DERIVATIONS ON BANACH ALGEBRAS

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Let $D$ be a derivation on a Banach algebra; by using the operator $D^2$, we give necessary and sufficient conditions for the separating ideal of $D$ to be nilpotent. We also introduce an ideal $M(D)$ and apply it to find out more equivalent conditions for the continuity of $D$ and for nilpotency of its separating ideal.

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1. Introduction. Let $A$ be a Banach algebra. By a derivation on $A$, we mean a linear mapping $D : A \rightarrow A$, which satisfies $D(ab) = aD(b) + D(a)b$ for all $a$ and $b$ in $A$. The separating space of $D$ is the set

\[ S(D) = \{ a \in A : \exists \{ a_n \} \subset A; a_n \rightarrow 0, D(a_n) \rightarrow a \}. \]  

(1.1)

The set $S(D)$ is a closed ideal of $A$ which, by the closed-graph theorem, is zero if and only if $D$ is continuous.

**Definition 1.1.** A closed ideal $J$ of $A$ is said to be a separating ideal if, for each sequence $\{ a_n \}$ in $A$, there is a natural $N$ such that

\[ (Ja_n \cdots a_1) = (Ja_N \cdots a_1) \quad (n \geq N). \]  

(1.2)

The separating space of a derivation on $A$ is a separating ideal [2, Chapter 5]; it also satisfies the same property for the left products.

The following assertions are of the most famous conjectures about derivations on Banach algebras:

- (C1) every derivation on a Banach algebra has a nilpotent separating ideal;
- (C2) every derivation on a semiprime Banach algebra is continuous;
- (C3) every derivation on a prime Banach algebra is continuous;
- (C4) every derivation on a Banach algebra leaves each primitive ideal invariant.

Clearly, if (C1) is true, then the same for (C2) and (C3). Mathieu and Runde in [5] proved that (C1), (C2), and (C3) are equivalent. The conjecture (C4) is known as the noncommutative Singer-Wermer conjecture, and it has been proved in [1] that if each of the conjectures (C1), (C2), or (C3) hold, then (C4) is also true. The conjectures (C1), (C2), and (C3) are still open even if $A$ is assumed
to be commutative, but (C4) is true in the commutative case, see [7]. These
conjectures are also related to some other famous open problems; the reader
is referred to [1, 3, 4, 5, 9] for more details.

In the next section, we deal with (C1), and although, for a derivation \( D \) on a
Banach algebra, the operators \( D^n, n = 2, 3, \ldots, \) are more complicated, by con-
sidering \( D^2 \), we easily give some equivalent conditions for \( S(D) \) to be nilpotent.
As a consequence, we reprove some of the results in [8]. At the end of the next
section, we introduce an ideal related to a derivation and apply it to obtain
some equivalent conditions for continuity of \( D \) and for nilpotency of \( S(D) \).

We recall that \( S(D) \) is nilpotent if and only if \( S(D) \cap R \) is nilpotent, see [1, Lemma 4.2].

2. The results. From now on, \( A \) is a Banach algebra, and \( R \) and \( L \) denote
the Jacobson radical and the nil radical of \( A \), respectively, (see [6, Chapter 4]
for definitions). Note that \( D \) is a derivation on \( A \), and \( S(D) \) is the separating
ideal of \( D \). If \( B_i \)'s, \( i = 1, 2, \ldots, n \), are subsets of \( A \), then \( B_1B_2 \cdots B_n \) denotes the
linear span of the set \( \{ b_1b_2 \cdots b_n : b_i \in B_i, \text{ for } i = 1, 2, \ldots, n \} \), and if all of \( B_i \)’s
coincide with each other, we denote this set by \( B^n \).

**Theorem 2.1.** Let \( J \) be a closed left ideal of \( A \). Then, \( S(D) \cap J \) is nilpotent if
and only if \( D^2 \mid \bigcap_{n=1}^{\infty} (S(D) \cap J)^n \) is continuous.

**Proof.** Suppose that \( D^2 \) is continuous on \( \bigcap_{n=1}^{\infty} (S(D) \cap J)^n \). Consider \( a \) in
\( S(D) \cap J \), then for each \( n \in \mathbb{N} \), \( a^n \in (S(D) \cap J)^n \), and since \( S(D) \) is a separating
ideal, there exists \( N \in \mathbb{N} \) such that
\[
S(D)an = S(D)a^n (n \geq N).
\]
Hence, by the Mittag-Leffler theorem [2, Theorem A.1.25] and the fact that
\( S(D)a^n \subseteq (S(D) \cap J)^n \), we have
\[
S(D)a^n = \bigcap_{n=1}^{\infty} S(D)a^n = \bigcap_{n=1}^{\infty} (S(D) \cap J)^n.
\]
Now, let \( \{ x_n \} \subseteq A, x_n \to 0, \) and \( D(x_n) \to a^{N+1} \). Take \( y_n = x_na^{N+1} \), then
\( y_n \in (S(D)a^n \subseteq \bigcap_{n=1}^{\infty} (S(D) \cap J)^n \), \( y_n \to 0 \), and \( D(y_n) \to a^{2(N+1)} \), and by the
hypothesis, \( D^2(y_n) \to 0 \) and \( D^2(y_n^2) \to 0 \). On the other hand,
\[
D^2(y_n^2) = y_nD^2(y_n) + 2D(y_n)^2 + D(y_n)y_n \to 2a^{4(N+1)}.
\]
Therefore, \( a^{4N+4} = 0 \), that is, \( S(D) \cap J \) is a nil and hence a nilpotent ideal by
closedness [6, Theorem 4.4.11]. The converse is trivial. \( \square \)
Remark 2.2. (i) Note that in Theorem 2.1, we can replace $J$ by a right ideal, see [2, Theorem 5.2.24].

(ii) The argument of Theorem 2.1 shows that if $J$ is not assumed to be closed and if $D^2$ is continuous on $\bigcap_{n=1}^{\infty} (S(D) \cap J)^n$, then $S(D) \cap J$ will be a nil ideal.

Corollary 2.3. The set $S(D)$ is nilpotent if and only if $D^2 \Big|_{\bigcap_{n=1}^{\infty} (S(D) \cap R)^n}$ is continuous.

Proof. If $S(D)$ is nilpotent, then the result is obvious. Conversely, by Theorem 2.1, $S(D) \cap R$ is nilpotent, and by [1, Lemma 4.2], $S(D)$ is nilpotent.

Corollary 2.4. If $\dim(\bigcap_{n=1}^{\infty} (S(D) \cap R)^n) < \infty$, then $S(D)$ is nilpotent.

The assertions of the following theorem were proved by Villena in [8], see also [9, Theorem 4.4]. Using Theorem 2.1, we can reprove them in a different way.

Theorem 2.5. The derivation $D$ is continuous if one of the following assertions hold:

(a) $A$ is semiprime and $\dim(R \cap (\bigcap_{n=1}^{\infty} A^n)) < \infty$;

(b) $A$ is prime and $\dim(\bigcap_{n=1}^{\infty} (aA \cap R)^n) < \infty$ for some $a \in A$ with $a^2 \neq 0$;

(c) $A$ is an integral domain and $\dim(\bigcap_{n=1}^{\infty} (aA \cap R)^n) < \infty$ for some nonzero $a \in A$.

Proof. (a) By Corollary 2.4, $S(D)$ is nilpotent, and since $A$ is semiprime, $D$ is continuous.

(b) Without loss of generality, we may assume that $A$ has an identity. By assumption, $\bigcap_{n=1}^{\infty} (aA \cap R \cap S(D))^n$ is finite dimensional; thus, $D^2$ is continuous on this space, and by Remark 2.2(ii), $aA \cap R \cap S(D)$ is a nil right ideal; therefore, $a(S(D) \cap R)$ is a nil right ideal, and by [6, Theorem 4.4.11], $a(S(D) \cap R) \subseteq L = \{0\}$. Thus, $AaA(S(D) \cap R) = \{0\}$, where $AaA$ is the ideal generated by $a$. Since $a^2 \neq 0$ and $A$ is prime, it follows that $S(D) \cap R = \{0\}$ and hence $S(D) \subseteq L = \{0\}$.

(c) The same argument as in (b) shows that $a(S(D) \cap R) = \{0\}$, and since $A$ is an integral domain, $S(D) \cap R = \{0\}$ and $D$ is continuous.

In the sequel, we give other equivalent conditions for $S(D)$ to be nilpotent, but first we introduce the set

$$M(D) = \{ x \in S(D) \cap R : D(x) \in R \}. \quad (2.4)$$

Obviously, $M(D)$ is an ideal of $A$ and $(S(D) \cap R)^2 \subseteq M(D)$. The following theorems show that this ideal can help us to study the continuity of a derivation or nilpotency of its separating ideal.

Theorem 2.6. The derivation $D$ is continuous if and only if $M(D) = \{0\}$. 


**Proof.** Clearly, if $D$ is continuous, then $M(D) = \{0\}$. Conversely, let $M(D) = \{0\}$; then, $(S(D) \cap R)^2 = \{0\}$. Therefore, $(S(D) \cap R)$ and hence $S(D)$ is a nilpotent ideal. Therefore, $S(D) \subseteq I$; we also have $D(L) \subseteq L$ by [1, Lemma 4.1]; thus, $D(S(D)) \subseteq R$, that is, $S(D) \subseteq M(D) = \{0\}$ and $D$ is continuous. \hfill \Box

**Theorem 2.7.** The following assertions are equivalent:
(a) $S(D)$ is nilpotent;
(b) $M(D)$ is a nil ideal;
(c) $\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$.

**Proof.** Clearly, (a) implies (b). Suppose that (b) holds, then $(S(D) \cap R)^2$ is a nil ideal; therefore, $S(D)$ is a nilpotent ideal and (a) holds. Now, if $S(D)$ is nilpotent, then $\bigcap_{n=1}^{\infty} (S(D)^n) = \{0\}$ and this implies (c). Finally, if $\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$, then by Theorem 2.1 and Remark 2.2 $M(D) = M(D) \cap S(D)$ is a nil ideal and (c) implies (b). \hfill \Box

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