( D \)-metric space with \( D \)-bound \( k \) satisfying
\[
\rho(y_n,y_{n+1},y_m) \leq \lambda^n k \quad (2.1)
\]
for all \( m > n \in \mathbb{N} \). Then, \( \{y_n\} \) is \( D \)-Cauchy.

Let \( A,B,S,T : X \to X \) be four maps such that
\[
A(X) \subseteq T(X), \quad B(X) \subseteq S(X). \quad (2.2)
\]
Condition (2.2) ensures that it is possible to define a sequence \( \{y_n\} \) in \( X \) as follows. Let \( x \in X \) be arbitrary. Then, in view of condition (2.2), there exists a sequence \( \{x_n\} \) such that
\[
x_0 = x, \quad Ax_{2n} = Tx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2}, \quad n \geq 0. \quad (2.3)
\]
Now, define \( \{y_n\} \) in \( X \) by
\[
y_0 = Sx_0, \quad y_{2n+1} = Tx_{2n+1}, \quad y_{2n+2} = Sx_{2n+2}, \quad n \geq 0. \quad (2.4)
\]
A point \( x \in X \) is called a coincidence point of two maps \( A,B : X \to X \) if \( Ax = Bx \), and in this case, the mappings \( A \) and \( B \) are called coincident on \( X \).
Similarly, a coincidence point of three or four maps on a $D$-metric space is defined.

**Lemma 2.2.** Let $A, B, S, T : X \rightarrow X$ satisfy (2.2), and let $\{y_n\} \subset X$ be defined by (2.4). Further, assume that $\{y_n\}$ is complete. Suppose that there exists a $\lambda \in [0, 1)$ such that

$$
\rho(y_n, y_{n+1}, z) \leq \lambda \rho(y_{n-1}, y_n, z) \tag{2.5}
$$

for all $n \in \mathbb{N}$ and $z \in \{y_n\}$. Then, either

(a) $A$ and $S$ have a coincidence point,

(b) $B$ and $T$ have a coincidence point,

(c) $A, S$, and $T$ have a coincidence point,

(d) $B, S$, and $T$ have a coincidence point, or

(e) $\{y_n\}$ converges to a point $u \in X$ and, for all $m > n \in \mathbb{N},$

$$
\rho(y_n, y_m, u) \leq 2 \sum_{j=n}^{m} \lambda^j k, \quad \rho(y_n, u, u) \leq 2 \frac{\lambda^n}{1-\lambda} k, \tag{2.6}
$$

where $k = \delta(\{y_n\}).$

**Proof.** Suppose that $y_{2n} = y_{2n+1}$ for some $n$. Then, $Sx_{2n} = Tx_{2n+1} = Ax_{2n}$ and (a) holds. Also, if $x_{2n} = x_{2n+1}$, then $Tx_{2n} = Tx_{2n+1}$ and so (c) holds. Similarly, if $y_{2n+1} = y_{2n+2}$ for some $n$, then it is shown analogously that (b) and (d) hold.

Suppose now that $y_n \neq y_{n+1}$ for each $n$. Then, from Proposition 1.1, it follows that

$$
\rho(y_n, y_{n+1}, y_m) \leq \lambda^n \rho(y_0, y_1, y_m) \leq \lambda^n k \tag{2.7}
$$

for all $m > n \in \mathbb{N}$. Now, an application of Lemma 2.1 yields that $\{y_n\}$ is $D$-Cauchy. Since $\{y_n\}$ is complete, there exists a point $u \in X$ such that $\lim_n y_n = u$.

Now, for any positive integers $m$ and $n$, $m > n$, by repeated application of the tetrahedral inequality,

$$
\rho(y_n, y_m, u) \leq \rho(y_n, y_{n+1}, y_m) + \rho(y_{n+1}, y_{n+1}, y_m) + \rho(y_{n+1}, y_{n+1}, u) + \rho(y_{n+1}, y_m, u) \\
\leq \lambda^n \rho(y_0, y_{n+1}, y_m) + \lambda^n \rho(y_0, y_1, u) + \rho(y_{n+1}, y_m, u) \\
\leq 2 \lambda^n k + \rho(y_{n+1}, y_m, u) \\
\leq 2 \lambda^n k + 2 \lambda^{n+1} k + \rho(y_{n+2}, y_m, u) \\
\vdots \\
\leq 2 (\lambda^n + \lambda^{n+1} + \cdots + \lambda^m) k \\
= 2 \sum_{j=n}^{m} \lambda^j k. \tag{2.8}
$$
The above inequality further gives that
\[
\rho(y_n, y_m, u) \leq 2 \sum_{j=n}^{m} \lambda^j k
= 2\lambda^n (1 + \lambda + \cdots + \lambda^{m-n}) k
= 2\lambda^n \left( \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) k.
\] (2.9)

Taking the limit as \( m \to \infty \) in the above inequality,
\[
\rho(y_n, u, u) \leq 2 \frac{\lambda^n}{1 - \lambda} k.
\] (2.10)

The proof of Lemma 2.2 is complete. \( \square \)

The three-maps version of Lemma 2.2 is obtained in two ways: one by setting \( S = T \) and the other by setting \( A = B \). In the situation when \( S = T \), condition (2.2) reduces to
\[
A(X) \subseteq S(X), \quad B(X) \subseteq S(X).
\] (2.11)

Then, it is possible to choose a sequence \( \{x_n\} \subset X \) such that
\[
x_0 = x, \quad Ax_{2n} = Sx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2}, \quad n \geq 0.
\] (2.12)

Now, define a sequence \( \{y_n\} \) in \( X \) as follows:
\[
y_0 = Sx_0, \quad y_{2n} = Sx_{2n}, \quad y_{2n+1} = Sx_{2n+1}, \quad n \in \mathbb{N}.
\] (2.13)

**Lemma 2.3.** Let \( A, B, S : X \to X \) satisfy (2.11). Suppose that there exists an \( x \in X \) such that the sequence \( \{y_n\} \subset X \) defined by (2.13) is complete. Further, suppose that
\[
\rho(y_n, y_{n+1}, z) \leq \lambda \rho(y_{n-1}, y_n, z)
\] (2.14)

for all \( n \in \mathbb{N} \) and \( z \in \{y_n\} \), where \( 0 \leq \lambda < 1 \). Then, either
(a) \( A \) and \( S \) have a coincidence point,
(b) \( B \) and \( S \) have a coincidence point, or
(c) \( \{y_n\} \) converges to a point \( u \in X \) and, for all positive integers \( m \) and \( n \), \( m > n \),
\[
\rho(y_n, y_m, u) \leq 2 \sum_{j=n}^{m} \lambda^j k, \quad \rho(y_n, u, u) \leq 2 \frac{\lambda^n}{1 - \lambda} k,
\] (2.15)

where \( k = \delta(\{y_n\}) \).

It is known that the fixed-point theorems for more than two maps require some sort of commutativity condition on the mappings under consideration.
Below, we will apply Lemma 2.2 for proving the common fixed-point theorem for four maps on a $D$-metric space under a suitable commutativity condition.

A sequence $\{x_n\} \subset X$ is called a sequence of coincidence for the maps $A, B : X \to X$ if $\lim_n Ax_n = \lim_n Bx_n$. In this case, the mappings $A$ and $B$ are called limit coincident on $X$. Similarly, two maps $A, B : X \to X$ are called commuting or commutative if $(AB)(x) = (BA)(x)$ for all $x \in X$ and limit commuting if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_n (AB)(x_n) = \lim_n (BA)(x_n). \quad (2.16)$$

Finally, two maps $A, B : X \to X$ are called limit coincidentally commuting if their limit coincidence implies the limit commuting on $X$, that is, for any sequence $\{x_n\} \subset X$ if

$$\lim_n Ax_n = \lim_n Bx_n \implies \lim_n (AB)(x_n) = \lim_n (BA)(x_n). \quad (2.17)$$

It is known that the limit coincidentally commuting mappings commute at their coincidence points. See, for details, Dhage [4].

Now, we are ready to give some applications of Lemma 2.2 for proving the existence of a common fixed point of four maps on a $D$-metric space $X$.

An orbit of four selfmaps $A, B, S, T$ of a $D$-metric space $X$ at a point $x \in X$ is a set $O_{A,B}(S,T : x)$ in $X$ defined by

$$O_{A,B}(S,T : x) = \{y_0 = Sx_0, y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} : n \geq 0\}. \quad (2.18)$$

Clearly, the orbit $O_{A,B}(S,T : x)$ is well defined if $A, B, S,$ and $T$ satisfy condition (2.2). By $\overline{O_{A,B}(S,T : x)}$, we denote the closure of the orbit $O_{A,B}(S,T : x)$ in $X$.

**Theorem 2.4.** Let $A, B, S, T : X \to X$ be four selfmaps of a $D$-metric space $X$ satisfying (2.2) and

$$\rho(Ax,By,z) \leq \lambda \max \{\rho(Sx,Ty,z),\rho(Sx,Ax,z),\rho(Ty,By,z)\} \quad (2.19)$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$. Assume further that

(a) $\overline{O_{A,B}(S,T : x)}$ is complete for each $x \in X$,
(b) $\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting,
(c) any one $A, B, S,$ or $T$ continuous.

Then, $A, B, S,$ and $T$ have a unique common fixed point.

**Proof.** Let $x \in X$ be arbitrary, and define a sequence $\{y_n\} \subset X$ by (2.4), which is possible in view of condition (2.4). Now, taking $x = x_{2n}$ and $y = x_{2n+1}$
in (2.19),
\[ \rho(y_{2n+1}, y_{2n+2}, z) \leq \lambda \max \{ \rho(y_{2n}, y_{2n+1}, z), \rho(y_{2n}, y_{2n+2}, z) \} \]
\[ = \lambda \rho(y_{2n}, y_{2n+1}, z) \]  
(2.20)

for all \( n \geq 0 \) and \( z \in \{ y_n \} \). Similarly, taking \( x = x_{2n} \) and \( y = x_{2n-1} \) in (2.19),
\[ \rho(y_{2n}, y_{2n+1}, z) \leq \lambda \max \{ \rho(y_{2n-1}, y_{2n}, z), \rho(y_{2n-1}, y_{2n}, z) \} \]
\[ = \lambda \rho(y_{2n-1}, y_{2n}, z) \]  
(2.21)

for all \( n \in \mathbb{N} \) and \( z \in \{ y_n \} \). Hence, in general,
\[ \rho(y_n, y_{n+1}, y_m) \leq \lambda \rho(y_{n-1}, y_n, y_m) \]  
(2.22)

for all \( m > n \in \mathbb{N} \) and \( 0 \leq \lambda < 1 \).

We prove the conclusion of our theorem in two cases.

**Case 1.** If \( y_n = y_{n+1} \), then \( y_n = y_{n+k} \) for all \( k \geq 0 \). If \( y_{n+1} \neq y_{n+2} \), then, replacing \( n \) in (2.22) by \( n+1 \),
\[ 0 < \rho(y_{n+1}, y_{n+2}, y_{n+1}) \leq \lambda \rho(y_{n}, y_{n+1}, y_{n+1}) = 0, \]  
(2.23)

which is a contradiction and \( y_{n+1} = y_{n+2} \), and, by induction, \( y_n = y_{n+k} \) for all \( k \geq 0 \). Therefore, by Lemma 2.2, there are points \( u \) and \( v \) in \( X \) such that \( w_1 = Au = Su \) and \( w_2 = Bv = Tv \).

We will show that \( w_1 = w_2 \). By (2.19),
\[ \rho(w_1, w_2, w_1) = \rho(Au, Bv, w_1) \]
\[ \leq \lambda \max \{ \rho(Su, Tv, w_1), \rho(Su, Au, w_1), \rho(Tv, Bv, w_1) \} \]
\[ = \lambda \max \{ \rho(w_1, w_2, w_1), \rho(w_1, w_1, w_1), \rho(w_2, w_2, w_1) \} \]
\[ = \lambda \rho(w_1, w_2, w_2). \]  
(2.24)

Again,
\[ \rho(w_1, w_2, w_2) = \rho(Au, Bv, w_2) \]
\[ \leq \lambda \max \{ \rho(Su, Tv, w_2), \rho(Su, Au, w_2), \rho(Tv, Bv, w_2) \} \]
\[ = \lambda \max \{ \rho(w_1, w_2, w_2), \rho(w_1, w_1, w_2), \rho(w_2, w_2, w_2) \} \]
\[ = \lambda \rho(w_1, w_2, w_1). \]  
(2.25)

Substituting (2.25) into (2.24),
\[ \rho(w_1, w_2, w_1) \leq \lambda^2 \rho(w_1, w_2, w_1), \]  
(2.26)
which is possible only when \( w_1 = w_2 \) since \( \lambda < 1 \). Hence, \( Au = Bu = Su = Tv = w \). Next, we show that \( w \) is a coincidence of \( A, B, S, \) and \( T \). Since \( \{ A, S \} \) and \( \{ B, T \} \) are limit coincidentally commuting, they commute at coincidence point. Therefore, \( Sw = SAu = ASu = Aw \) and \( Tw = TBu = BTu = Bw \). Now,

\[
\rho(Aw, Bw, Aw) \leq \lambda \max \{ \rho(Sw, Tw, Aw), \rho(Sw, Aw, Aw), \rho(Tw, Bw, Aw) \} \\
= \lambda \max \{ \rho(Aw, Bw, Aw), \rho(Bw, Bw, Aw) \} \\
= \lambda \rho(Aw, Bw, Bw). 
\]

(2.27)

Similarly,

\[
\rho(Aw, Bw, Bw) \leq \lambda \rho(Aw, Bw, Aw). 
\]

(2.28)

Substituting (2.28) into (2.27),

\[
\rho(Aw, Bw, Aw) \leq \lambda^2 \rho(Aw, Bw, Aw), 
\]

(2.29)

which is possible only when \( Aw = Bw \). Hence, \( Aw = Sw = Tw = Bw \) is a coincidence point of the four maps \( A, B, S, \) and \( T \). Finally, we prove that \( w \) is a common fixed point of \( A, B, S, \) and \( T \). If \( w \neq Aw \), then, by (2.19),

\[
\rho(Aw, w, w) = \rho(Aw, Bw, w) \\
\leq \lambda \max \{ \rho(Sw, Tw, w), \rho(Sw, Aw, w), \rho(Tw, Bw, w) \} \\
= \lambda \rho(Aw, w, Aw). 
\]

(2.30)

Similarly,

\[
\rho(Aw, w, Aw) \leq \lambda \rho(Aw, w, w). 
\]

(2.31)

From (2.30) and (2.31),

\[
\rho(Aw, w, w) \leq \lambda^2 \rho(Aw, w, w), 
\]

(2.32)

which is a contradiction to \( Aw = w \) and hence, \( w = Aw = Sw = Tw = Bw \).

**Case 2.** Suppose that \( y_n \neq y_{n+1} \) for each \( n \). Then, by Lemma 2.2, there exists a point \( w \in X \) such that \( \lim_n y_n = w \). By definition of \( \{ y_n \} \),

\[
\lim_n y_{2n} = \lim_n Sx_{2n} = \lim_n Ax_{2n} = \lim_n Tx_{2n+1} = w, \\
\lim_n y_{2n+1} = \lim_n Tx_{2n+1} = \lim_n Bx_{2n+1} = \lim_n y_{2n+2} = w. 
\]

(2.33)

Since \( \{ A, S \} \) and \( \{ B, T \} \) are limit coincidentally commuting, then

\[
\lim_n ASx_{2n} = \lim_n SAx_{2n}, \\
\lim_n BTx_{2n+1} = \lim_n TBx_{2n+1}. 
\]

(2.34)
Suppose first that $S$ is continuous on $X$. Then,

$$\lim_n SSx_{2n} = \lim_n SAx_{2n} = \lim_n ASx_{2n} = Sw. \quad (2.35)$$

First, we show that $w$ is a fixed point of $S$. If $w \neq Sw$, then, by (2.19),

$$\rho(Sw,w,w) = \lim_n \rho(ASx_{2n},Bx_{2n+1},w) \leq \lambda \max_n \{\rho(SSx_{2n},Tx_{2n+1},w),\rho(SBx_{2n},ASx_{2n},w),\rho(Tx_{2n+1},Bx_{2n+1},w)\}$$

$$= \lambda \max \{\rho(Sw,w,w),\rho(Sw,Sw,w)\} = \lambda \rho(Sw,Sw,w). \quad (2.36)$$

Similarly,

$$\rho(Sw,w,Sw) \leq \lambda \rho(Sw,w,w). \quad (2.37)$$

Substituting (2.37) into (2.36),

$$\rho(Sw,w,w) \leq \lambda^2 \rho(Sw,w,w), \quad (2.38)$$

which is a contradiction and hence, $Sw = w$. Similarly,

$$\rho(Aw,w,w) \leq \lambda \lim_n \rho(Aw,Bx_{2n+1},w) \leq \lambda \max_n \{\rho(Sw,Tx_{2n+1},w),\rho(Tx_{2n+1},Bx_{2n+1},w),\rho(Sw,Aw,w)\}$$

$$= \lambda \max \{0,0,\rho(Aw,w,w)\} = \lambda \rho(Aw,w,w), \quad (2.39)$$

which implies that $Aw = w$, since $\lambda < 1$. From the condition $A(X) \subseteq T(X)$, it follows that there is a point $p \in X$ such that $w = Aw = Tp$. We show that $Bp = Tp$. If not, then

$$\rho(Tp,Bp,w) = \rho(Aw,Bp,w) \leq \lambda \max \{\rho(Sw,Tp,w),\rho(Sw,Aw,w),\rho(Tp,Bp,w)\} = \lambda \rho(Tp,Bp,w), \quad (2.40)$$

which is a contradiction. Hence, $Bp = Tp$. Since $\{B,T\}$ are limit coincidentally commuting, we obtain $Bw = BTp = TBP = Bw$. Now,

$$\rho(Aw,Bw,w) \leq \lambda \max \{\rho(Sw,Tw,w),\rho(Sw,Aw,w),\rho(Tw,Bw,w)\} = \lambda \rho(Aw,Bw,Bw). \quad (2.41)$$
Similarly,

\[ \rho(Aw, Bw, Bw) \leq \lambda \rho(Aw, Bw, w). \tag{2.42} \]

Substituting (2.42) into (2.41),

\[ \rho(Aw, Bw, w) \leq \lambda^2 \rho(Aw, Bw, w), \tag{2.43} \]

which is possible only when \( Aw = Bw \). Thus, \( w \) is a common fixed point of \( A, B, S, \) and \( T \).

Similarly, if \( T \) is continuous, then it is proved in an analogous way that \( A, B, S, \) and \( T \) have a common fixed point.

Next, suppose that \( A \) is continuous. Then, we have

\[ \lim_n AAx_{2n} = \lim_n ASx_{2n} = \lim_n SAx_{2n} = Aw. \tag{2.44} \]

First, we show that \( Aw = w \). If \( Aw \neq w \), then

\[
\rho(Aw, w, w) = \lim_n \rho(AAx_{2n}, Bx_{2n+1}, w) \\
\leq \lambda \lim_n \max \{ \rho(SAx_{2n}, Tx_{2n+1}, w), \rho(SAx_{2n}, AAx_{2n}, w), \rho(Tx_{2n+1}, Bx_{2n+1}, w) \} \\
= \lambda \max \{ \rho(Aw, w, w), \rho(Aw, Aw, w) \} \\
= \lambda \rho(Aw, Aw, w). \tag{2.45}
\]

Similarly,

\[ \rho(Aw, w, Aw) \leq \lambda \rho(Aw, w, w). \tag{2.46} \]

Substituting (2.46) into (2.45),

\[ \rho(Aw, w, w) \leq \lambda^2 \rho(Aw, w, w), \tag{2.47} \]

which is a contradiction. Hence, \( Aw = w \). Using condition (2.2), there exists a point \( p \in X \) such that \(Tp = Aw = w \). We show that \( Bp = Tp \). Now,

\[
\rho(Aw, Bp, w) = \lim_n \rho(AAx_{2n}, Bp, w) \\
\leq \lambda \lim_n \max \{ \rho(SAx_{2n}, Tp, w), \rho(SAx_{2n}, AAx_{2n}, w), \rho(Tp, Bp, w) \} \\
= \lambda \rho(w, Bp, w), \tag{2.48}
\]
which gives that $Bp = Tp$. Since $\{B,T\}$ are limit coincidentally commuting, they commute at coincidence point. Hence, $Tw = TBp = BTp = Bw$. Now,

$$\rho(Ax_{2n},Bw,w) \leq \lambda \max \{\rho(Sx_{2n},Tw,w),\rho(Sx_{2n},Ax_{2n},w),\rho(Tw,Bw,w)\}. \quad (2.49)$$

Taking the limit as $n \to \infty$,

$$\rho(w,Bw,w) \leq \lambda \rho(w,Bw,w). \quad (2.50)$$

Similarly,

$$\rho(w,Bw,Bw) \leq \lambda \rho(w,Bw,w). \quad (2.51)$$

Substituting (2.51) into (2.50),

$$\rho(w,Bw,w) \leq \lambda^2 \rho(w,Bw,w), \quad (2.52)$$

which implies that $w = Bw = Tw = Aw$. Since $B(X) \subseteq S(X)$, there is a point $q \in X$ such that $w = Bw = Sq$. We show that $Aq = Sq$. Now,

$$\rho(Aq,Sq,w) = \rho(Aq,Bw,w)$$
$$\leq \lambda \max \{\rho(Sq,Tw,w),\rho(Sq,Aq,w),\rho(Tw,Bw,w)\}$$
$$= \lambda \rho(Sq,Aq,w), \quad (2.53)$$

which implies that $Aq = Sq$. Since $\{A,S\}$ are limit coincidentally commuting $Sw = SAq = ASq = Aw = w$. Thus, $Aw = Sw = Tw = Bw$, that is, $w$ is a common fixed point of $A, B, S, \text{and } T$. Similarly, if $B$ is continuous, it is proved that $A, B, S, \text{and } T$ have a common fixed point.

To prove the uniqueness, let $w^*(\neq w)$ be common fixed point of $A, B, S, \text{and } T$. Then,

$$\rho(w,w^*,w^*) = \rho(Aw,Bw^*,w^*)$$
$$\leq \lambda \max \{\rho(Sw,Tw^*,w^*),\rho(Sw,Aw,w^*),\rho(Tw^*,Bw^*,w^*)\}$$
$$= \lambda \rho(w,w^*,w^*). \quad (2.54)$$

Similarly,

$$\rho(w,w,w^*) \leq \lambda \rho(w,w^*,w^*). \quad (2.55)$$

Substituting (2.54) into (2.55),

$$\rho(w,w,w^*) \leq \lambda^2 \rho(w,w,w^*), \quad (2.56)$$

which is a contradiction. Hence, $w = w^*$. This completes the proof. \qed
Letting $S = T$ in Theorem 2.4, we obtain the following corollary.

**Corollary 2.5.** Let $A$, $B$, and $S$ be three selfmappings of a $D$-metric space satisfying (2.11) and

$$
\rho(Ax,By,z) \leq \lambda \max \{\rho(Sx,Sy,z),\rho(Sx,Ax,z),\rho(Sy,By,z)\} \tag{2.57}
$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$. Further assume that

(a) $\overline{O_{A,B}(Sx)}$ is complete for each $x \in X$,
(b) $\{A,S\}$ and $\{B,S\}$ are limit coincidentally commuting,
(c) any one of $A$, $B$, and $S$ is continuous.

Then $A$, $B$, and $S$ have a unique common fixed point.

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