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SOME VERSIONS OF ANDERSON'S AND MAHER'S INEQUALITIES II

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We are interested in the investigation of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel.

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1. Introduction. Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H into itself. Given $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \mapsto B(H)$ by $\delta_{A,B}(X) = AX - XB$ and the elementary operator derivation $\Delta_{A,B} : B(H) \mapsto B(H)$ by $\Delta_{A,B}(X) = AXB - X$. Denote $\delta_{A,A} = \delta_A$, $\Delta_{A,A} = \Delta_A$.

In [1, Theorem 1.7], Anderson shows that if A is normal and commutes with T, then, for all $X \in B(H)$,

$$\left\|T + \delta_A(X)\right\| \geq \|T\|. \tag{1.1}$$

It is shown in [9] that if the pair (A, B) has the Fuglede-Putnam property (in particular, if A and B are normal operators) and $AT = TB$, then, for all $X \in B(H)$,

$$\left\|T + \delta_{A,B}(X)\right\| \geq \|T\|. \tag{1.2}$$

Duggal [3] showed that the above inequality (1.2) is also true when $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. The related inequality (1.1) was obtained by the author [10], showing that if the pair (A, B) has the Fuglede-Putnam property $(\mathrm{FP})_{C_p}$, then

$$\left\|T + \delta_{A,B}(X)\right\|_p \geq \|T\|_p \tag{1.3}$$

for all $X \in B(H)$, where C_p is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\|\cdot\|_p$ is its norm for all $X \in B(H)$ and for all $T \in C_p \cap \ker \delta_{A,B}$. In all of the above results, A was not arbitrary. In fact, certain normality-like assumptions have been imposed on A. A characterization of $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\delta_A | C_p)$ (the range of $\delta_A | C_p$) for a general operator A, has been carried out by Kittaneh [6], showing that if T has the polar decomposition $T = U|T|$, then

$$\|T + \delta_A(X)\|_p \geq \|T\|_p \tag{1.4}$$

for all $X \in C_p$ $(1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker\delta_A$. By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if T has the polar decomposition $T = U|T|$, then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \tag{1.5}$$

for all $X \in C_p$ $(1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker\delta_{B,A}$. In Sections 1, 2, 3, and 4, we prove these results in the case where we consider $E_{A,B}$ instead of $\delta_{A,B}$, which leads us to prove that if $T \in C_p$ and $\ker E_{A,B} \subseteq \ker E^*_{A,B}$, then

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \tag{1.6}$$

for all $X \in C_p$ $(1 < p < \infty)$ if and only if $T \in \ker E_{A,B}$. In Sections 5, 6, and 7, we minimize the map $\|S + E_{A,B}(X)\|_p$ and we classify its critical points.

2. Preliminaries. Let $T \in B(H)$ be compact and let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular values of T, that is, the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_j(T)^p\right]^{1/p} = \left[\operatorname{tr}(T)^p\right]^{1/p}, \quad 1 \leq p < \infty, \tag{2.1}$$

where tr denotes the trace functional. Hence, C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\| \tag{2.2}$$

denoting the usual operator norm. For the general theory of the Schatten p-classes, the reader is referred to [7, 11].

Recall that the norm $\|\cdot\|$ of the B-space V is said to be Gateaux differentiable at nonzero elements $x \in V$ if

$$\lim_{t\to 0,\, t\in\mathbb{R}} \frac{\|x+ty\| - \|x\|}{t} = \Re D_x(y) \tag{2.3}$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of reals, $\Re$ denotes the real part, and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at x implies that x is a smooth point of the sphere of radius $\|x\|$.

It is well known (see [7] and the references therein) that, for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore, every nonzero $T \in C_p$ is a smooth point and, in this case, the support functional of T is given by

$$D_T(X) = \operatorname{tr}\left[\frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}}\right] \tag{2.4}$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T.

Definition 2.1. Let E be a complex Banach space. We define the orthogonality in E. We say that $b \in E$ is orthogonal to $a \in E$ if, for all complex λ, there holds

$$\|a + \lambda b\| \geq \|a\|. \tag{2.5}$$

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in C\}$ is disjoint with the open ball $K(0, \|a\|)$, that is, if and only if this complex line is a tangent one. Note that if b is orthogonal to a, then a needs not be orthogonal to b. If E is a Hilbert space, then from (2.5), it follows that $\langle a, b\rangle = 0$, that is, orthogonality in the usual sense.

3. The elementary operators $AXB - CXD$

Lemma 3.1. *Let $A, B \in B(H)$. The following statements are equivalent:*

(1) *the pair (A, B) has the property $(FP)_{C_p}$, $1 \leq p < \infty$;*

(2) *if $AT = TB$, where $T \in C_p$, then $\overline{R(T)}$ reduces A, $\ker(T)^\perp$ reduces B, and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are normal operators.*

Proof. (1)⇒(2). Since C_p is a bilateral ideal and $T \in C_p$, then $AT \in C_p$. Hence as $AT = TB$ and (A, B) satisfies $(FP)_{C_p}$, $A^*T = TB^*$, and so, $\overline{R(T)}$ and $\ker(T)^\perp$ are reducing subspaces for A and B, respectively. Since $A(AT) = (AT)B$ implies that $A^*(AT) = (AT)B^*$ by $(FP)_{C_p}$ and the equality $A^*T = TB^*$ implies that $A^*AT = AA^*T$, thus we see that $A|_{\overline{R(T)}}$ is normal. Clearly, (B^*, A^*) satisfies $(FP)_{C_p}$ and $B^*T^* = T^*A^*$. Therefore, it follows from the above argument that $B^*|_{\overline{R(T^*)}} = B|_{\ker(T)^\perp}$ is normal.

(2)⇒(1). Let $T \in C_p$ such that $AT = TB$. Taking the two decompositions of H, $H_1 = H = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $H_2 = H = \ker(T)^\perp \oplus \ker T$, then we can write A and B on H_1 into H_2, respectively:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{3.1}$$

where A_1 and B_1 are normal operators. Also we can write T and X on H_2 into H_1:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{3.2}$$

It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since A_1 and B_1 are normal operators, then, by applying the Fuglede-Putnam theorem, we obtain $A_1^*T_1 = T_1B_1^*$, that is, $A^*T = TB^*$. □

Theorem 3.2. *Let $A,B \in B(H)$. If A and B are normal operators, then*

$$||S-(AX-XB)||_p \geq \|S\|_p \tag{3.3}$$

for all $X \in C_p$ and for all $S \in \ker\delta_{A,B} \cap C_p$ $(1 \leq p < \infty)$.

Proof. Let $S = U|S|$ be the polar decomposition of S, where U is an isometry such that $\ker U = \ker|S|$. Since

$$||U^*S||_p \leq ||U^*||_p \|S\|_p = \|S\|_p \tag{3.4}$$

for all $S \in C_p$, then

$$||S-(AX-XB)||_p^p \geq ||U^*[S-(AX-XB)]||_p^p = |||S|-U^*(AX-XB)||_p^p, \tag{3.5}$$

and we have

$$|||S|-U^*(AX-XB)||_p^p \geq \sum_n |\langle[|S|-U^*(AX-XB)]\varphi_n,\varphi_n\rangle|^p \tag{3.6}$$

for any orthonormal basis $\{\varphi_n\}_{n\geq 1}$ of H. Since $AS = SB$, and A and B are normal operators, it follows from the Fuglede-Putnam theorem that $S^*A = BS^*$. Consequently, $S^*AS = BS^*S$ or $S^*SB = BS^*S$, that is, $B|S| = |S|B$. Since $|S|$ is a compact normal operator and commutes with B, there exists an orthonormal basis $\{f_k\} \cup \{g_m\}$ of H such that $\{f_k\}$ consists of common eigenvectors of B and $|S|$, and $\{g_m\}$ is an orthonormal basis of $\ker|S|$. Since $\{f_k\}$ is an orthonormal basis of the normal operator B, then there exists a scalar α_k such that $Bf_k = \alpha_k f_k$ and $B^*f_k = \overline{\alpha}_k f_k$. Consequently,

$$\begin{aligned}\langle U^*(AX-XB)f_k,|S|f_k\rangle &= \langle S^*(AX-XB)f_k,f_k\rangle \\ &= \langle (B(S^*X)-(S^*X)B)f_k,f_k\rangle = 0,\end{aligned} \tag{3.7}$$

that is, $\langle U^*(AX-XB)f_k,f_k\rangle = 0$.

In (3.6) take $\{\varphi_n\} = \{f_k\} \cup \{g_m\}$ as an orthonormal basis of H, then

$$\begin{aligned}&\sum_n |\langle[|S|-U^*(AX-XB)]\varphi_n,\varphi_n\rangle|^p \\ &\quad \geq \sum_k |\langle |S|f_k,f_k\rangle|^p + \sum_m |\langle U^*(AX-XB)g_m,g_m\rangle|^p \\ &\quad \geq \sum_k |\langle |S|f_k,f_k\rangle|^p = \|S\|_p^p.\end{aligned} \tag{3.8}$$

□

Lemma 3.3. *Let $A,B \in B(H)$ satisfying $(FP)_{C_p}$. Then*

$$\|S + AX - XB\|_p^p \geq \|S\|_p^p \tag{3.9}$$

for every operator $S \in \ker\delta_{A,B} \cap C_p$ $(1 < p < \infty)$ and for all $X \in C_p$.

Proof. If the pair (A,B) satisfies the $(FP)_{C_p}$ property, then $\overline{R(S)}$ reduces A, $\ker^{\perp} S$ reduces B, and $A|_{\overline{R(S)}}$ and $B|_{\ker^{\perp} S}$ are normal operators. Letting $S_0 : \ker^{\perp} S \to \overline{R(S)}$ be the quasiaffinity defined by setting $S_0 x = Sx$ for each $x \in \ker^{\perp} S$, it results that $\delta_{A_1,B_1}(S_0) = \delta_{A_1^*,B_1^*}(S_0) = 0$. Let $A = A_1 \oplus A_2$, with respect to $H = \overline{R(S)} \oplus \overline{R(S)}^{\perp}$, $A = B_1 \oplus B_2$, with respect to $H = \ker(S)^{\perp} \oplus \ker S$, and $X : \overline{R(S)} \oplus \overline{R(S)}^{\perp} \to \ker(S)^{\perp} \oplus \ker S$ have the matrix representation

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{3.10}$$

Then we have

$$||S - (AX - XB)||_p = \left|\left| \begin{bmatrix} S_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{bmatrix} \right|\right|_p. \tag{3.11}$$

The result of Gohberg and Kreĭn [4] guarantees that

$$||S - (AX - XB)||_p \geq ||S_1 - (A_1X_1 - X_1B_1)||_p. \tag{3.12}$$

Since A_1 and B_1 are two normal operators, then it results from Theorem 3.5 that

$$||S_1 - (A_1X_1 - X_1B_1)||_p \geq ||S_1||_p = \|S\|_p. \tag{3.13}$$

□

Lemma 3.4 [6]. *Let u and v be two elements of a Banach space V with norm $\|\cdot\|$. If u is a smooth point, then $D_u(v) = 0$ if and only if*

$$\|u + zv\| \geq \|u\| \tag{3.14}$$

for all $z \in \mathbb{C}$ (the complex numbers).

Theorem 3.5. *Let $A,B \in B(H)$ and $T \in C_p$ $(1 < p < \infty)$. Then*

$$||T + \delta_{A,B}(X)||_p \geq \|T\|_p \tag{3.15}$$

for all $X \in B(H)$ with $\Delta_{A,B}(X) \in C_p$ if and only if

$$\operatorname{tr}\left(|T|^{p-1}U^*\delta_{A,B}(X)\right) = 0 \tag{3.16}$$

for all such X.

Proof. The theorem is an immediate consequence of equality (2.4) and Lemma 3.4. $\square$

Theorem 3.6. *Let $A, B \in B(H)$ and $T \in C_p$ $(1 < p < \infty)$. Then*

$$\left|\left|T + \delta_{A,B}(X)\right|\right|_p \geq \|T\|_p \tag{3.17}$$

for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^* \in \ker\delta_{B,A}$.

Proof. By virtue of Theorem 3.5, it is sufficient to show that $\operatorname{tr}(\tilde{T}\delta_{A,B}(X)) = 0$ for all $X \in C_p$ if and only if $\tilde{T} \in \ker\delta_{B,A}$.

Choose X to be the rank-one operator $f \otimes g$ for some arbitrary elements f and g in H. Then $\operatorname{tr}(\tilde{T}(AX - XB)) = \operatorname{tr}(B\tilde{T} - \tilde{T}A)X = 0$ implies that $\langle\delta_{B,A}(\tilde{T})f, g\rangle = 0 \Leftrightarrow \tilde{T} \in \ker\delta_{B,A}$.

Conversely, assume that $\tilde{T} \in \ker\delta_{B,A}$, that is, $B\tilde{T} = \tilde{T}A$. Since $\tilde{T}X$ and $\tilde{T}\delta_{B,A}$ are trace classes, then for all $X \in C_p$, we get

$$\begin{aligned} \operatorname{tr}\left(\tilde{T}(AX - XB)\right) &= \operatorname{tr}\left(\tilde{T}AX - \tilde{T}XB\right) = \operatorname{tr}\left(XB\tilde{T} - X\tilde{T}A\right) \\ &= \operatorname{tr}\left(X\delta_{B,A}\left(\tilde{T}\right)\right) = 0. \end{aligned} \tag{3.18}$$

$\square$

Lemma 3.7. *Let $A, B \in B(H)$ and $S \in C_p$ such that $\delta_{A,B}(T) = 0 = \delta^*_{A,B}(T)$. If $A|S|^{p-1}U^* = |S|^{p-1}U^*B$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S, then $A|S|U^* = |S|U^*B$.*

Proof. If $T = |S|^{p-1}$, then

$$ATU^* = TU^*B. \tag{3.19}$$

We prove that

$$AT^nU^* = T^nU^*B \tag{3.20}$$

for all $n \geq 1$. If $S = U|S|$, then

$$\begin{aligned} \ker U = \ker|S| &= \ker|S|^{p-1} = \ker T, \\ (\ker U)^\perp &= (\ker T)^\perp = \overline{R(T)}. \end{aligned} \tag{3.21}$$

This shows that the projection U^*U onto $(\ker T)^\perp$ satisfies $U^*UT = T$ and $TU^*UT = T^2$. By taking the adjoints of (3.19) and since A and B are normal operators applying Fuglede-Putnam theorem, we get $BUT = UTA$ and $AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A$.

Since A commutes with the positive operator T^2, A commutes with its square roots, that is,

$$AT = TA. \tag{3.22}$$

By (3.19) and (3.22) we obtain (3.20). Let $f(t)$ be the map defined on $\sigma(T) \subset \mathbb{R}^+$ by $f(t) = t^{1/(p-1)}$ $(1 < p < \infty)$. Since f is the uniform limit of a sequence

(P_i) of polynomials without constant term (since $f(0) = 0$), it follows from (3.20) that $AP_i(T)U^* = P_i(T)U^*B$. Therefore, $AT^{1/(p-1)}U^* = U^*T^{1/(p-1)}B$. □

Theorem 3.8. *Let A and B be operators in $B(H)$ such that $\delta_{A,B}(T) = 0 = \delta^*_{A,B}(T)$. Then $T \in \ker\Delta_{A,B} \cap C_p$ if and only if*

$$\left\|S + \delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.23}$$

for all $X \in C_p$.

Proof. If $S \in \ker\Delta_{A,B}$, then it follows from Lemma 3.3 that

$$\left\|S + \delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.24}$$

for all $X \in C_p$. Conversely, if

$$\left\|S + \delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.25}$$

for all $X \in C_p$, then, from Theorem 3.6,

$$A|S|^{p-1}U^* = |S|^{p-1}U^*B. \tag{3.26}$$

Since $\delta_{A,B}(S) = 0 = \delta_{A,B}{}^*(S)$,

$$A^*|S|^{p-1}U^* = |S|^{p-1}U^*B^*. \tag{3.27}$$

By taking adjoints, we get

$$AU|S|^{p-1} = U|S|^{p-1}B. \tag{3.28}$$

From Lemma 3.7, it follows that $AU|S| = U|S|B$, that is, $S \in \ker\Delta_{A,B}$. □

Remark 3.9. (1) It is well known that the Hilbert-Schmidt class C_2 is a Hilbert space under the inner product $\langle Y, Z\rangle = \operatorname{tr} Z^*Y$.

We remark here that for the Hilbert-Schmidt norm $\|\cdot\|_2$, the orthogonality result in Theorem 3.8 is to be understood in the usual Hilbert-space sense. Note in the case where $I = C_2$ that

$$\left\|T + \delta_{A,B}(X)\right\|_2^2 = \left\|\delta_{A,B}(X)\right\|_2^2 + \|T\|_2^2 \tag{3.29}$$

for all $X \in C_2$ if and only if $AT^* = T^*B$. This can be seen as an immediate consequence of the fact that

$$R(\delta_{A,B}|C_2)^{\perp} = \ker(\delta_{A,B}|C_2)^* = \ker(\delta_{B^*,A^*}|C_2). \tag{3.30}$$

(2) It is known [2] that if A and B are contractions and $S \in C_p$, then $\delta_{A^*,B^*}(S) = \delta_{A,B}(S) = 0$. Hence

$$\left\|S + \delta_{A,B}(X)\right\|_p \geq \|S\|_p \tag{3.31}$$

holds for all $X \in C_p$ if and only if $S \in \ker(\delta_{A,B}|C_p)$.

(3) If $A = B$, then the following counterexample shows that Theorem 3.8 does not hold if $p < 1$. Take $p = 1/2$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \tag{3.32}$$

where α is real such that $0 < \alpha < 1$. We have

$$S - (AX - XA) = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \tag{3.33}$$

and, for eigenvectors $\beta_1 = 1 - \alpha$, $\beta_2 = 1 + \alpha$. Then

$$\|S - (AX - XA)\|_{1/2} = \left[(1-\alpha)^{1/2} + (1+\alpha)^{1/2}\right]^2 < 4 = \|S\|_{1/2}. \tag{3.34}$$

Corollary 3.10. *Let $A, B \in L(H)$. Then*

$$\|S + AX - XB\|_p \geq \|S\|_p \tag{3.35}$$

if and only if $S \in \ker \delta_{A,B} \cap C_p$ and for all $X \in C_p$, in each of the following cases:

(1) *if $A, B \in L(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$,*

(2) *if A is invertible and B is such that $\|A^{-1}\| \|B\| \leq 1$.*

Proof. The result of Tong [13, Lemma 1] guarantees that the above condition implies that, for all $T \in \ker(\delta_{A,B}|K(H))$, $\overline{R(T)}$ reduces A, $\ker(T)^\perp$ reduces B, and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are unitary operators. Hence it results from Lemma 3.1 that the pair (A,B) has the property $(FP)_{K(H)}$ and the results hold by Theorem 3.8. Here $K(H)$ is the ideal of compact operators.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in H$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$, $B_1 = \|B\|^{-1}B$.

Then $\|A_1 x\| \geq \|x\| \geq \|B_1 x\|$ and the result holds by (1) for all $x \in H$. □

4. Orthogonality and the elementary operators $AXB - CXD$. Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H into itself. Given A, B, C, and D normal operators in $B(H)$ such that $AC = CA$, $BD = DB$, we define the elementary operator $\Psi : B(H) \mapsto B(H)$ by $\Psi(X) = AXB - CXD$. We prove that if $T \in C_p$ $(1 < p < \infty)$, then $\|T + \Phi(X)\|_p \geq \|T\|_p$ if and only if $T \in \ker \Phi$ for all $X \in C_p$.

By the same argument used in the proofs of Theorems 3.5 and 3.6, we prove the following theorems.

Theorem 4.1. *Let $A, B, C, D \in B(H)$ and $T \in C_p$ $(1 < p < \infty)$. Then*

$$\|T + \Psi(X)\|_p \geq \|T\|_p \tag{4.1}$$

for all $X \in B(H)$ *with* $\Psi(X) \in C_p$ *if and only if*

$$\operatorname{tr}\left(|T|^{p-1}U^*\Psi(X)\right) = 0 \tag{4.2}$$

for all such X*.*

Theorem 4.2. *Let* $A,B,C,D \in B(H)$ *and* $T \in C_p$ $(1 < p < \infty)$*. Then*

$$\left\|T+\Psi(X)\right\|_p \geq \|T\|_p \tag{4.3}$$

for all $X \in C_p$ *if and only if* $\tilde{T} = |T|^{p-1}U^* \in \ker\Psi$*.*

Lemma 4.3. *Let* $A,B \in B(H)$ *be normal operators and* $AB = BA$*. Suppose that* $ASB = BSA$*,* $S \in C_p$ $(1 < p < \infty)$*. If*

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \tag{4.4}$$

then

$$AU|S|B = BU|S|A. \tag{4.5}$$

Proof. Assume that $B^{-1} \in B(H)$. Then, from $ASB = BSA$ and $AB = BA$, we get $AB^{-1}S = SB^{-1}A$. Hence, applying the above lemma to the operators AB^{-1}, $B^{-1}A$, and S, we get

$$AB^{-1}U|S|^{p-1} = U|S|^{p-1}B^{-1}A, \tag{4.6}$$

which implies that

$$AB^{-1}U|S| = U|S|B^{-1}A. \tag{4.7}$$

Multiply (4.6) and (4.7) at right and left by B to obtain

$$BAB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}AB \tag{4.8}$$

or

$$ABB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}BA, \tag{4.9}$$

that is,

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \tag{4.10}$$

which implies that

$$AU|S|B = BU|S|A. \tag{4.11}$$

Consider now the case when B is injective, that is, $\ker B = \{0\}$. Let

$$\Delta_n = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n}\right\} \tag{4.12}$$

and let $E_B(\Delta_n)$ be the corresponding spectral projector.

Putting

$$P_n = I - E_B(\Delta_n), \tag{4.13}$$

the subspace P_nH reduces both operators A and B (since they commute and are normal). Hence, with respect to the decomposition

$$H = (I - P_n)H \oplus P_nH,$$

$$A = \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2^{(n)} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1^{(n)} & 0 \\ 0 & B_2^{(n)} \end{bmatrix}, \tag{4.14}$$
$$S = \begin{bmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11}(n) & X_{12}(n) \\ X_{21}(n) & X_{22}(n) \end{bmatrix},$$

it is easy to see that $B_2^{(n)}$ acting on P_nH is invertible. Then, from $ASB = BSA$, it follows that

$$A_2^{(n)} S_{22}(n) B_2^{(n)} = B_2^{(n)} S_{22}(n) A_2^{(n)}, \tag{4.15}$$

and, from $AB = BA$, we get $A_2B_2 = B_2A_2$. Since

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \tag{4.16}$$

according to the first part of the proof, it follows that

$$A_2^{(n)} U \,|\, S_{22}(n) \,|^{\,p-1} B_2^{(n)} = B_2^{(n)} U \,|\, S_{22}(n) \,|^{\,p-1} A_2^{(n)}, \tag{4.17}$$

which implies that

$$A_2^{(n)} U \,|\, S_{22}(n) \,|\, B_2^{(n)} = B_2^{(n)} U \,|\, S_{22}(n) \,|\, A_2^{(n)}, \tag{4.18}$$

so we have $AU|S|B = BU|S|A$. Assume now $\ker A \cap \ker B = \{0\}$.

Then $\ker B$ reduces A and $P_{\ker B} A P_{\ker B}$ is injective. Let $H = \ker B \oplus H_1$ ($H_1 = H \ominus \ker B$). Then we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{4.19}$$

where A_1, B_2 are injective and their ranges are dense in subspaces they act on. We have

$$ASB - BSA = \begin{bmatrix} 0 & A_1S_{12}B_2 \\ -B_2S_{21}A_1 & A_2S_{22}B_2 - B_2S_{22}A_2 \end{bmatrix}. \tag{4.20}$$

Now, if $ASB = BSA$, then $A_2S_{22}B_2 = B_2S_{22}A_2$, $B_2S_{21}A_1 = 0$, and $A_1S_{12}B_2 = 0$, that is, $S_{21} = S_{12} = 0$. It follows that

$$S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}. \tag{4.21}$$

Since $A_2B_2 = B_2A_2$, $A_2S_{22}B_2 = B_2S_{22}A_2$, and B_2 is injective, and we have already proved that

$$A_2U|S_{22}|^{p-1}B_2 = B_2U|S_{22}|^{p-1}A_2 \tag{4.22}$$

implies

$$A_2U|S_{22}|B_2 = B_2U|S_{22}|A_2, \tag{4.23}$$

so we have $AU|S|B = BU|S|A$. □

Let $\Phi(X) = AXB - BXA$. We prove the following theorem.

Theorem 4.4. *Let $A, B \in B(H)$ be normal operators, $AB = BA$, and $S \in C_p$ $(1 < p < \infty)$. Then $S \in \ker\Phi$ if and only if*

$$\|S - (AXB - BXA)\|_p \geq \|S\|_p \tag{4.24}$$

for all $X \in C_p$.

Proof. If $S \in \ker\Phi$, then, from [13, Theorem 3.4], it follows that

$$\|S + \Phi(X)\|_p \geq \|S\|_p \tag{4.25}$$

for all $X \in C_p$. Conversely, if

$$\|S + \Phi(X)\|_p \geq \|S\|_p \tag{4.26}$$

for all $X \in C_p$, then, from Theorem 4.2,

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A. \tag{4.27}$$

Since A and B are normal operators applying Fuglede-Putnam theorem, we get $A^*|S|^{p-1}U^*B^* = B^*|S|^{p-1}U^*A^*$. By taking adjoints, we get $AU|S|^{p-1}B = BU|S|^{p-1}A$.

From Lemma 4.3, it follows that $AU|S|B = BU|S|A$, that is, $S \in \ker\Phi$. □

Let $\Psi(X) = AXB - CXD$.

Theorem 4.5. *Let $A, B, C, D \in B(H)$ be normal operators, $AC = CA$, $BD = DB$, and $S \in C_p$ $(1 < p < \infty)$. Then $S \in \ker\Psi$ if and only if*

$$\|S - (AXB - CXD)\|_p \geq \|S\|_p \tag{4.28}$$

for all $X \in C_p$.

Proof. It suffices to take the Hilbert space $H \oplus H$ and the operators

$$A^{\sim} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \qquad B^{\sim} = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix},$$
$$S^{\sim} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \qquad X^{\sim} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \tag{4.29}$$

and apply Theorem 4.4. □

Remark 4.6. The results of the above theorems can be obtained when the normality of A and B is replaced by some other condition, in particular, if $|A| = |B|$, $|A^*| = |B^*|$. In this case, it suffices to take

$$A^{\sim} = \begin{bmatrix} 0 & A^* \\ B & 0 \end{bmatrix}, \qquad B^{\sim} = \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix},$$
$$S^{\sim} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \qquad X^{\sim} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \tag{4.30}$$

and apply Lemma 4.3 and Theorem 4.4.

5. On minimizing $\|AX - XB - T\|_p^p$.

Maher [8, Theorem 3.2] shows that if A is normal and $S \in \ker\delta_A \cap C_p$ $(1 \le p < \infty)$, then the map F_p defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VA = 0$.

In this section, we prove that if the pair (A,B) has the property $(FP)_{C_p}$ (i.e., $AT = TB$, where $T \in C_p$, implies $A^*T = TB^*$), $1 \le p < \infty$, and $S \in \ker\delta_{A,B} \cap C_p$, then the map F_p defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$||S - (AX - XB)||_p^p \ge \|T\|_p^p \tag{5.1}$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. Thus in Halmos' terminology [5], the zero commutator is the commutator approximant in C_p of T. Additionally, we show that if the pair (A,B) has the property $(FP)_{C_p}$ and $S \in \ker\delta_{A,B} \cap C_p$ $(1 < p < \infty)$, then the map F_p has a critical point at W if and only if $AW - WB = 0$, that is, if $\mathcal{D}_W F_p$ is the Frechet derivative at W of F_p, the set $\{W \in B(H) : \mathcal{D}_W F_p = 0\}$ coincides with $\ker\delta_{A,B}$ (the kernel of $\delta_{A,B}$).

Theorem 5.1 [9]. *If $1 < p < \infty$, then the map $F_p : C_p \mapsto \mathbb{R}^+$ defined by $X \mapsto \|X\|_p^p$ is differentiable at every $X \in C_p$ with derivative $\mathcal{D}_X F_p$ given by $\mathcal{D}_X F_p(T) = p\operatorname{Re}\operatorname{tr}(|X|^{p-1}U^*T)$, where tr denotes trace, $\operatorname{Re} z$ is the real part of a complex number z, and $X = U|X|$ is the polar decomposition of X. If $\dim H < \infty$, then the same result holds for $0 < p \le 1$ at every invertible X.*

Theorem 5.2 [9]. *If $\mathcal{U}$ is a convex set of C_p, $1 < p < \infty$, then the map $X \mapsto \|X\|_p^p$, where $X \in \mathcal{U}$, has at most a global minimizer.*

Definition 5.3. Let $\mathcal{U}(A,B) = \{X \in B(H) : AX - XB \in C_p\}$ and let $F_p : \mathcal{U} \mapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AX - XB)\|_p^p$, where $T \in \ker \delta_{A,B} \cap C_p$, $1 \le p < \infty$.

6. Main results. By simple modifications in the proof of Lemma 3.7, we can prove the following lemma.

Lemma 6.1. *Let $A, B \in B(H)$ and $C \in B(H)$ such that the pair (A,B) has the property $(FP)_{B(H)}$. If $A|S|^{p-1}U^* = |S|^{p-1}U^*B$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S, then $A|S|U^* = |S|U^*B$.*

Theorem 6.2. *Let $A, B \in \mathcal{L}(H)$. If the pair (A,B) has the property $(FP)_{C_p}$ and $S \in C_p$ such that $AS = SB$, then*

(1) *for $1 \le p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AW - WB = 0$;*
(2) *for $1 < p < \infty$, the map F_p has a critical point at W if and only if $AW - WB = 0$;*
(3) *for $0 < p \le 1$ $\dim \mathcal{H} < \infty$ and $S - (AW - WB)$ is invertible, then F_p has a critical point at W if $AW - WB = 0$.*

Proof. Since the pair (A,B) has the property $(FP)_{C_p}$, it follows from Lemma 3.3 that

$$||S - (AX - XB)||_p^p \ge \|S\|_p^p, \tag{6.1}$$

that is, $F_p(X) \ge F_p(W)$.

Conversely, if F_p has a minimum, then

$$||S - (AW - WB)||_p^p = \|S\|_p^p. \tag{6.2}$$

Since $\mathcal{U}$ is convex, the set $\mathcal{V} = \{S - (AX - XB);\ X \in \mathcal{U}\}$ is also convex. Thus Theorem 5.2 implies that

$$S - (AW - WB) = S. \tag{6.3}$$

(2) Let $W, S \in \mathcal{U}$ and let ϕ and φ be two maps defined, respectively, by $\phi : X \mapsto S - (AX - XB)$ and $\varphi : X \mapsto \|X\|_p^p$.

Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h}, \tag{6.4}$$

it follows that

$$\mathcal{D}_W F_p(T) = [\mathcal{D}_{S-(AW-WB)}](TB - AT). \tag{6.5}$$

If W is a critical point of F_p, then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 5.1, we get

$$\begin{aligned}\mathcal{D}_W F_p(T) &= p\operatorname{Re}\operatorname{tr}\left[\left|S-(AW-WB)\right|^{p-1}W^*(TB-AT)\right] \\ &= p\operatorname{Re}\operatorname{tr}\left[Y(TB-AT)\right] = 0,\end{aligned} \tag{6.6}$$

where $S-(AW-WB) = W|S-(AW-WB)|$ is the polar decomposition of the operator $S-(AW-WB)$ and $Y = |S-(AW-WB)|^{p-1}W^*$. An easy calculation shows that $BY - YA = 0$, that is,

$$A\left|S-(AW-WB)\right|^{p-1}W^* = \left|S-(AW-WB)\right|^{p-1}W^*B. \tag{6.7}$$

It follows from Lemma 6.1 that

$$A\left|S-(AW-WB)\right|W^* = \left|S-(AW-WB)\right|W^*B. \tag{6.8}$$

By taking adjoints and since the pair (A,B) has the property $(FP)_{C_p}$, we get $A(T-(AW-WB)) = (T-(AW-WB))B$. Then $A(AW-WB) = (AW-WB)B$. Hence

$$AW - WB \in R(\delta_{A,B}) \cap \ker\delta_{A,B}. \tag{6.9}$$

By the same argument used in the proof of Lemma 6.1 we can prove that

$$\left\|S-(AX-XB)\right\| \geq \|S\| \tag{6.10}$$

for all $X \in B(H)$ and for all $T \in B(H)$ and it results that $AW - WB = 0$.

Conversely, if $AW = WB$, then W is a minimum, and since F_p is differentiable, then W is a critical point.

(3) Suppose that $\dim H < \infty$. If $AW - WB = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$ taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of S. Since $AS = SB$ implies that $S^*A = BS^*$, then $S^*AS = BS^*S$, and this implies that $|S|^2B = B|S|^2$ and $|S|B = B|S|$.

Since $S^*A = BS^*$, that is, $|S|U^*A = B|S|U^*$, then $|S|(U^*A - BU^*) = 0$, and since $B|S|^{p-1} = |S|^{p-1}B$, then

$$BY - YA = B|S|^{p-1}U^* - |S|^{p-1}U^*A = |S|^{p-1}(BU^* - U^*A) \tag{6.11}$$

so that $BY - YA = 0$ and $\operatorname{tr}[(BY-YA)T] = 0$ for every $T \in B(H)$. Since $S = S-(AW-WB)$, then

$$\begin{aligned}0 &= \operatorname{tr}[YTB - YAT] = \operatorname{tr}\left[Y(TB-AT)\right] \\ &= p\operatorname{Re}\operatorname{tr}\left[Y(TB-AT)\right] = p\operatorname{Re}\operatorname{tr}\left[|S|^{p-1}U^*(TB-AT)\right] \\ &= (\mathcal{D}_T\phi)(TB-AT) = (\mathcal{D}_W F_p)(T).\end{aligned} \tag{6.12}$$

□

Remark 6.3. In Theorem 6.2, the implication "W is a critical point implies $AW - WB = 0$" does not hold in the case $0 < p \le 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \le t < \infty$, is only valid for $1 < p < \infty$.

7. On minimizing $\|T - (AXB - CXD)\|_p^p$. In this section, we consider the elementary operator $\Phi(X) = AXB - CXD$ and we prove that if $AC = CA$, $BD = DB$, and $ASB = CSD$, $S \in C_p$, then, for $1 < p < \infty$, the map F_p defined by $F_p(X) = \|T - (AXB - CXD)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AVB - CVD = 0$. In other words, we have $\|T - (AXB - CXD)\|_p^p \ge \|T\|_p^p$ if, and for $1 < p < \infty$ only if, $AVB - CVD = 0$. Additionally, we show that if $AC = CA$, $BD = DB$, and $T \in \ker\Delta_{A,B} \cap C_p$, $1 < p < \infty$, then the map F_p has a critical point at W if and only if $AWB - CWD = 0$, that is, if $\mathfrak{D}_W F_p$ is the Frechet derivative at W of F_p, the set $\{W \in B(H) : \mathfrak{D}_W F_p = 0\}$ coincides with $\ker\Phi$ (the kernel of Φ).

Definition 7.1. Let $\mathcal{U}(A,B) = \{X \in B(H) : AXB - CXD \in C_p\}$ and let $F_p : \mathcal{U} \mapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AXB - CXD)\|_p^p$, where $T \in \ker\Phi \cap C_p$, $1 \le p < \infty$.

The proof of the following lemma is similar to the proof of Lemma 4.3.

Lemma 7.2. *Let $A, B \in B(H)$ be normal commuting operators. Suppose that $ASB = BSA$, $S \in C_p$ $(1 < p < \infty)$. If*

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A, \tag{7.1}$$

then

$$A|S|U^*B = B|S|U^*A. \tag{7.2}$$

Theorem 7.3. *Let $A, B, C, D \in B(H)$ be normal operators such that $AC = CA$ and $BD = DB$. Assume that $ASB = CSD$, $S \in C_p$ $(1 < p < \infty)$. If $A|S|^{p-1}U^*B = C|S|^{p-1}U^*D$, then $A|S|U^*B = C|S|U^*D$.*

Proof. It suffices to take the Hilbert space $H \oplus H$ and the operators

$$A^{\sim} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \qquad B^{\sim} = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \qquad S^{\sim} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \tag{7.3}$$

and apply Lemma 7.2. □

Theorem 7.4. *Let $A, B, C, D \in B(H)$ be normal operators, $AC = CA$, and $BD = DB$. Suppose that $ASB = CSD$, $S \in C_p$. Then, for $1 \le p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AWB - CWD = 0$.*

Proof. If $AC = CA$, $BD = DB$, and $ASB = CSD$, $S \in C_p$, then, for $1 < p < \infty$, the result of Turnšek [14, Theorem 3.4] guarantees that

$$||T - (AXB - CXD)||_p^p \geq \|T\|_p^p, \tag{7.4}$$

that is, $F_p(X) \geq F_p(W)$. Conversely, if F_p has a minimum, then

$$||T - (AWB - CWD)||_p^p = \|S\|_p^p. \tag{7.5}$$

Since $\mathcal{U}$ is convex, then the set $\mathcal{V} = \{T - (AXB - CXD);\ X \in \mathcal{U}\}$ is also convex. Thus Theorem 5.2 implies that $S - (AWB - CWD) = S$. □

Theorem 7.5. *Let A, B, C, and D be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker \Phi \cap C_p$, then, for $1 < p < \infty$, the map F_p has a critical point at W if and only if $AWB - CWD = 0$.*

Proof. Let $W, S \in \mathcal{U}$ and let ϕ and φ be two maps defined, respectively, by $\phi : X \mapsto S - (AXB - CXD)$ and $\varphi : X \mapsto \|X\|_p^p$. Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h}, \tag{7.6}$$

it follows that $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S-(AWB-CWD)}](BTA - DTC)$. If W is a critical point of F_p, then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 5.1, we get

$$\begin{aligned} \mathcal{D}_W F_p(T) &= p \operatorname{Re} \operatorname{tr}\left[|S - (AWB - CWD)|^{p-1} W^* (BTA - DTC) \right] \\ &= p \operatorname{Re} \operatorname{tr}\left[Y(BTA - DTC) \right] = 0, \end{aligned} \tag{7.7}$$

where $S - (AWB - CWD) = W|S - (AWB - CWD)|$ is the polar decomposition of the operator $S - (AWB - CWD)$ and $Y = |S - (AWB - CWD)|^{p-1} W^*$. An easy calculation shows that $BYA - DYC = 0$, that is,

$$A|S - (AWB - CWD)|^{p-1} W^* B = C|S - (AWB - CWD)|^{p-1} W^* D. \tag{7.8}$$

It follows from Theorem 7.3 that

$$A|S - (AWB - CWD)| W^* B = C|S - (AWB - CWD)| W^* D. \tag{7.9}$$

By taking adjoints and since A and B are normal operators, applying Fuglede-Putnam theorem, we get $A(T - (AWB - CWD))B = C(T - (AWB - CWD))D$. Then $A(AW - WB)B = C(AWB - CWD)D$. Hence $AWB - CWD \in R(\Phi) \cap \ker \Phi$. By the same argument used in the proof of [13, Theorem 3.4], we can prove that

$$||T - (AXB - CXD)|| \geq \|T\| \tag{7.10}$$

for all $T \in B(H)$. Hence $AWB - CWD = 0$.

Conversely, if $AWB = CWD$, then W is a minimum, and since F_p is differentiable, then W is a critical point. $\square$

Theorem 7.6. *Let A, B, C, and D be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker\Phi \cap C_p$, $0 < p \le 1$, $\dim H < \infty$, and $S - (AWB - CWD)$ is invertible, then F_p has a critical point at W if $AWB - CWD = 0$.*

Proof. Suppose that $\dim H < \infty$. If $AWB - CWD = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \le 1$. Taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of S, choose X to be the rank-one operator $f \otimes g$ for some arbitrary elements f and g in $H \oplus H$. Then $\operatorname{tr}(Y(AXB - CXD)) = \operatorname{tr}(AYB - CYD)X = 0$ implies that $\langle \Psi(Y)f, g\rangle = 0 \Leftrightarrow Y \in \ker\Phi$, that is, $AYB - CYD = 0$ and $\operatorname{tr}[(DYC - AYB)T] = 0$ for every $T \in B(H)$. Since $S = S - (AWB - CWD)$, then

$$\begin{aligned}
0 &= \operatorname{tr}[YDTC - YATB] = \operatorname{tr}\left[Y(DTC - ATB)\right] \\
&= p\operatorname{Re}\operatorname{tr}\left[Y(DTC - ATB)\right] = p\operatorname{Re}\operatorname{tr}\left[|S|^{p-1}U^*(DTC - ATB)\right] \\
&= (\mathcal{D}_T\phi)(DTC - ATB) = (\mathcal{D}_W F_p)(T).
\end{aligned} \tag{7.11}$$

$\square$

Remark 7.7. The set $\mathcal{S} = \{X : AXB - CXD \in C_p\}$ contains C_p; if $X \in C_p$, then $X \in \mathcal{S}$ and, for example, $I \in \mathcal{S}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems 7.3, 7.4, 7.5, and 7.6 hold for all $X \in B(H)$.

For $n > 2$ the generalization of the above results to the elementary operators $\sum_{i=1}^{n} A_iXB_i$ is not possible. In [12], Shul'man stated that there exists a normally represented elementary operator of the form $\sum_{i=1}^{n} A_iXB_i$ with $n > 2$ such that $\operatorname{asc} E > 1$, that is, the range and kernel have no trivial intersection.

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