The Bessel-Muirhead hypergeometric system (or $0F_1$-system) in two variables (and three variables) is solved using symmetric series, with an explicit formula for coefficients, in order to express the $K$-Bessel function as a linear combination of the J-solutions. Limits of this method and suggestions for generalizations to a higher rank are discussed.

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1. Introduction. The Bessel functions (of the first kind) defined on the space of real symmetric matrices appeared in the work of James [5] as an ingredient in the expression of some densities in multivariate statistics. At the same time, more systematic treatment was done by Herz [4]. In [8], Muirhead proved that they are solutions of a system of differential operators which will be designated here as Bessel-Muirhead operators following [6]. We can see [1, 3] for the generalization of this set of functions to a Jordan algebra. In what follows, we explicitly write down a fundamental set of solutions when the rank equals 2 or 3. Our approach is slightly different from [7] in the final form of the coefficients. Then (and this is our main result), we express the $K$-Bessel function defined in this context as a linear combination of the J-solutions in the rank-2 case, so answering a question in [4].

**Definition 1.1.** Bessel-Muirhead operators are defined by

$$B_i = x_i \frac{\partial^2}{\partial x_i^2} + (\nu + 1) \frac{\partial}{\partial x_i} + 1 + \frac{d}{2} \sum_{j \neq i} \frac{1}{x_i - x_j} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right), \quad 1 \leq i \leq r, \quad (1.1)$$

where $r$ is the rank of the system. A symmetric function $f$ is said to be a Bessel function if it is a solution of $B_i f = 0$, $i = 1, 2, \ldots, r$.

Denote by $t_1, t_2, \ldots, t_r$ the elementary symmetric functions, that is,

$$t_p = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq r} x_{i_1} x_{i_2} \cdots x_{i_p} \quad (1.2)$$
with \( t_0 = 1 \) and \( t_p = 0 \) if \( p < 0 \) or \( p > r \). The Bessel-Muirhead system is then equivalent to the system \( Z_k g = 0, \ 1 \leq k \leq r \), (see [1, 5]) where

\[
Z_k = \sum_{i,j=1}^{r} A_{ij}^k \frac{\partial^2}{\partial t_i \partial t_j} + \left( \nu + 1 + \frac{r - k}{2} \right) \frac{\partial}{\partial t_k} + \delta_1^k,
\]  

(1.3)

\[
A_{ij}^k = \begin{cases} 
  t_i + j - k & \text{if } i, j \geq k, \\
  -t_i + j - k & \text{if } i, j < k, i + j \geq k, \\
  0 & \text{elsewhere.}
\end{cases} 
\]  

(1.4)

Here, \( \delta_1^k \) is the Kronecker symbol and \( g(t_1, t_2, \ldots, t_r) = f(x_1, x_2, \ldots, x_r) \).

2. Case \( r = 2 \). In this case, we have \( A_1^1 = \begin{pmatrix} t_1 & t_2 \\ 0 & 0 \end{pmatrix} \), and \( A_2^2 = \begin{pmatrix} -1 & 0 \\ 0 & t_2 \end{pmatrix} \), and the operators in the modified system (1.3) can be written as follows:

\[
t_1 Z_1 = \theta_1 \left( \partial_1 + 2 \theta_2 + \nu + \frac{\nu}{2} \right) + t_1,
\]

\[
t_2 Z_2 = \theta_2 (\theta_2 + \nu) - t_2 \frac{\partial^2}{\partial t_1^2},
\]  

(2.1)

where \( \theta_1 = t_1 (\partial/\partial t_1) \) and \( \theta_2 = t_2 (\partial/\partial t_2) \). The operators \( \theta_i \) are used because their action on powers is easily checked by the rule \( \theta_i t_i^\alpha = \alpha t_i^\alpha \). Now, putting in the system (2.1) a series of the form \( S(\lambda_1, \lambda_2)(t_1, t_2) = \sum_{m_1, m_2 \geq 0} c(m_1, m_2) t_1^{m_1+\lambda_1} t_2^{m_2+\lambda_2} \), we can write the following system of recurrence formulas:

\[
(m_1 + \lambda_1) (m_1 + 2m_2 + \lambda_1 + 2\lambda_2 + \nu + \frac{\nu}{2}) c(m_1, m_2) + c(m_1 - 1, m_2) = 0,
\]

\[
(m_2 + \lambda_2) (m_2 + \lambda_2 + \nu) c(m_1, m_2)
\]

\[
- (m_1 + 2 + \lambda_1) (m_1 + 1 + \lambda_1) c(m_1 + 2, m_2 - 1) = 0.
\]

(2.2)

Then, we first obtain the system of critical exponents \((\lambda_1, \lambda_2)\) when \((m_1, m_2) = (0, 0)\):

\[
\lambda_1 \left( \lambda_1 + 2 \lambda_2 + \nu + \frac{\nu}{2} \right) = 0,
\]

\[
\lambda_2 (\lambda_2 + \nu) = 0,
\]  

(2.3)

which admits, as solutions, the set

\[
\Lambda_{2, \nu} = \left\{ (0, 0); (0, -\nu); \left( -\nu - \frac{\nu}{2}, 0 \right); \left( \nu - \frac{\nu}{2}, -\nu \right) \right\}.
\]  

(2.4)

Now, with the help of the second equation of (2.2), we can express \( c(m_1, m_2) \) in terms of \( c(m_1 + 2m_2, 0) \) and then in terms of \( c(0, 0) \) thanks to the first
The modified system (1.3) takes the form
\[ c(m_1, m_2) = \frac{(-1)^{m_1+2m_2}c(0,0)}{(1 + \lambda_1)m_1 (1 + \lambda_2)m_2 (1 + \lambda_2 + \nu)m_2 (1 + \lambda_1 + 2\lambda_2 + \nu + d/2)m_1+2m_2}. \]

(2.5)

**Theorem 2.1.** For generic \( \nu \) (i.e., \( \nu \notin \mathbb{Z} \) and \( \nu \pm d/2 \notin \mathbb{Z} \)), the series \( S_{(\lambda_1, \lambda_2)}(t_1, t_2) \) with \( c(m_1, m_2) \) as in (2.5) and \( (\lambda_1, \lambda_2) \in \Lambda_{2, \nu} \) form a fundamental set of solutions of system (2.1).

**Remark 2.2.** The convergence of this series is obvious.

3. **Case** \( r = 3 \). As in the previous case, we have
\[
A^1 = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t_2 & t_3 \\ 0 & t_3 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -t_1 & 0 \\ 0 & 0 & t_3 \end{pmatrix},
\]
(3.1)

the modified system (1.3) takes the form
\[
t_1 Z_1 = \theta_1(\theta_1 + 2\theta_2 + 2\theta_3 + \nu + d) + t_1 t_3 \frac{\partial^2}{\partial t_1^2},
\]
\[
t_2 Z_2 = \theta_2(\theta_2 + 2\theta_3 + \nu + d/2) - t_2 \frac{\partial^2}{\partial t_1 \partial t_2},
\]
\[
t_3 Z_3 = \theta_3(\theta_3 + \nu) - 2t_3 \frac{\partial^2}{\partial t_1 \partial t_2} - t_1 t_3 \frac{\partial^2}{\partial t_2^2},
\]
(3.2)

and we obtain the following system of recurrence formulas for the coefficients of a series of the form \( \sum_{m_1, m_2, m_3 \geq 0} c(m_1, m_2, m_3)t_1^{m_1 + \lambda_1}t_2^{m_2 + \lambda_2}t_3^{m_3 + \lambda_3} \):
\[
I_1(\lambda + m)c(m) + c(m - e_1) + (m_2 + 2 + \lambda_2)(m_2 + 1 + \lambda_2)c(m - e_1 + 2e_2 - e_3) = 0,
\]
\[
I_2(\lambda + m)c(m) - (m_1 + 2 + \lambda_1)(m_1 + 1 + \lambda_1)c(m + 2e_1 - e_2) = 0,
\]
\[
I_3(\lambda + m)c(m) - 2(m_1 + 1 + \lambda_1)(m_2 + 1 + \lambda_2)c(m + e_1 + e_2 - e_3)
\]
\[- (m_2 + 2 + \lambda_2)(m_2 + 1 + \lambda_2)c(m - e_1 + 2e_2 - e_3) = 0,
\]
(3.3)

where \( m = (m_1, m_2, m_3) \), \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \), \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \), and
\[
I_1(s) = s_1(s_1 + 2s_2 + 2s_3 + \nu + d),
\]
\[
I_2(s) = s_2(s_2 + 2s_3 + \nu + d/2),
\]
\[
I_3(s) = s_3(s_3 + \nu).
\]
(3.4)
The critical exponents set \( \Lambda_{3, \nu} \) is obtained after solving \( I_1(\lambda) = I_2(\lambda) = I_3(\lambda) = 0 \). Then we have

\[
\Lambda_{3, \nu} = \begin{cases} 
(0,0,0); & (0,0,-\nu), \\
(0,-\nu-d,0,0); & (\nu-d,0,-\nu), \\
(0,-\nu-d/2,0); & (0,-\nu-d/2,-\nu), \\
(v,-\nu-d/2,0); & (-\nu,v-d/2,-\nu).
\end{cases} \tag{3.5}
\]

Now, by the second equation of (3.3), we can express \( c(m) \) in terms of \( c(m_1 + 2m_2, 0, m_3) \). The third equation of (3.3) allows us to express \( c(m_1 + 2m_2 + 3m_3, 0, 0) \), and finally, by the first equation, we regress to \( c(0,0,0) \). After all reductions, we obtain

\[
c(m) = \frac{(-1)^{m_1 + 2m_2 + 3m_3} c(0)}{(1+\lambda_1)_{m_1} (1+\lambda_2)_{m_2} (1+\lambda_3)_{m_3} (1+\lambda_2+2\lambda_3+\nu+d/2)_{m_2+2m_1}} \times \frac{(1+\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\nu + d)_{m_1+2m_2+4m_3}}{(1+\lambda_1 + 2\lambda_2 + 2\lambda_3 + \nu + d)_{m_1+2m_2+3m_3}} \times \frac{1}{(1+\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\nu + d)_{m_1+2m_2+3m_3}} \tag{3.6}
\]

and all ingredients to write a theorem like Theorem 2.1.

4. \( K \)-Bessel function. As an application, we derive, in the case \( r = 2 \), the expansion of the \( K \)-Bessel function in the previous basis (J-functions) of the Bessel system. Recall the one-variable situation (small letters refer to special functions of one variable); the \( k \)-Bessel function can be defined by

\[
k_\nu(x) = \int_0^{+\infty} \exp \left( -y - \frac{x}{y} \right) y^{-\nu-1} \, dy. \tag{4.1}
\]

If we put

\[
j_\nu(x) = \, _0f_1 \left( \begin{array}{c} - \\
\nu+1 \end{array}; x \right) = \sum_{n \geq 0} \frac{(-1)^n}{n!(\nu+1)n} x^n, \tag{4.2}
\]

we have the formula

\[
k_\nu(x) = \Gamma(-\nu) j_\nu(-x) + \Gamma(\nu) x^{-\nu} j_{-\nu}(-x). \tag{4.3}
\]

Recall also the Mellin transform of \( k_\nu(x) \),

\[
M(k_\nu)(s) = \int_0^{+\infty} k_\nu(x) x^{s-1} \, dx = \Gamma(s) \Gamma(s - \nu). \tag{4.4}
\]
Now, we write the two-variable situation in a Jordan algebra context. Take an \( n \)-dimensional Jordan algebra \( A \) of a rank 2, the generic case is \( A = \mathbb{R} \times \mathbb{R}^{n-1} \), endowed with the product

\[
x \cdot y = (\xi \eta + \langle u, v \rangle, \xi v + \eta u)
\]  

(4.5)

if \( x = (\xi, u) \), \( y = (\eta, v) \), and \( \langle u, v \rangle = \sum_{1 \leq i \leq n-1} u_i v_i \). The unit is obviously \( e = (1, 0) \). Then we have a Cayley-Hamilton-like theorem \( x^2 - 2\xi x + (\xi^2 - \|u\|^2)e = 0 \), and we can put \( \text{tr}(x) = 2\xi \) and \( \det(x) = \xi^2 - \|u\|^2 \). We consider the following scalar product on \( A \):

\[
(x, y) = \text{tr}(x \cdot y) = 2\xi \eta + 2\langle u, v \rangle.
\]  

(4.6)

We can show that each \( x \) has a spectral decomposition \( x = x_1 \hat{e}_1 + x_2 \hat{e}_2 \), with \( x_1, x_2 \in \mathbb{R} \) and \( \{\hat{e}_1, \hat{e}_2\} \) is a pair of primitive strongly orthogonal idempotents. More precisely, \( \hat{e}_1 = (1/2, u/2\|u\|) \) and \( \hat{e}_2 = (1/2, -u/2\|u\|) \). Observe that \( \sigma_x = u/\|u\| \in S^{n-2} \). Any element \( y \) can be decomposed as follows: \( y = k \cdot (y_1 \hat{e}_1 + y_2 \hat{e}_2) \) with \( k \in \text{SO}(n-1) \) acting on \( \hat{e}_1 \), for example, by \( k \cdot \hat{e}_1 = (1/2, (1/2)k \cdot \sigma_x) \), where \( k \cdot \sigma_x \) is the standard action of \( \text{SO}(n-1) \) on \( \mathbb{R}^{n-1} \). The scalar product takes the form

\[
(x, y) = \frac{1}{2} (x_1 + x_2)(y_1 + y_2) + \frac{1}{2} (x_1 - x_2)(y_1 - y_2) \langle \sigma_x, k \cdot \sigma_x \rangle.
\]  

(4.7)

Now, the \( K \)-Bessel function can be defined by

\[
K_\nu(x) = \int_\Omega e^{-\text{tr}(y^{-1}) - \langle x, y \rangle} (\det y)^{\nu - n/2} dy,
\]  

(4.8)

where \( \Omega = \{x \in A/ \text{tr}(x) > 0 \text{ and } \det x > 0 \} \) is the cone of positivity of \( A \). After a change of variables, we can show that

\[
K_\nu(x) = (\det x)^{-\nu} K_{-\nu}(x).
\]  

(4.9)

So, following [1], where it is proved that \( K_\nu \) is a solution of a differential system similar to (1.1), we can write

\[
K_\nu(x) = a_\nu S_{(0,0)}(-t_1, t_2) + b_\nu S_{(0,-\nu)}(-t_1, t_2)
+ c_\nu S_{(-\nu-d/2,0)}(-t_1, t_2) + d_\nu S_{(\nu-d/2,-\nu)}(-t_1, t_2)
\]  

(4.10)
(here, \( d = n - 2 \)). According to (4.9), we have \( a_\nu = b_{-\nu} \) and \( c_\nu = d_{-\nu} \). For suitable \( \nu \), the following limit holds (see [2] for more information on \( \Gamma_\Omega \), the gamma function of the cone \( \Omega \)):

\[
\lim_{x \to 0} K_\nu(x) = \Gamma_\Omega(-\nu) = (2\pi)^{(n-2)/2} \Gamma(-\nu) \Gamma \left( -\nu - \frac{n-2}{2} \right),
\]

so

\[
a_\nu = b_{-\nu} = (2\pi)^{(n-2)/2} \Gamma(-\nu) \Gamma \left( -\nu - \frac{n-2}{2} \right)
\]

according to the behaviour of the solutions \( S(\lambda_1, \lambda_2) \). To determine \( c_\nu \) (and then \( d_\nu \)), we take \( x \neq 0 \) on the boundary of \( \Omega \). If \( x = 2\xi \hat{e}_1 \), then the integral representation of \( K_\nu \) takes the explicit form

\[
K_\nu(2\xi \hat{e}_1) = C \int_{SO(n-1)} \int_{y_1 > y_2 > 0} e^{-(1/y_1 + 1/y_2 + \xi(y_1 + y_1)) - \xi(y_1 - y_2)(\sigma_x \cdot \sigma_x)} \times (y_1 y_2)^{\nu - n/2} (y_1 - y_2)^{n-2} dk \, dy_1 \, dy_2,
\]

where \( C \) is a constant (see [2, Theorem VI.2.3, page 104] for the integration formula in polar coordinates in \( \Omega \)). In the particular case of rank-2 Jordan algebras, we have \( C = 2^{2-n/2} \pi^{(n-1)/2} / \Gamma((n-1)/2) \). Now, after integration over \( SO(n-1) \), we obtain

\[
K_\nu(2\xi \hat{e}_1) = C \int_{y_1 > y_2 > 0} e^{-(1/y_1 + 1/y_2 + \xi(y_1 + y_1))} (y_1 y_2)^{\nu - n/2} \times (y_1 - y_2)^{n-2} \ \left( -\frac{n-1}{2}; \frac{\xi^2 (y_1 - y_2)^2}{4} \right) dy_1 \, dy_2.
\]

Then, the evaluation of the (one variable) Mellin transform of \( K_\nu(2\xi \hat{e}_1) \) gives

\[
M(K_\nu(2(\cdot) \hat{e}_1))(s) = \int_0^\infty K_\nu(2\xi \hat{e}_1) \xi^{s-1} \, d\xi
\]

\[
= C \Gamma(s) \int_{y_1 > y_2 > 0} e^{-(1/y_1 + 1/y_2)} (y_1 y_2)^{\nu - n/2} (y_1 - y_2)^{n-2} \times (y_1 + y_2)^{-s} \ \left( -\frac{s}{2}, \frac{s+1}{2}; \frac{y_1 - y_2}{y_1 + y_2} \right) dy_1 \, dy_2.
\]
This last integral can be computed after making the change \( y_1 = re^{\theta} \) and \( y_2 = re^{-\theta} \) with \( r, \theta > 0 \); so

\[
M(K_\nu)(s) = 2^{n-1-s} C \Gamma(s) \int_0^\infty \int_0^\infty e^{2\cosh \theta/r} r^{2\nu-s-1} (\sinh \theta)^{n-2} (\cosh \theta)^{-s} \times \frac{s\cdot s+1}{2}\cdot \frac{2}{n-1} + \frac{1}{2}; (\tanh \theta)^2 d\theta \]

\[
= 2^{n+2(\nu-s)} C \Gamma(s) \Gamma(s-2\nu) \int_0^\infty (\sinh \theta)^{n-2} (\cosh \theta)^{2(\nu-s)} \times \left( \frac{s\cdot s+1}{2}\cdot \frac{2}{n-1} + \frac{1}{2}; (\tanh \theta)^2 d\theta \right)
\]

\[
= 2^{n+2(\nu-s)} C \frac{\Gamma(s) \Gamma(s-2\nu) \Gamma(s-\nu+1-n/2) \Gamma((n-1)/2)}{\Gamma(s-\nu+1/2)} \times \left( \frac{s\cdot s+1}{2}\cdot \frac{2}{n-1} + \frac{1}{2}; (\tanh \theta)^2 d\theta \right)
\]

(4.16)

and finally

\[
M(K_\nu(2(\cdot\hat{e}_1)))(s) = (2\pi)^{(n-2)/2} \Gamma(-\nu) \frac{\Gamma(s)\Gamma(s-\nu-(n-2)/2)}{2^s}.
\]

(4.17)

So, we can write

\[
K_\nu(\xi\hat{e}_1) = (2\pi)^{(n-2)/2} \Gamma(-\nu) k_{\nu+(n-2)/2}(\xi)
\]

(4.18)

according to (4.4). Finally, using (4.3) and the expression of \( S(\lambda_1, \lambda_2) \) in terms of \( j_\nu \) when \( x = 2\xi\hat{e}_1 \), we obtain

\[
c_\nu = d_{-\nu} = (2\pi)^{(n-2)/2} \Gamma(-\nu) \Gamma(\nu + \frac{n-2}{2}).
\]

(4.19)

5. Conclusion. The resolution of the recurrence systems was possible because each one contains at least one equation with two coefficients of the series. Unfortunately, in the higher rank, such a situation does not occur. But we conjecture that a recurrence on the rank exists. We expect also that a similar situation is possible for the systems satisfied by the multivariate hypergeometric functions \( \, _{1}F_{1} \) and \( \, _{2}F_{1} \).

For the \( K \)-Bessel function in the case \( r = 3 \), there is four nonequivalent classes of the Euclidean Jordan algebra. So, we think that we have to perform case-by-case calculations, and the essential difficulty arises in the evaluation of
the integral over the automorphism group of the Jordan algebra-like formulas (4.13) and (4.14). This will be the subject of another paper.

REFERENCES


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