SPECTRAL INCLUSIONS AND STABILITY RESULTS FOR STRONGLY CONTINUOUS SEMIGROUPS

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We prove some spectral inclusions for strongly continuous semigroups. Some stability results are also established.

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1. Introduction and preliminaries. Let X be a complex Banach space and let A be a closed linear operator with domain $D(A)$, kernel $N(A)$, and range $R(A)$ in X . We will say that A is *semiregular* if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$, where $R^\infty(A) := \bigcap_{n=1}^\infty R(A^n)$.

Denote the regular spectrum by

$$\sigma_y(A) := \{\lambda \in \mathbb{C}, \lambda - A \text{ is not semiregular}\}. \quad (1.1)$$

The set $\sigma_y(A)$ was studied (under various names) by several authors, see for instance [11, 12, 13, 14] and the references therein.

An operator A is said to be *essentially semiregular* if $R(A)$ is closed and there exists a finite-dimensional subspace $G \subseteq X$ such that $N(A) \subseteq R^\infty(A) + G$. Define further the essential regular spectrum of A by

$$\sigma_{ye}(A) := \{\lambda \in \mathbb{C}, \lambda - A \text{ is not essentially semiregular}\}. \quad (1.2)$$

This concept was introduced and studied for bounded operators in [13, 17].

We say that A is *upper semi-Fredholm* if $R(A)$ is closed and $\dim N(A) < \infty$. The left essential spectrum is given by

$$\sigma_\pi(A) := \{\lambda \in \mathbb{C}, \lambda - A \text{ is not upper semi-Fredholm}\}. \quad (1.3)$$

Let X^* denote the dual space of X and A^* the adjoint operator of A . Define the *reduced minimum modulus* $\gamma(A)$ by setting

$$\gamma(A) := \inf \left\{ \frac{\|Au\|}{d(u, N(A))}, u \in D(A) \setminus N(A) \right\}. \quad (1.4)$$

It is well known (see [9]) that $\gamma(A) = \gamma(A^*)$ and $\gamma(A) > 0$ if and only if $R(A)$ is closed. Let $H_0(A)$ denote the *quasinilpotent part* of A given by

$$H_0(A) := \left\{ x \in \bigcap_{n \geq 0} D(A^n) : \lim_{n \rightarrow \infty} \|A^n x\|^{1/n} = 0 \right\}. \quad (1.5)$$

Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup with generator A on X . We will denote the *type* (growth bound) of $\mathcal{T}$ by ω_0 :

$$\begin{aligned} \omega_0 &:= \lim_{t \rightarrow \infty} \frac{\ln \|T(t)\|}{t} \\ &= \inf \{ \omega \in \mathbb{R} : \text{there exists } M \text{ such that } \|T(t)\| \leq M e^{\omega t}, t \geq 0 \}. \end{aligned} \quad (1.6)$$

Following [6], the semigroup $\mathcal{T}$ is called bounded if there exists $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. Basic materials on semigroups may be found in [4, 6, 15].

In [5], we have studied the regular spectrum for strongly continuous semigroups. As a continuation of [5], the present paper deals with the essentially regular spectrum. Moreover, we establish some stability results for strongly continuous semigroups.

The present paper is organized as follows. In [Section 2](#), we first prove that the spectral inclusion for semigroups remains true for the regular spectrum, the left essential spectrum, and the essentially regular spectrum ([Theorem 2.1](#)). Secondly, we give necessary and sufficient conditions for the generator of a strongly continuous semigroup to be semiregular ([Theorem 2.3](#)) and essentially semiregular ([Theorem 2.5](#)).

In [Section 3](#), we derive some stability results for strongly continuous semigroups. Among other results, we give necessary and sufficient conditions for the generator of a bounded strongly continuous semigroup to have no pure imaginary point in its spectrum ([Theorem 3.3](#)). This, in particular, provides us with a spectral characterization of the strong stability of the ultrapower extension of a given semigroup. Finally, we discuss the strong stability of a strongly continuous semigroup via the behavior of the resolvent of its generator, on the imaginary axis.

Throughout this paper, we let $\sigma(A)$, $\rho(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, and $\sigma_{su}(A)$ denote, respectively, the spectrum, the resolvent set, the point spectrum, the approximative spectrum, and the surjective spectrum of an operator A . For $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes the resolvent operator $(\lambda - A)^{-1} \in \mathcal{B}(X)$ of A , where $\mathcal{B}(X)$ stands for the algebra of bounded linear operators on X .

For later use, we introduce the following operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$:

$$I(\lambda, t)x := \int_0^t e^{\lambda(t-s)} T(s)x \, ds, \quad x \in X. \quad (1.7)$$

It is well known (see [15]) that $I(\lambda, t)$ is a bounded linear operator on X and we have

$$\begin{aligned} e^{\lambda t}x - T(t)x &= (\lambda - A)I(\lambda, t)(x) \quad (x \in X) \\ &= I(\lambda, t)(\lambda - A)(x) \quad (x \in D(A)). \end{aligned} \quad (1.8)$$

We conclude this section by the following result which we need in the sequel.

LEMMA 1.1 [10]. *If $A \in \mathcal{B}(X)$ is essentially semiregular, then $R^\infty(A)$ is closed and the operator $\hat{A} : X/R^\infty(A) \rightarrow X/R^\infty(A)$ induced by A is upper semi-Fredholm.*

2. Spectral inclusions. In this section, we study the regular spectrum and the essentially regular spectrum of the generator of a strongly continuous semigroup. We begin with the following spectral inclusions.

THEOREM 2.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$, there exist the spectral inclusions*

$$e^{tv(A)} \subseteq v(T(t)) \setminus \{0\}, \quad \forall t \geq 0, \quad (2.1)$$

where $v \in \{\sigma_\gamma, \sigma_\pi, \sigma_{ye}\}$.

To prove this result, we need the following lemma.

LEMMA 2.2. *Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$. Then, for all $\lambda \in \mathbb{C}$, $t \geq 0$, and $n \in \mathbb{N}$,*

(i)

$$\begin{aligned} (e^{\lambda t} - T(t))^n(x) &= (\lambda - A)^n I(\lambda, t)^n(x) \quad (x \in X) \\ &= I(\lambda, t)^n (\lambda - A)^n(x) \quad (x \in D(A^n)); \end{aligned} \quad (2.2)$$

(ii) $R^\infty(e^{\lambda t} - T(t)) \subseteq R^\infty(\lambda - A)$;

(iii) $N((\lambda - A)^n) \subseteq N((e^{\lambda t} - T(t))^n)$;

(iv) $H_0(\lambda - A) \subseteq H_0(e^{\lambda t} - T(t))$.

PROOF OF LEMMA 2.2. As mentioned before, $I(\lambda, t)$ is a bounded linear operator on X and we have

$$\begin{aligned} e^{\lambda t}x - T(t)x &= (\lambda - A)I(\lambda, t)(x) \quad (x \in X) \\ &= I(\lambda, t)(\lambda - A)(x) \quad (x \in D(A)). \end{aligned} \quad (2.3)$$

Proceeding by induction, we get the desired result. The assertions (ii), (iii), and (iv) follow easily from (i). $\square$

PROOF OF THEOREM 2.1

THE REGULAR SPECTRUM. See [5].

THE LEFT ESSENTIAL SPECTRUM. Let $t_0 > 0$ be fixed and suppose that $e^{\lambda t_0} \notin \sigma_\pi(T(t_0))$ for some $\lambda \in \mathbb{C}$. We show that $\lambda \notin \sigma_\pi(A)$. Using Lemma 2.2(iii), together with $\dim N(e^{\lambda t_0} - T(t_0)) < \infty$, we infer that $N(\lambda - A)$ is finite dimensional. Now, we prove that $R(\lambda - A)$ is closed. Since $N(e^{\lambda t_0} - T(t_0))$ is finite dimensional, there exists a closed subspace Y of X such that $N(e^{\lambda t_0} - T(t_0)) \oplus Y = X$. But, $(\lambda - A)(N(e^{\lambda t_0} - T(t_0)) \cap D(A))$ is finite dimensional and therefore closed. Then, we need only to show that $(\lambda - A)(Y \cap D(A))$ is closed. From the closed-graph theorem and the closedness of $R(e^{\lambda t_0} - T(t_0))$, it follows that there is a constant $C > 0$ such that

$$\|e^{\lambda t_0} x - T(t_0)x\| \geq C\|x\|, \quad \forall x \in Y. \quad (2.4)$$

From Lemma 2.2(i), we obtain that, for every $x \in D(A)$,

$$\|e^{\lambda t_0} x - T(t_0)x\| \leq M\|\lambda x - Ax\| \quad (2.5)$$

for some positive constant M . The combination of inequalities (2.4) and (2.5) gives us

$$\|\lambda x - Ax\| \geq \frac{C}{M}\|x\|, \quad x \in Y \cap D(A). \quad (2.6)$$

From the fact that $\lambda - A$ is closed, the result follows.

THE ESSENTIAL REGULAR SPECTRUM. Let $t_0 > 0$ be fixed and suppose that $e^{\lambda t_0} - T(t_0)$ is essentially semiregular for some $\lambda \in \mathbb{C} \setminus \{0\}$. We show that $\lambda - A$ is essentially semiregular. To this end, consider the closed $(T(t))_{t \geq 0}$ -invariant subspace $M := R^\infty(e^{\lambda t_0} - T(t_0))$ of X and the quotient semigroup $(\hat{T}(t))_{t \geq 0}$ defined on X/M by

$$\hat{T}(t)\hat{x} := \widehat{T(t)x}, \quad \text{for } \hat{x} \in X/M, \quad (2.7)$$

with generator $\hat{A}$ defined by

$$D(\hat{A}) := \{\hat{x}, x \in D(A)\}, \quad \hat{A}\hat{x} := \widehat{Ax}, \quad \forall \hat{x} \in D(\hat{A}). \quad (2.8)$$

From Lemma 1.1, it follows that the operator $e^{\lambda t_0} - \hat{T}(t_0)$ is upper semi-Fredholm. Thus, $e^{\lambda t_0} \notin \sigma_\pi(\hat{T}(t_0))$. By virtue of the precedent case, we get $\lambda \notin \sigma_\pi(\hat{A})$. In consequence, the operator $\lambda - \hat{A}$ is upper semi-Fredholm. Next, let $\pi : X \rightarrow X/M$ be the canonical projection. Using Lemma 2.2(ii), together with $\dim(N(\lambda - \hat{A})) < \infty$, it can be verified that

$$N(\lambda - A) \subseteq \pi^{-1}N(\lambda - \hat{A}) \subseteq M + G \subseteq R^\infty(\lambda - A) + G \quad (2.9)$$

for a finite-dimensional subspace G of X . Now, we show that $R(\lambda - A)$ is closed. To do this, consider a sequence $(u_n)_n$ of elements of $R(\lambda - A)$, which converges to u . Then, there exists a sequence $(v_n)_n$ of elements of $D(A)$ such that

$(\lambda - A)v_n = u_n \rightarrow u$. Since $R(\lambda - \hat{A})$ is closed, there exists $\hat{w} \in D(\hat{A})$ such that $\hat{u} = (\lambda - \hat{A})\hat{w}$. Hence,

$$u - (\lambda - A)w \in R^\infty(e^{\lambda t_0} - T(t_0)) \subseteq R^\infty(\lambda - A) \subseteq R(\lambda - A). \quad (2.10)$$

Accordingly, $u \in R(\lambda - A)$. Consequently, the operator $\lambda - A$ is essentially semiregular. This proves the theorem. $\square$

The next theorem gives, under suitable assumptions, necessary and sufficient conditions for the generator of a strongly continuous semigroup to be semiregular. The proof can be found in [5].

THEOREM 2.3. *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup with generator A and type ω_0 . If $(T(t))_{t \geq 0}$ satisfies any of the following conditions:*

- (a) $\lim_{t \rightarrow \infty} (1/t) \|T(t)\| = 0$;
- (b) $|\omega_0| < \gamma(A)$,

then the following assertions are equivalent:

- (i) A is semiregular;
- (ii) $0 \in \rho(A)$;
- (iii) $H_0(A) = \{0\}$ and $R(A)$ is closed.

The following example shows that conditions (a) and (b) in [Theorem 2.3](#) are needed for the conclusion.

EXAMPLE 2.4. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^\infty$. Let A be the operator on H defined by $Ae_n = e_{n+1}$, $n = 1, 2, \dots$, and let $T(t) = e^{tA}$ be the semigroup generated by A . It is well known (see, e.g., [16, Chapter 2, Theorems 4 and 6]) that $\sigma(A) = \{\lambda \in \mathbb{C}, |\lambda| \leq 1\}$ and $\sigma_{\text{ap}}(A) = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$. Thus, A is semiregular but $0 \notin \rho(A)$.

We conclude this section by the following result.

THEOREM 2.5. *Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ satisfying $\lim_{t \rightarrow \infty} (1/t) \|T(t)\| = 0$. The following assertions are equivalent:*

- (i) A is essentially semiregular;
- (ii) A is upper semi-Fredholm.

PROOF. (i) $\Rightarrow$ (ii). Since A is essentially semiregular, there exists a finite-dimensional subspace G of X such that $N(A) \subseteq R^\infty(A) + G$. As noticed in [13], we may assume that $G \subseteq N(A)$. Let $y \in N(A)$ and let $x \in D(A)$ and $g \in G$ such that $y = Ax + g$. Using [Lemma 2.2\(i\)](#), we infer that

$$T(t)x = x + \int_0^t T(s)(y - g)ds = x + t(y - g), \quad \forall t \geq 0. \quad (2.11)$$

Since $\lim_{t \rightarrow \infty} (1/t) \|T(t)\| = 0$, then $y = g$. In consequence, $N(A) = G$. This is the desired result.

(ii) $\Rightarrow$ (i). Obvious. $\square$

REMARK 2.6. Theorems 2.3 and 2.5 follow under weaker assumptions (e.g., $\lim_{t \rightarrow \infty} t^{-n} \|T(t)\|$, $n \in \mathbb{N}$). This follows essentially from the fact that the spectrum of A is contained in the left half-plane $\{\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) \leq 0\}$.

3. Stability results. In this section, we give some stability results for strongly continuous semigroups. First, we introduce some relevant notations and terminologies. By $\mathbb{C}^-$ we denote the open left half of the complex plane, that is, the set of all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) < 0$. A closed operator A is called *stable* if $\sigma(A) \subseteq \mathbb{C}^-$. A strongly continuous semigroup $(T(t))_{t \geq 0}$ is said to be *strongly stable* if $\|T(t)x\| \rightarrow 0$ as $t \rightarrow \infty$ for all $x \in X$. We say that $(T(t))_{t \geq 0}$ is *uniformly stable* if $\|T(t)\| \rightarrow 0$ as $t \rightarrow \infty$. Recall that a strongly stable semigroup is necessarily bounded and has no pure imaginary point in the point spectrum of its generator. For a recent account of stability results of strongly continuous semigroups, we refer the reader to [6, Chapter V].

We begin with the following stability results.

THEOREM 3.1 [5]. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$. If $\sigma_y(A) \cap i\mathbb{R} = \emptyset$, then $(T(t))_{t \geq 0}$ is strongly stable.*

THEOREM 3.2 [5]. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$. Then, the following assertions are equivalent:*

- (i) $(T(t))_{t \geq 0}$ is uniformly stable;
- (ii) $\sigma_y(T(t)) \cap \Gamma = \emptyset$,

where Γ stands for the unit circle of $\mathbb{C}$.

In order to state the next result, we need to introduce the ultrapower semigroup $\tilde{\mathcal{T}}$ of a given semigroup $\mathcal{T} = (T(t))_{t \geq 0}$.

Following [20, page 35] (see also [1, 8]), we define the space $\ell_0^\infty(X)$ as the set of all bounded sequences $(x_n)_n \subseteq X$ such that

$$\lim_{t \downarrow 0} \left(\sup_n \|T(t)x_n - x_n\| \right) = 0. \quad (3.1)$$

The ultrapower semigroup $\tilde{\mathcal{T}} = (\tilde{T}(t))_{t \geq 0}$ is defined on the quotient space

$$\tilde{X} := \ell_0^\infty(X) / C_0(X) \quad (3.2)$$

by

$$\tilde{T}(t)((x_n)_n + C_0(X)) = (T(t)x_n)_n + C_0(X), \quad (3.3)$$

where $C_0(X)$ stands for the space of all sequences in X that converge to 0.

The semigroup $\tilde{\mathcal{T}}$ is, by construction, strongly continuous. Its generator $\tilde{A}$ is given by

$$\begin{aligned} D(\tilde{A}) &= \left\{ (x_n)_n + C_0(X), (x_n)_n \in \ell_0^\infty(X), x_n \in D(A) \ \forall n, (Ax_n)_n \in \ell_0^\infty(X) \right\}, \\ \tilde{A}\left((x_n)_n + C_0(X)\right) &= (Ax_n)_n + C_0(X). \end{aligned} \quad (3.4)$$

The spectra of A and $\tilde{A}$ are related as follows:

$$\sigma(\tilde{A}) = \sigma(A), \quad \sigma_p(\tilde{A}) = \sigma_{\text{ap}}(\tilde{A}) = \sigma_{\text{ap}}(A). \quad (3.5)$$

THEOREM 3.3. *Let A be the generator of a bounded strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$. Then the following assertions are equivalent:*

- (i) $\sigma_y(A) \cap i\mathbb{R} = \emptyset$;
- (ii) $\sigma(A) \cap i\mathbb{R} = \emptyset$;
- (iii) A is stable;
- (iv) for every $x^* \in X^*$ and for every $\beta \in \mathbb{R}$, $\|R(\lambda + i\beta, A^*)x^*\| = O(1)$ (as $\lambda \rightarrow 0$);
- (v) $\tilde{\mathcal{T}}$ is strongly stable.

PROOF. (i) $\Rightarrow$ (ii). It suffices to apply [Theorem 2.3](#) to the rescaled semigroup $(e^{-i\lambda t}T(t))_{t \geq 0}$ whose generator is $A - i\lambda$.

(ii) $\Rightarrow$ (i). Obvious.

(ii) $\Leftrightarrow$ (iii). This is an immediate consequence of the Hille-Yosida theorem [6, Chapter II, Theorem 3.8].

(ii) $\Leftrightarrow$ (iv). Applying [18, Theorem 3] to the rescaled semigroup $S(t) = e^{-i\beta t}T(t)$, $\beta \in \mathbb{R}$, whose generator is $A - i\beta$, we can assert that condition (iv) is equivalent to $\sigma_{\text{su}}(A^*) \cap i\mathbb{R} = \emptyset$. This is equivalent to $\sigma_{\text{ap}}(A) \cap i\mathbb{R} = \emptyset$. Using [Theorem 2.3](#), we infer that $\sigma(A) \cap i\mathbb{R} = \emptyset$, which is the desired result.

(v) $\Rightarrow$ (ii). Since $\tilde{\mathcal{T}}$ is strongly stable, then $\sigma_p(\tilde{A}) \cap i\mathbb{R} = \emptyset$. In consequence, $\sigma_{\text{ap}}(A) \cap i\mathbb{R} = \emptyset$. Arguing as above, we get $\sigma(A) \cap i\mathbb{R} = \emptyset$.

(ii) $\Rightarrow$ (v). This follows from [Theorem 3.1](#) on the basis of (3.5). $\square$

REMARK 3.4. (1) It was shown in [3], under the hypothesis of [Theorem 3.3](#), that the condition $\sigma(A) \subseteq \mathbb{C}^-$ is equivalent to

$$\sup_{t > 0} \left\| \int_0^t e^{i\mu s} T(s) ds \right\| < \infty, \quad \forall \mu \in \mathbb{R}, \forall x \in X. \quad (3.6)$$

(2) In the general case, the condition $\sigma(A) \cap i\mathbb{R} = \emptyset$ does not characterize, even in Hilbert spaces, the strong stability of the semigroup generated by A . The translation semigroup $T(t)f(x) := f(x+t)$, $t \geq 0$, on $L^2(\mathbb{R}_+)$ shows that this condition is not necessary for strong stability. In fact, this semigroup has the generator $A = d/dx$, and the spectrum of A is the left half plane $\{\lambda \in \mathbb{C} : \text{Re } \lambda \leq 0\}$, see [1, A.III, 2.4, page 66]. Hence, $\sigma(A) \cap i\mathbb{R} = i\mathbb{R}$ but

$\lim_{t \rightarrow \infty} \|T(t)f\| = 0$ for every $f \in L^2(\mathbb{R}_+)$. However, [Theorem 3.3](#) shows that the condition $\sigma(A) \cap i\mathbb{R} = \emptyset$ characterizes completely the strong stability of the ultrapower extension of the semigroup generated by A .

COROLLARY 3.5. *Let A be the generator of a bounded strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on a reflexive Banach space X . Then, conditions (i), (ii), (iii), (iv), and (v) of [Theorem 3.3](#) are equivalent to*

(vi) for every $x \in X$ and for every $\beta \in \mathbb{R}$, $\|R(\lambda + i\beta, A)x\| = O(1)$ (as $\lambda \rightarrow 0$).

PROOF. It is well known [[19](#), Corollary 1.3.2] that the adjoint semigroup of a strongly continuous semigroup on a reflexive Banach space is again strongly continuous. It suffices to apply [Theorem 3.3](#) to the adjoint semigroup whose generator is A^* . $\square$

Note that, in reflexive Banach spaces, condition (vi) implies the strong stability of the bounded semigroup generated by A (this follows from [Corollary 3.5](#) and [Theorem 3.1](#)). In general Banach spaces setting, we have the following proposition.

PROPOSITION 3.6. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$, satisfying condition (vi). Then,*

$$\lim_{t \rightarrow \infty} \|T(t)x\| = 0, \quad \forall x \in \overline{\bigcap_{\beta \in \mathbb{R}} R(i\beta - A)}. \quad (3.7)$$

PROOF. Condition (vi) implies that $\lim_{\lambda \rightarrow 0} \lambda R(\lambda, A - i\beta)x = 0$ for all $x \in X$. By the abelian mean ergodic theorem [[7](#), page 520], it follows that $R(i\beta - A)$ is dense in X for all $\beta \in \mathbb{R}$. Hence,

$$\bigcap_{\beta \in \mathbb{R}} R(i\beta - A) = \bigcap_{\beta \in \mathbb{R}} (i\beta - A) \overline{(R(i\beta - A) \cap D(A))}. \quad (3.8)$$

Using [[2](#), Theorem 6.3(ii)] and the strong continuity of the semigroup, we get the desired result. $\square$

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