SOME CHARACTERIZATIONS OF SPECIALLY MULTIPLICATIVE FUNCTIONS

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A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that $f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)f_A(d)$ for all m and n . For example, the divisor functions and Ramanujan's τ -function are specially multiplicative functions. Some characterizations of specially multiplicative functions are given in the literature. In this paper, we provide some further characterizations of specially multiplicative functions.

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1. Introduction. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(m)f(n) = f(mn) \quad (1.1)$$

whenever $(m, n) = 1$. If (1.1) holds for all m and n , then f is said to be completely multiplicative. A multiplicative function is known if the values $f(p^n)$ are known for all prime numbers p and positive integers n . A completely multiplicative function is known if the values $f(p)$ are known for all prime numbers p .

A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that

$$f(m)f(n) = \sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right)f_A(d) \quad (1.2)$$

for all m and n , or equivalently

$$f(mn) = \sum_{d|(m,n)} f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)\mu(d)f_A(d) \quad (1.3)$$

for all m and n , where μ is the Möbius function. If $f_A = \delta$, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$, then (1.2) reduces to (1.1). Therefore, the class of completely multiplicative functions is a subclass of the class of specially multiplicative functions.

The study of specially multiplicative functions was initiated in [7], and arose in an effort to understand the identity

$$\sigma_\alpha(mn) = \sum_{d|(m,n)} \sigma_\alpha\left(\frac{m}{d}\right) \sigma_\alpha\left(\frac{n}{d}\right) \mu(d) d^\alpha, \quad (1.4)$$

where $\sigma_\alpha(n)$ denotes the sum of the α th powers of the positive divisors of n . Vaidyanathaswamy used the term “quadratic function,” while the term “specially multiplicative function” was coined by Lehmer [3]. For more background information, reference is made to the books by McCarthy [4] and Sivaramakrishnan [6].

The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right). \quad (1.5)$$

The function δ serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1) \neq 0$.

We next review some basic characterizations of specially multiplicative functions, see [4, 6].

PROPOSITION 1.1. *The following statements are equivalent.*

- (1) *The function f is a specially multiplicative function.*
- (2) *The function f is the Dirichlet convolution of two completely multiplicative functions a and b . (In this case $f_A = ab$, the usual product of a and b .)*
- (3) *The function f is a multiplicative function, and for each prime number p ,*

$$f^{-1}(p^n) = 0, \quad n \geq 3. \quad (1.6)$$

(In this case $f_A(p) = f^{-1}(p^2)$ for all prime numbers p .)

- (4) *The function f is a multiplicative function, and for each prime number p , there exists a complex number $g(p)$ such that*

$$f(p^{n+1}) = f(p)f(p^n) - g(p)f(p^{n-1}), \quad n \geq 1. \quad (1.7)$$

(In this case $f_A(p) = g(p)$ for all prime numbers p .)

- (5) *The function f is a multiplicative function, and for each prime number p , there exists a complex number $g(p)$ such that*

$$f(p^n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} [f(p)]^{n-2k} [g(p)]^k, \quad n \geq 0. \quad (1.8)$$

(In this case $f_A(p) = g(p)$ for all prime numbers p .)

REMARK 1.2. Completely multiplicative functions a and b in part 2 need not be unique. The usual product ab , however, is unique. For example, let a, b, c , and d be completely multiplicative functions such that $a(p) = 1$ and $b(p) = 2$ for all prime numbers p , and $c(2) = 2, c(p) = 1, d(2) = 1$, and $d(p) = 2$ for all prime numbers $p \neq 2$. Then $a * b = c * d$, but $a, b \neq c$ and $a, b \neq d$. However, $ab = cd$.

The purpose of this paper is to provide some further characterizations of specially multiplicative functions. As applications, we obtain formulas for the usual products $\sigma_\alpha \phi_\beta$, $\sigma_\alpha \sigma_\beta$, and $\sigma_\alpha \tau$, where ϕ_β is a generalized Euler totient function and τ is Ramanujan's τ -function. The function ϕ_β is given by $\phi_\beta = N^\beta * \mu$, where $N^\beta(n) = n^\beta$ for all n . In particular, we denote $N^1 = N, N^0 = \zeta$, and $\phi_1 = \phi$, where ϕ is the Euler totient function. Ramanujan's τ -function is a specially multiplicative function with $\tau_A = N^{11}$.

In the characterizations, we need the concepts of the unitary convolution and the k th convolute. The unitary convolution of two arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \parallel n} f(d)g\left(\frac{n}{d}\right), \quad (1.9)$$

where $d \parallel n$ means that $d \mid n, (d, n/d) = 1$. The k th convolute of an arithmetical function f is defined as $\Omega_k(f)(n) = f(n^{1/k})$ if n is a k th power, and $\Omega_k(f)(n) = 0$ otherwise.

2. Characterizations

THEOREM 2.1. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$h * f(g * \mu) = fg, \quad (2.1)$$

where h is the specially multiplicative function such that

$$h(p) = f(p), \quad h_A(p) = g(p)f_A(p) \quad (2.2)$$

for all prime numbers p . Conversely, if $f(1) = 1$ and there exist completely multiplicative functions a, b, g , and k such that

$$a * b * f(g * \mu) = fg, \quad (2.3)$$

where

$$a(p) + b(p) = f(p), \quad a(p)b(p) = g(p)k(p), \quad (g * \mu)(n) \neq g(n) \quad (2.4)$$

for all prime numbers p and integers $n (\geq 2)$, then f is a specially multiplicative function with $f_A = k$.

PROOF. By multiplicativity, it suffices to show that (2.1) holds at prime powers, that is,

$$[f(g * \mu)](p^e) = (fg * h^{-1})(p^e) \quad (2.5)$$

for all prime powers p^e . If $e = 1$, then both sides of (2.5) are equal to $f(p)g(p) - f(p)$. Assume that $e \geq 2$. Then

$$\begin{aligned} (fg * h^{-1})(p^e) &= f(p^e)g(p^e) + f(p^{e-1})g(p^{e-1})h^{-1}(p) \\ &\quad + f(p^{e-2})g(p^{e-2})h^{-1}(p^2) \\ &= f(p^e)g(p^e) - f(p^{e-1})g(p^{e-1})f(p) \\ &\quad + f(p^{e-2})g(p^{e-2})g(p)f_A(p). \end{aligned} \quad (2.6)$$

By (1.7), we obtain

$$(fg * h^{-1})(p^e) = f(p^e)g(p^e) - f(p^e)g(p^{e-1}) = f(p^e)(g * \mu)(p^e). \quad (2.7)$$

Thus we have proved (2.5).

To prove the converse, we write (2.3) in the form

$$(f(g * \mu))(n) = (fg * a^{-1} * b^{-1})(n). \quad (2.8)$$

We write $n = p^{e+1}$ ($e \geq 1$) and, after some simplifications, obtain

$$f(p^{e+1}) = f(p^e)f(p) - f(p^{e-1})k(p). \quad (2.9)$$

Therefore, by (1.7), it remains to prove that f is multiplicative. Denote $n = p_1^{e_1} \cdots p_r^{e_r} p_{r+1} \cdots p_{r+s}$, where $e_i > 1$ ($i = 1, 2, \dots, r$). We proceed by induction on $e_1 + \cdots + e_r + s$ to prove that

$$f(n) = f(p_1^{e_1}) \cdots f(p_r^{e_r})f(p_{r+1}) \cdots f(p_{r+s}). \quad (2.10)$$

If $e_1 + \cdots + e_r + s = 1$, then (2.10) holds. Suppose that (2.10) holds when $e_1 + \cdots + e_r + s < m$. Then for $e_1 + \cdots + e_r + s = m$, we have after some manipulation

$$\begin{aligned} f(n)(g * \mu)(n) &= (fg * a^{-1} * b^{-1})(n) \\ &= f(n)g(n) + \sum_{\substack{d|n \\ d>1}} f\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right)(a^{-1} * b^{-1})(d) \\ &= f(n)g(n) - \prod_{p^e \| n} f(p^e)g(p^e) + \prod_{p^e \| n} (fg * a^{-1} * b^{-1})(p^e) \end{aligned}$$

$$\begin{aligned}
&= f(n)g(n) - \prod_{p^e \parallel n} f(p^e)g(p^e) \\
&\quad + \prod_{i=1}^r \left[f(p_i^{e_i})g(p_i^{e_i}) - f(p_i^{e_i-1})f(p_i)g(p_i^{e_i-1}) + f(p_i^{e_i-2})k(p_i)g(p_i^{e_i-1}) \right] \\
&\quad \times \prod_{i=1}^s (f(p_{r+i})g(p_{r+i}) - f(p_{r+i})).
\end{aligned} \tag{2.11}$$

Using (2.9), we obtain

$$f(n)(g * \mu)(n) = f(n)g(n) - g(n) \prod_{p^e \parallel n} f(p^e) + (g * \mu)(n) \prod_{p^e \parallel n} f(p^e). \tag{2.12}$$

This gives (2.10). $\square$

REMARK 2.2. The converse part of [Theorem 2.1](#) can also be written as follows. If $f(1) = 1$ and there exist completely multiplicative functions g and k , and a specially multiplicative function h such that

$$h * f(g * \mu) = fg, \tag{2.13}$$

where

$$h(p) = f(p), \quad h_A(p) = g(p)k(p), \quad (g * \mu)(n) \neq g(n) \tag{2.14}$$

for all prime numbers p and integers $n (\geq 2)$, then f is a specially multiplicative function with $f_A = k$.

COROLLARY 2.3. If f is a specially multiplicative function, then

$$h * f\phi = fN, \tag{2.15}$$

where h is the specially multiplicative function such that

$$h(p) = f(p), \quad h_A(p) = pf_A(p) \tag{2.16}$$

for all prime numbers p . Conversely, if $f(1) = 1$ and if there exist completely multiplicative functions a , b , and k such that

$$a * b * f\phi = fN, \tag{2.17}$$

where

$$a(p) + b(p) = f(p), \quad a(p)b(p) = pk(p) \tag{2.18}$$

for all prime numbers p , then f is a specially multiplicative function with $f_A = k$.

COROLLARY 2.4. *If f and g are completely multiplicative functions, then*

$$f * f(g * \mu) = fg. \quad (2.19)$$

Conversely, if $f(1) = 1$ and if there exists a completely multiplicative function g such that

$$f * f(g * \mu) = fg, \quad (2.20)$$

where

$$(g * \mu)(n) \neq g(n) \quad (2.21)$$

for all integers n (≥ 2), then f is a completely multiplicative function.

COROLLARY 2.5 (Sivaramakrishnan [5]). *If $f(1) = 1$, then f is a completely multiplicative function if and only if*

$$f * f\phi = fN. \quad (2.22)$$

EXAMPLE 2.6. We have

$$\sigma_\alpha \phi_\beta = \sigma_\alpha N^\beta * h^{-1}, \quad (2.23)$$

where h is the specially multiplicative function such that

$$h(p) = \sigma_\alpha(p) = p^\alpha + 1, \quad h_A(p) = p^\beta p^\alpha = p^{\alpha+\beta} \quad (2.24)$$

for all prime numbers p .

THEOREM 2.7. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 f_A g)). \quad (2.25)$$

Conversely, if $f(1) \neq 0$ and if there exist completely multiplicative functions g and k such that

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 k g)), \quad (2.26)$$

where

$$(g * \mu)(n) \neq g(n) \quad (2.27)$$

for all n , then f is a specially multiplicative function with $f_A = k$.

PROOF. We observe that

$$\begin{aligned} (\mu f \oplus \Omega_2(\mu^2 f_A g))(p) &= -f(p), \\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^2) &= f_A(p)g(p), \\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^n) &= 0 \end{aligned} \quad (2.28)$$

for all prime numbers p and integers $n (\geq 3)$. Therefore $\mu f \oplus \Omega_2(\mu^2 f_A g) = h^{-1}$, where h is the specially multiplicative function in [Theorem 2.1](#). Thus (2.25) follows from (2.1).

The converse follows from [Theorem 2.1](#) since $\mu f \oplus \Omega_2(\mu^2 g k) = a^{-1} * b^{-1}$, where a and b are completely multiplicative functions as given in [Theorem 2.1](#). $\square$

THEOREM 2.8. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \mu) = fg * (f^{-1} \oplus \Omega_2(\mu^2 f_A(g \oplus \mu))). \quad (2.29)$$

Conversely, if $f(1) = 1$ and there exist completely multiplicative functions c , d , and g such that

$$f(g * \mu) = fg * ((c * d)^{-1} \oplus \Omega_2(\mu^2 cd(g \oplus \mu))), \quad (2.30)$$

where

$$c(p) + d(p) = f(p), \quad (g * \mu)(n) \neq g(n) \quad (2.31)$$

for all prime numbers p and integers $n (\geq 2)$, then f is the specially multiplicative function given as $f = c * d$.

PROOF. Proof of [Theorem 2.8](#) is similar to that of [Theorem 2.7](#). $\square$

EXAMPLE 2.9. We have

$$\begin{aligned} \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\mu \sigma_\alpha \oplus \Omega_2(\mu^2 N^{\alpha+\beta})), \\ \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\sigma_\alpha^{-1} \oplus \Omega_2(\mu^2 N^\alpha (N^\beta \oplus \mu))). \end{aligned} \quad (2.32)$$

LEMMA 2.10. *Suppose that f is an arithmetical function such that $f(1) = 1$ and $f^{-1}(p^i) = 0$ for $3 \leq i < k$ ($k \geq 4$). Then*

$$f(p^k) = f(p)f(p^{k-1}) - f^{-1}(p^2)f(p^{k-2}) - f^{-1}(p^k). \quad (2.33)$$

PROOF. [Lemma 2.10](#) follows from the equation

$$\sum_{i=0}^k f^{-1}(p^i)f(p^{k-i}) = 0. \quad (2.34)$$

$\square$

THEOREM 2.11. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \zeta) = fg * f * \Omega_2(f_A g)^{-1}. \quad (2.35)$$

Conversely, if f is a multiplicative function such that

$$f(g * \zeta) = fg * f * \Omega_2(hg)^{-1}, \quad (2.36)$$

where g is a completely multiplicative function with $g(p)(g * \zeta)(p^e) \neq 0$ for all prime powers p^e and where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A = h$.

PROOF. Let $f = a * b$, where a and b are completely multiplicative functions. It is known [7] that

$$f(g * \zeta) = (a * b)(g * \zeta) = ag * a\zeta * bg * b\zeta * \Omega_2(abg\zeta)^{-1}. \quad (2.37)$$

Using elementary properties of arithmetical functions, we obtain

$$f(g * \zeta) = (a * b)g * (a * b) * \Omega_2(f_A g)^{-1} = fg * f * \Omega_2(f_A g)^{-1}. \quad (2.38)$$

This proves (2.35).

Assume that (2.36) holds. Then (2.36) at p^2 gives

$$h(p) = f(p)^2 - f(p^2). \quad (2.39)$$

Since $f^{-1}(p^2) = f(p)^2 - f(p^2)$ for all multiplicative functions, we obtain

$$h(p) = f^{-1}(p^2). \quad (2.40)$$

We next prove that

$$f^{-1}(p^i) = 0 \quad \forall i \geq 3. \quad (2.41)$$

We proceed by induction on i . Calculating (2.36) at p^3 and using (2.40) gives

$$f(p^3) = f(p)f(p^2) - f(p)f^{-1}(p^2). \quad (2.42)$$

Since

$$f(p^3) - f(p^2)f(p) + f(p)f^{-1}(p^2) + f^{-1}(p^3) = 0, \quad (2.43)$$

we see that $f^{-1}(p^3) = 0$.

Suppose that $f^{-1}(p^i) = 0$ for all $3 \leq i < k$ ($k > 3$). We write (2.36) as

$$f(g * \zeta) * f^{-1} = fg * \Omega_2(hg)^{-1}. \quad (2.44)$$

Suppose that k is even, say $k = 2e$ ($e > 1$). At p^{2e} , the left-hand side of (2.44) becomes

$$\begin{aligned}
 & \sum_{i=0}^{2e} f(p^i)(g * \zeta)(p^i)f^{-1}(p^{2e-i}) \\
 &= f^{-1}(p^{2e}) + f(p^{2e-2})(g * \zeta)(p^{2e-2})f^{-1}(p^2) \\
 & \quad + f(p^{2e-1})(g * \zeta)(p^{2e-1})f^{-1}(p) + f(p^{2e})(g * \zeta)(p^{2e}) \\
 &= f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g * \zeta)(p^{2e-2}) \\
 & \quad - f(p)f(p^{2e-1})g(p^{2e-1}) + f(p^{2e})g(p^{2e-1}) + f(p^{2e})g(p^{2e}) \\
 &= f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g * \zeta)(p^{2e-1}) \\
 & \quad - f^{-1}(p^2)f(p^{2e-2})g(p^{2e-1}) + f(p^{2e})g(p^{2e}),
 \end{aligned} \tag{2.45}$$

where the last two equations are derived by Lemma 2.10. Further, at p^{2e} , the right-hand side of (2.44) becomes

$$\begin{aligned}
 & \sum_{i=0}^{2e} f(p^{2e-i})g(p^{2e-i})\Omega_2(hg)^{-1}(p^i) \\
 &= \sum_{i=0}^e f(p^{2(e-i)})g(p^{2(e-i)})\mu(p^i)h(p^i)g(p^i) \\
 &= f(p^{2e})g(p^{2e}) - f(p^{2(e-1)})g(p^{2(e-1)})h(p)g(p).
 \end{aligned} \tag{2.46}$$

Now, we see that $f^{-1}(p^{2e}) = 0$, that is, $f^{-1}(p^k) = 0$.

If k is odd, a similar argument applies. Thus (2.41) holds and therefore, by (1.6), f is a specially multiplicative function with $f_A = h$. $\square$

COROLLARY 2.12. *If f is a specially multiplicative function, then*

$$f\sigma_0 = f * f * \Omega_2(f_A)^{-1}. \tag{2.47}$$

Conversely, if f is a multiplicative function such that

$$f\sigma_0 = f * f * \Omega_2(h)^{-1}, \tag{2.48}$$

where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A = h$.

COROLLARY 2.13 (Apostol [1]). *If f and g are completely multiplicative functions, then*

$$f(g * \zeta) = fg * f. \tag{2.49}$$

Conversely, if f is a multiplicative function such that

$$f(g * \zeta) = fg * f, \tag{2.50}$$

where g is a completely multiplicative function with $g(p)(g * \zeta)(p^e) \neq 0$ for all prime powers p^e , then f is a completely multiplicative function.

COROLLARY 2.14 (Carlitz [2]). *Suppose that f is a multiplicative function. Then f is a completely multiplicative function if and only if*

$$f\sigma_0 = f * f. \quad (2.51)$$

COROLLARY 2.15. *There exist*

$$\begin{aligned} \tau\sigma_\alpha &= \tau N^\alpha * \tau * \Omega_2(N^{\alpha+11})^{-1}, \\ \sigma_\alpha\sigma_\beta &= \sigma_\alpha N^\beta * \sigma_\alpha * \Omega_2(N^{\alpha+\beta})^{-1}. \end{aligned} \quad (2.52)$$

REFERENCES

- [1] T. M. Apostol, *Some properties of completely multiplicative arithmetical functions*, Amer. Math. Monthly **78** (1971), 266–271.
- [2] L. Carlitz, *Completely multiplicative function*, Amer. Math. Monthly **78** (1971), 1140.
- [3] D. H. Lehmer, *Some functions of Ramanujan*, Math. Student **27** (1959), 105–116.
- [4] P. J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer-Verlag, New York, 1986.
- [5] R. Sivaramakrishnan, *Multiplicative function and its Dirichlet inverse*, Amer. Math. Monthly **77** (1970), 772–773.
- [6] ———, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 126, Marcel Dekker, New York, 1989.
- [7] R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Amer. Math. Soc. **33** (1931), no. 2, 579–662.

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