IJMMS 2003:41, 2609–2617
PII. S0161171203204233
http://ijmms.hindawi.com

α-FUZZY COMPACTNESS IN I-TOPOLOGICAL SPACES

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Received 28 April 2002

Using a gradation of openness in a (Chang fuzzy) I-topological space, we introduce degrees of compactness that we call α-fuzzy compactness (where α belongs to the unit interval), so extending the concept of compactness due to C. L. Chang. We obtain a Baire category theorem for α-locally compact spaces and construct a one-point α-fuzzy compactification of an I-topological space.

2000 Mathematics Subject Classification: 54A40, 54E52, 54D35.

1. Introduction. In 1968, Chang [1] introduced the concept of a fuzzy topology on a set X. However, some authors criticized that his notion did not really describe fuzziness with respect to the concept of openness of a fuzzy set. In the light of this difficulty, Šostak [9, 10] began his study on fuzzy structures of topological type. Subsequently, by means of some variant of a Šostak fuzzy topology (compare [2]), the authors of [5] developed a theory of α-gradation of open sets (i.e., they introduced the concept of an α-open set where α belongs to the unit interval) for a fuzzy topological space in the sense of Chang. Their theory of gradation of openness makes it possible to introduce degrees of fuzzy topological concepts, which generalize the corresponding ones in general topology on the one hand, and allow one to work with points of X instead of fuzzy points on the other hand. In particular, they proved that the family of all α-neighborhoods (α-nbhds for short) has similar properties as in the classical case; furthermore, they compared their α-T_i separation axioms with those discussed in [3].

We would like to draw the attention of the reader to the fact that in the present literature fuzzy topologies in Chang's sense are called I-topologies and gradations of openness are called I-fuzzy topologies (see, e.g., [6, 8]). On the other hand, our study is mainly based on [5] and in the present paper the authors see no need to extend their results to L-(fuzzy) topologies.

In our paper, we first introduce a gradation of compactness, namely, α-fuzzy compactness, based both on the aforementioned concept of an α-open set due to [5], as well as, on the notion of compactness due to Chang. Then we investigate the newly defined concepts by establishing analogues of classical topological results related to the concept of compactness.

We note that Gantner et al. [4] have introduced a concept of α-compactness based on their notion of an α-shading.

The structure of our paper is as follows: after the preliminary Section 2, in Section 3, the concept of α-fuzzy compactness is defined and its basic properties are studied. In Section 4, we present a Baire category theorem for α-locally compact α-quasiregular spaces. Finally, in Section 5, we construct a one-point α-fuzzy compactification for I-topologies. All the results mentioned generalize the corresponding ones from general topology.

2. Preliminaries. Let X be a nonempty set and I the closed real unit interval. A fuzzy set of X is a map $M : X \to I$. Here $M(x)$ is interpreted as the degree of membership of a point $x \in X$ in the fuzzy set M.

We then define the union, intersection, and complement of fuzzy sets as follows: for any $x \in X$,

(i) $(\bigcup_i A_i)(x) = \bigvee_i A_i(x)$,

(ii) $(\bigcap A_i)(x) = \bigwedge_i A_i(x)$,

(iii) $A^c(x) = 1 - A(x)$.

As usual, for $A, B \in I^X$, we write $A \subseteq B$ if $A(x) \le B(x)$ whenever $x \in X$.

Šostak [9] defined a fuzzy topology on X as a function $\tau : I^X \to I$ satisfying the following axioms:

(i) $\tau(\mathbf{0}) = \tau(\mathbf{1}) = 1$;

(ii) $\mu, \nu \in I^X$ implies that $\tau(\mu \cap \nu) \ge \tau(\mu) \wedge \tau(\nu)$;

(iii) $\mu_i \in I^X$ whenever $i \in I$ implies that $\tau(\bigcup_i \mu_i) \ge \wedge_i \tau(\mu_i)$;

where $\mathbf{0}, \mathbf{1}$ are the constant functions with values 0 and 1, respectively.

Chattopadhyay et al. [2] rediscovered the concept of fuzzy topology introduced by Šostak and called the function τ gradation of openness. Similarly, by interchanging intersection with union and vice versa in the aforementioned axioms and applying the modified axioms to a function $F : I^X \to I$, these authors defined the concept of a gradation of closedness on X.

In the following, a fuzzy topology in Šostak's sense (or a gradation of openness) will be called an I-fuzzy topology, and we define an I-topological space as a pair $(X, \mathcal{X})$ where $\mathcal{X}$ is an I-topology (topology in Chang's sense) on X, that is, $\mathcal{X}$ is a collection of fuzzy sets of X containing $\mathbf{0}, \mathbf{1}$ and closed under arbitrary unions and finite intersections. A set is called *open* if it belongs to $\mathcal{X}$ and closed if its complement belongs to $\mathcal{X}$. The closure of A, $\operatorname{cl} A$, is the smallest closed set containing A.

The *crisp subsets* of X are the characteristic functions of the (ordinary) subsets of X, and we will identify a subset of X with its associated crisp subset of X.

Recall that the support of a fuzzy set A is defined by $\operatorname{supp} A = \{x \in X : A(x) > 0\}$. We will write $x \mathrel{\hat{\in}} A$ if $x \in \operatorname{supp} A$, and then say that A contains the point x or that x is in A.

The following results and definitions can be found in [5].

Proposition 2.1. *Let X be a nonempty set. The map $\sigma : I^X \to I$ given by $\sigma(\mathbf{0}) = 1$ and $\sigma(A) = \inf\{A(x) : x \in \operatorname{supp} A\}$ if $A \neq \mathbf{0}$ is an I-fuzzy topology*

on X; it satisfies both the axioms of gradation of openness and the axioms of gradation of closedness.

The real number $\sigma(A)$ is the degree of openness [9] of the fuzzy set A; clearly $\sigma(A) = \alpha$ implies that the degree of membership of each point in the support of A, that is, in the fuzzy set A, is at least α. We observe that $\sigma(A) = 1$ if and only if A is a crisp subset of X, and $\sigma(A) = 0$ if and only if there is a sequence $\{x_n\}$ in X such that $A(x_n) > 0$, $n \in \mathbb{N}$ and $\lim_n A(x_n) = 0$. (Here $\mathbb{N}$ denotes the set of positive integers.)

Definition 2.2. The fuzzy set A of X is an α-set (where $\alpha \in [0,1]$) of X if $\sigma(A) \geq \alpha$. If the α-set A is open (closed), then A is said to be α-open (α-closed, resp.).

Clearly, each $A \in I^X$ is a 0-set, and the 1-sets are exactly the crisp subsets of X. Observe also that the union and intersection of α-sets is an α-set.

Remark 2.3. Let K be an ordinary subset of X. If $A \in I^X$ is an α-set of X, then the restriction $A_{|_K}$ of A to K is an α-set of K, but the converse does not hold in general.

Definition 2.4 [2]**.** Let $(X,\mathcal{X})$ be an I-topological space and $\alpha \in [0,1]$. The family $\mathcal{X}_\alpha = \{A \in \mathcal{X} : \sigma(A) \geq \alpha\}$ is an I-topology on X, called the α-level I-topology of X. Clearly $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}_1$ is an ordinary topology on X.

Definition 2.5. Let $(X,\mathcal{X})$ be an I-topological space. The fuzzy set A of X is called an α-neighborhood (α-nbhd for short) of $p \in X$ if there exists $C \in \mathcal{X}_\alpha$ such that $p \,\hat{\in}\, C \subseteq A$. (The family $\mathcal{N}_\alpha(p)$ of all α-nbhds of p satisfies similar properties as the set of neighborhoods at a point in general topology.)

Definition 2.6. The I-topological space $(X,\mathcal{X})$ is called α-fuzzy Hausdorff if for $x, y \in X$, $x \neq y$, there are $G, H \in \mathcal{X}_\alpha$ such that $x \,\hat{\in}\, G$, $y \,\hat{\in}\, H$, and $G \cap H = \mathbf{0}$.

Recall that the empty set is identified with $\mathbf{0}$. The following definitions and result are due to [1].

A family $\mathcal{A}$ of fuzzy sets of X is a cover of a fuzzy set $B \in I^X$ provided that $B \subseteq \bigcup\{A : A \in \mathcal{A}\}$. It is an open cover if each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover.

An I-topological space $(X,\mathcal{X})$ is compact if each open cover of X has a finite subcover. A family $\mathcal{A}$ of fuzzy sets has the finite intersection property if the intersection of the members of each finite subfamily of $\mathcal{A}$ is nonempty. An I-topological space is compact if and only if each family of closed sets which has the finite intersection property has a nonempty intersection.

3. Gradation of compactness in I-topological spaces. Next we introduce the following new concepts.

Definition 3.1. Let $(X,\mathcal{X})$ be an I-topological space and $\alpha \in [0,1]$. A cover $\mathcal{A}$ of a fuzzy set is an *α-cover* if each member of $\mathcal{A}$ is an *α-set*. It is called an

α-open cover provided that each member of $\mathcal{A}$ is an α-open set. Moreover, X is said to be *α-fuzzy compact* if each α-open cover of X has a finite subcover of X, that is, $(X,\mathcal{X}_\alpha)$ is compact (in Chang's sense).

Observations. An I-topological space X is compact if and only if X is 0-fuzzy compact. For $\alpha<\beta$, α-fuzzy compactness implies β-fuzzy compactness. If X is a topological space, then X is compact if and only if X is 1-fuzzy compact (or 0-fuzzy compact, because the open sets are all 1-sets).

Definition 3.2. Let K be a crisp subset of the I-topological space $(X,\mathcal{X})$. Then K is called α-fuzzy compact if the subspace $(K,\mathcal{X}_{|_K})$ of X is α-fuzzy compact, that is, $(K,(\mathcal{X}_{|_K})_\alpha)$ is compact.

Remark 3.3. For $\alpha\neq 0$, note that the α-level I-topology $(\mathcal{X}_{|_K})_\alpha$ of K does not agree, in general, with the family $(\mathcal{X}_\alpha)_{|K}$ of the restrictions of $\mathcal{X}_\alpha$ to K (see Remark 2.3). However, observe that the inclusion $(\mathcal{X}_\alpha)_{|_K}\subseteq(\mathcal{X}_{|_K})_\alpha$ is always satisfied.

Indeed, if $A\in\mathcal{X}_{\alpha|_K}$, then there is an α-open C of X such that $A=C_{|_K}$; thus $A\in\mathcal{X}_{|_K}$ and obviously $A\in(\mathcal{X}_{|_K})_\alpha$. But the other inclusion only holds, in general, for $\alpha=0$ (since $\mathcal{X}_0=\mathcal{X}$ and $(\mathcal{X}_{|_K})_0=\mathcal{X}_{|_K}$). Therefore, it is easy to verify that if K is α-fuzzy compact, then each α-open cover of K formed by α-open sets of X has a finite subcover of K. The converse, however, (which is true for $\alpha=0$) is, in general, false as the following example shows.

Example 3.4. Let X be the real interval $[-1,1]$ and take $K=[0,1]$. For each $a\in[0,1]$ consider the fuzzy set M_a of X given by $M_a(x)=ax+a$ if $x\in[-1,0]$ and $M_a(x)=a$ if $x\in K$. Then, $\mathcal{X}=\{M_a:a\in[0,1]\}\cup\{\mathbf{1}\}$ is an I-topology on X, and the family $\mathcal{X}|_K$ of open sets of K is formed by all the constant functions on K with values in $[0,1]$ (i.e., $\mathcal{X}_{|_K}$ is the indiscrete fuzzy topology on K in the sense of Lowen [7]). Now, for $\alpha\neq 0$, each α-open cover $\mathcal{A}$ of K formed by α-open sets of X has a finite subcover of K (in fact, if $\mathbf{0}\notin\mathcal{A}$, then the only possibility is $\mathcal{A}=\{\mathbf{1}\}$), but clearly K is not α-fuzzy compact for $\alpha\in[0,1[$ (in fact, $\{M_{a|_K}:a\in[\alpha,1[\}$ is an α-open cover of K, without any finite subcover of K). Note that K and X are 1-fuzzy compact and X is α-fuzzy compact for $\alpha\neq 0$.

We omit the proof of the following proposition.

Proposition 3.5. *Let B and C be two crisp subsets of the I-topological space X. If B and C are α-fuzzy compact, then $B\cup C$ is α-fuzzy compact.*

Theorem 3.6. *Let F be a closed crisp subset of the I-topological space $(X,\mathcal{X})$. If X is α-fuzzy compact, then F is α-fuzzy compact as a subspace of X.*

Proof. Let $\mathcal{U}$ be an α-open cover, in the subspace F, of F, and put $\mathcal{V}=\{V\in\mathcal{X}:V_{|_F}\in\mathcal{U}\}$. Consider $\mathcal{V}^*=\{V\cup F^c:V\in\mathcal{V}\}$. Clearly, $\mathcal{V}^*$ is an α-open

cover of X. So $\mathcal{V}^*$ has a finite subcover $V_1 \cup F^c, \dots, V_n \cup F^c$ ($V_i \in \mathcal{V}$, $i = 1, \dots, n$) of X. Then, $(V_1 \cup F^c)_{|_F}, \dots, (V_n \cup F^c)_{|_F}$ is a finite subcover of $\mathcal{U}$ and of F, since $(V_i \cup F^c)_{|_F} = V_i$, $i = 1, \dots, n$. □

Definition 3.7. An I-topological space $(X, \mathcal{X})$ is α-strongly Hausdorff, or α-sT_2 for short, if for all points $x, y \in X$ with $x \neq y$ there exist $G, H \in \mathcal{X}_\alpha$ such that $x \hat{\in} G$, $y \hat{\in} H$, $G(x) = H(y) = 1$, and $G \cap H = \mathbf{0}$.

Theorem 3.8. *Let S be a crisp subspace of an α-sT_2 I-topological space $(X, \mathcal{X})$. If S is α-fuzzy compact, then S is closed in X.*

Proof. Let $x \in X \setminus S$. We will show that there exists $U \in \mathcal{X}$ with $U(x) = 1$ and $U \subseteq X \setminus S$. For each $y \in S$ we can find $U_y, V_y \in \mathcal{X}_\alpha$ with $U_y(x) = V_y(y) = 1$ and $U_y \cap V_y = \mathbf{0}$. Thus $\{V_{y_{|S}} : y \in S\}$ is an α-cover of S, in the subspace S of X; so it has a finite subcover $\{V_{y_1|S}, \dots, V_{y_n|S}\}$ of S. Let $U = U_{y_1} \cap \dots \cap U_{y_n}$. Then $U(x) = 1$ and $U \cap (V_{y_1} \cup \dots \cup V_{y_n}) = \mathbf{0}$. For each $z \in S$, there exists a $k \in \{1, \dots, n\}$ with $V_{y_k}(z) = 1$, so $U(z) = 0$. Hence $U \subseteq X \setminus S$. □

4. A Baire category theorem. Let $\alpha \in [0,1]$. We introduce the following new concepts.

Definition 4.1. The I-topological space $(X, \mathcal{X})$ is said to be *α-locally compact* if each $x \in X$ has a crisp set K of X which is an α-nbhd of x, in X, and α-fuzzy compact as a subspace of X. Moreover X is called locally compact if it is 0-locally compact.

Clearly, if X is α-fuzzy compact, then it is α-locally compact.

Example 4.2 (an α-locally compact space which is not α-fuzzy Hausdorff)**.** For each $n \in \mathbb{N}$ define the fuzzy set A_n on $\mathbb{N}$ as follows: $A_n(x) = 1$ if $x < n$, $A_n(x) = 1/2$ if $x = n$ and $A_n(x) = 0$ if $x > n$, where $x \in \mathbb{N}$. Put $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$. Now $\mathcal{X} = \mathcal{A} \cup \{\mathbf{0}\}$ is an I-topology on $\mathbb{N}$, and clearly $(\mathbb{N}, \mathcal{X})$ is not α-fuzzy Hausdorff, for $\alpha \in [0,1]$.

On the other hand, the family $\mathcal{A}$ is an α-open cover of $\mathbb{N}$ which has not any finite subcover and then $(\mathbb{N}, \mathcal{X})$ is not an α-fuzzy compact space for $\alpha \in [0, 1/2]$.

Finally, for $t \in \mathbb{N}$, the crisp set $K_t = \{x \in \mathbb{N} : x \leq t\}$ of $\mathbb{N}$ is an α-neighborhood of t in $(\mathbb{N}, \mathcal{X})$, for $\alpha \in [0, 1/2]$, and obviously K_t is α-fuzzy compact in $(K_t, \mathcal{X}_{|K_t})$, for $\alpha \in [0,1]$.

Example 4.3 (an α-locally compact space which is α-fuzzy Hausdorff)**.** Let $\mathbb{Z}$ denote the set of integers. For each $p \in \mathbb{Z}$ consider the fuzzy set A_p on $\mathbb{Z}$, given by $A_p(x) = 1$ if $x = p$, $A_p(x) = 1/2$ if $x = p+1$, and $A_p(x) = 0$ elsewhere, where $x \in \mathbb{Z}$.

Then, $\mathcal{A} = \{A_p : p \in \mathbb{Z}\}$ is a subbase for an I-topology $\mathcal{X}$ on $\mathbb{Z}$, which is clearly α-fuzzy Hausdorff, and since $\mathcal{A}$ is an α-open cover of $\mathbb{Z}$ without any finite subcover, $(\mathbb{Z}, \mathcal{X})$ is not α-fuzzy compact; finally, for $t \in \mathbb{Z}$, the crisp set

$K_t = \{t, t+1\}$ of $\mathbb{Z}$ is an α-neighborhood of t in $(\mathbb{Z}, \mathscr{X})$, for $\alpha \in [0, 1/2]$. Obviously, K_t is α-fuzzy compact in $(K_t, \mathscr{X}_{|K_t})$ for $\alpha \in [0, 1]$.

Definition 4.4. Let $(X, \mathscr{X})$ be an I-topological space. The fuzzy set D of X is called *α-dense* in X if $D \cap C$ is nonempty for each nonempty α-open set C of X. Then D is called dense in X if it is 0-dense in X. (Then, D is α-dense in X if and only if D is dense in $(X, \mathscr{X}_a)$.)

Example 4.5 (α-dense fuzzy sets)**.** For each $i \in I$ define the fuzzy set A_i on I as follows: $A_i(x) = i$ if $x \leq i$, and $A_i(x) = 0$ elsewhere, where $x \in I$.

It is obvious that $\{A_i : i \in I\}$ is an I-topology on I. If B is a fuzzy set on I such that $B(x) \neq 0$ for all $x \in I$, then B is α-dense for $\alpha \in [0, 1]$. Now, consider the fuzzy set D on I defined by $D(x) = 0$ if $x \leq 1/2$ and $D(x) = 1/2$ if $x > 1/2$, where $x \in I$. Then, D is α-dense if and only if $\alpha > 1/2$.

Definition 4.6. The I-topological space $(X, \mathscr{X})$ is said to be α-quasiregular if for each nonempty $C \in \mathscr{X}_\alpha$ there exists a nonempty $H \in \mathscr{X}_\alpha$ such that $H \subseteq \mathscr{X}_\alpha - \mathrm{cl}(H) \subseteq C$, where $\mathscr{X}_\alpha - \mathrm{cl}(H)$ denotes the closure in $\mathscr{X}_\alpha$ of H. Furthermore, X is called quasiregular if it is 0-quasiregular. (Then X is α-quasiregular if and only if $(X, \mathscr{X}_\alpha)$ is quasiregular.)

Example 4.7 (an α-quasiregular space)**.** Define the I-topology $\mathscr{L}$ on $\mathbb{N}$ as follows: $A \in \mathscr{L}$ if and only if $A(x) \in \{0, 1/2, 1\}$ for all $x \in \mathbb{N}$. Take two elements a, b which do not belong to $\mathbb{N}$, and put $X = \mathbb{N} \cup \{a, b\}$. Consider the I-topology $\mathscr{X}$ on X defined as follows: let $G \in I^X$. Then $G \in \mathscr{X}$ if and only if G satisfies one of the following two conditions:

(1) $G|_{\mathbb{N}} \in \mathscr{L}$ and $G(a) = G(b) = 0$;
(2) $G(x) = 1$ for all $x \in X$ except, at most, on a finite set of X where G takes the value $1/2$.

We claim that $(X, \mathscr{X})$ is an α-quasiregular space for $\alpha \in [0, 1/2]$. Indeed, if $G \in \mathscr{X}_\alpha$ with $G \neq \mathbf{0}$, then there exists $y \in \mathbb{N}$ such that $G(y) \in \{1/2, 1\}$. Now, consider the fuzzy set H_y of X given by $H_y(x) = 0$ if $x \neq y$, and $H_y(x) = 1/2$ if $x = y$, where $x \in X$.

It is obvious that $H_y \in \mathscr{X}_\alpha$ and $H_y \subseteq G$. On the other hand, $(1 - H_y)(x) = 1$ if $x \neq y$ and $(1 - H_y)(x) = 1/2$ if $x = y$, and so $1 - H_y \in \mathscr{X}$; moreover, $1 - H_y \in \mathscr{X}_\alpha$ and clearly $H_y \subseteq \mathscr{X}_\alpha - \mathrm{cl}(H_y) = H_y \subseteq G$ and hence $(X, \mathscr{X})$ is α-quasiregular, for $\alpha \in [0, 1/2]$.

Definition 4.8. The I-topological space $(X, \mathscr{X})$ is said to be α-Baire if the intersection of every sequence $\{G_n : n \in \mathbb{N}\}$ of α-dense α-open sets of X is α-dense in X. We say that X is Baire if it is 0-Baire. (Then X is α-Baire if and only if $(X, \mathscr{X}_\alpha)$ is Baire.)

Now, if X is an ordinary topological space, then X is locally compact, quasiregular, and Baire or D is dense in X, if it is so in the above defined fuzzy sense (equivalently, if and only if X is α-locally compact, α-quasiregular, and α-Baire

or D is α-dense in X, respectively, for some $\alpha \in [0,1]$, since the open sets of X are 1-sets).

Theorem 4.9 (Baire's category theorem). *Let $(X,\mathscr{X})$ be an I-topological space and $\alpha \in [0,1]$. If X is α-quasiregular and α-locally compact, then X is α-Baire.*

Proof. Let $\{D_n : n \in \mathbb{N}\}$ be a sequence of α-open α-dense sets in X. Let U be a nonempty α-open set in X. Take $x \,\hat{\in}\, U$; then there exists an α-nbhd K of x in X which is an α-fuzzy compact subspace of X, that is, there is $H \in \mathscr{X}_\alpha$ such that $x \,\hat{\in}\, H \subseteq K$. Furthermore, $H \cap U$ is a nonempty α-open set of X. Then, $D_1 \cap (H \cap U)$ is a nonempty α-open set of X. Choose a nonempty $V_1 \in Z_\alpha$, such that $V_1 \subseteq \mathscr{X}_\alpha - \mathrm{cl}(V_1) \subseteq D_1 \cap (H \cap U)$. Next by induction we construct a sequence $\{V_n : n \in \mathbb{N}\}$ such that $\mathscr{X}_\alpha - \mathrm{cl}(V_n) \subseteq D_n \cap \mathscr{X}_\alpha - \mathrm{cl}(V_{n-1})$.

It follows that for each $n \in \mathbb{N}$, $\mathscr{X}_\alpha - \mathrm{cl}(V_n)$ is a closed set in $(X,\mathscr{X}_\alpha)$ and by Remark 3.3 $\mathscr{X}_\alpha - \mathrm{cl}(V_n)_{|_K}$ is closed in the compact space $(K,(\mathscr{X}_{|_K})_\alpha)$.

Clearly, $\{\mathscr{X}_\alpha - \mathrm{cl}(V_n) : n \in \mathbb{N}\}$ is a family of closed sets in $(X,\mathscr{X}_\alpha)$ which has the finite intersection property. Now, $\mathscr{X}_\alpha - \mathrm{cl}(V_n) \subseteq K$, whenever $n \in \mathbb{N}$, and then $\{\mathscr{X}_\alpha - \mathrm{cl}(V_n)_{|_K} : n \in \mathbb{N}\}$ is a family of closed sets in the compact space $(K,(\mathscr{X}_{|_K})_\alpha)$ which also satisfies the finite intersection property and therefore has a nonempty intersection. Clearly, $(\bigcap_{n=1}^{\infty} D_n) \cap U \neq \mathbf{0}$. □

5. One-point α-fuzzy compactifications

Definition 5.1. Let $(X,\mathscr{X})$ be an I-topological space and ω a point that does not belong to X. Set $X^* = X \cup \{\omega\}$. A fuzzy set G in X^* is called open if either $G(\omega) = 0$ and $G_{|X}$ is open in X, or $G^c_{|X}$ is closed in X and $\{x \in X^* : G(x) \neq 1\}$ is contained in the support of an α-fuzzy compact subspace of X. Then, X^* is called the one-point α-fuzzy compactification of X.

Theorem 5.2. *The one-point α-fuzzy compactification X^* of $(X,\mathscr{X})$ is an α-fuzzy compact I-topological space and X is an I-subspace of X^*.*

Proof. Suppose that G and H are open in X^*. We distinguish three cases:

(1) $G(\omega) = H(\omega) = 0$. Then $(G \cap H)_{|X} = G_{|X} \cap H_{|X}$ is open in X, and thus $G \cap H$ is open in the space X^*;

(2) $G(\omega) = 1$, $H(\omega) = 0$. So $G^c_{|X}$ is closed in X; then $G_{|X}$ is open in X. Now, $(G \cap H)(\omega) = 0$ and $G_{|X}$, $H_{|X}$ are open in X. We have $(G \cap H)_{|X} = G_{|X} \cap H_{|X}$ is open in X and therefore $G \cap H$ is open in X^*;

(3) $G(\omega) = H(\omega) = 1$. Suppose $\{x \in X^* : G(x) \neq 1\} \subseteq K$ and $\{x \in X^* : H(x) \neq 1\} \subseteq C$, where K and C are α-fuzzy compact subspaces of X.

Now, $(G \cap H)^c_{|X} = (G^c \cup H^c)_{|X} = G^c_{|X} \cup H^c_{|X}$ is closed in X and $\{x \in X^* : (G \cap H)(x) \neq 1\} \subseteq K \cup C$, but $K \cup C$ is α-fuzzy compact by Proposition 3.5.

Let $\mathscr{C}$ be a collection of open sets in X^*. We distinguish two cases.

(1) If $G(\omega) = 0$ whenever $G \in \mathscr{C}$, then clearly $\bigcup\{G : G \in \mathscr{C}\}$ is open in X^*.

(2) Now, suppose that there is $G \in \mathscr{C}$ such that $G(\omega) = 1$, and $\{x \in X^* : G(x) \neq 1\} \subseteq K$ where K is an α-fuzzy compact subspace of X. Therefore, $(\bigcup_{G\in\mathscr{C}} G)(\omega) = 1$ and $G^c_{|X}$ is closed in X for each $G \in \mathscr{C}$. Moreover, $(\bigcap_{G\in\mathscr{C}} G)^c_{|X} = \bigcap\{G^c_{|X} : G \in \mathscr{C}\}$ is closed in X and $\{x \in X^* : (\bigcup_{G\in\mathscr{C}} G)(x) \neq 1\} \subseteq K$. Thus $\bigcup\{G : G \in \mathscr{C}\}$ is open in X^*.

Clearly X is an I-topological subspace of X^*. (If G is open in X, then G is the restriction of the open set G^* in X^*, to X, where $(G^*)(x) = 0$ if $x = \omega$ and $(G^*)(x) = G(x)$ if $x \neq \omega$. On the other hand, if G is relatively open, say $G = H_{|X}$ where H is open in X^*, then G is open in X. In fact, if $H(\omega) = 0$, then $H(x) = 0$ if $x = \omega$, and $H(x) = G(x)$ if $x \neq \omega$ where G is open in X. If $H(\omega) = 1$, then $H^c_{|X}$ is closed in X and G is open in X.)

Now, let $\mathcal{U}$ be an α-open cover of X^*. Then $U(\omega) = 1$ for some $U \in \mathcal{U}$ and $\{x \in X^* : U(x) \neq 1\}$ is contained in some α-fuzzy compact subspace K of X. Then by Remark 3.3 K is covered by $U_1, \dots, U_n$ where $U_i \in \mathcal{U}$ $(i = 1, \dots, n)$ and hence $U_1, \dots, U_n, U$ is a finite subcover of $\mathcal{U}$, of X^*. □

Theorem 5.3. *Let X^* be the one-point α-fuzzy compactification of $(X, \mathscr{L})$.*

(i) *If X^* is α-fuzzy Hausdorff, then X is α-fuzzy Hausdorff and α-locally compact.*

(ii) *If X is α-sT_2 and α-locally compact, then X^* is α-sT_2.*

(iii) *Moreover, $\{\omega\}$ is open in X^* if and only if X is α-fuzzy compact.*

(iv) *Furthermore, X is dense in X^* if and only if X is not α-fuzzy compact.*

Proof. (i) If X^* is α-fuzzy Hausdorff, then, obviously, X is α-fuzzy Hausdorff. Let $x \in X$. There exist two α-open sets U and V which are nbhds of ω and x, respectively, such that $U \cap V = \mathbf{0}$. Suppose that $\{x \in X^* : U(x) \neq 1\} \subseteq K$ where K is an α-fuzzy compact subspace of X. Then, $x \,\hat{\in}\, V \subseteq \{x \in X^* : U(x) = 0\} \subseteq K$.

Now, K is α-fuzzy compact and V is α-open and therefore X is α-locally compact.

(ii) Let $x \in X$. Then, there exists an α-open set U of X such that $x \,\hat{\in}\, U \subseteq K$ where K is an α-fuzzy compact subspace of X. By Theorem 3.8, K is closed in X and then $X^* \setminus K$ is a 1-open nbhd of ω such that $(X^* \setminus K) \cap U = \mathbf{0}$.

(iii) If $\{\omega\}$ is open in X^*, then X is closed in X^* and, by Theorem 3.6, α-fuzzy compact. If X is α-fuzzy compact, then $\{\omega\}$ is open in X^*, since X is closed in X.

(iv) The assertion is a consequence of (iii). □

Remark 5.4. We say that X^* is a one-point compactification of X if it is the one-point 0-fuzzy compactification. Now, if X is an ordinary topological space, then the defined (fuzzy) one-point compactification (or one-point α-fuzzy compactification for some $\alpha \in [0,1]$) agrees with the classical one.

Acknowledgments. While working on this paper, the first author was partially supported by a Grant from Generalitat Valenciana CTIDIB/2002/59. The

second author acknowledges the partial support by a URC grant of the University of Cape Town.

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