Let $K$ be a pure cubic field. Let $L$ be the normal closure of $K$. A relative integral basis (RIB) for $L$ over $\mathbb{Q}(\sqrt{-3})$ is given. This RIB simplifies and completes the one given by Haghighi (1986).

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1. Introduction. Let $K$ be the pure cubic field $\mathbb{Q}(d^{1/3})$, where $d$ is a cube-free integer, and let $L$ be the normal closure of $K$ so that $\mathbb{Q} \subset K \subset L$, $[L : K] = 2$, and $[K : \mathbb{Q}] = 3$. Let $k$ be the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ so that $\mathbb{Q} \subset k \subset L$, $[L : k] = 3$, and $[k : \mathbb{Q}] = 2$. The ring of all algebraic integers is denoted by $\Omega$. The rings of integers of $K, k, L$ are $\mathcal{O}_K = K \cap \Omega$, $\mathcal{O}_k = k \cap \Omega$, $\mathcal{O}_L = L \cap \Omega$, respectively. As $\mathcal{O}_k$ is a principal ideal domain, $L/k$ possesses a relative integral basis (RIB) [3, Corollary 3, page 401]. Haghighi [2, Theorems 5.1, 5.3, 5.6] has given a RIB for $L/k$. However, Haghighi’s RIB for $L/k$ contains two difficulties. The first is that in certain cases the RIB makes use of an element of norm 3 in a pure cubic field, a quantity which is not easy to determine, see [2, Theorem 5.1]. The second problem is that the RIB is not completely general, see [2, Theorem 5.3]. In this note, we give a simple and completely general RIB for $L/k$.

2. Preliminary remarks. As $d$ is a cube-free integer, we can define integers $a$ and $b$ by

$$d = ab^2, \quad (a, b) = 1, \ a, b \text{ square-free.} \quad (2.1)$$

If $a^2 \not\equiv b^2 (\bmod 9)$, an integral basis for $K$ is

$$\left\{1, (ab^2)^{1/3}, (a^2b)^{1/3}\right\}, \quad (2.2)$$

and if $a^2 \equiv b^2 (\bmod 9)$, an integral basis is

$$\left\{1, (ab^2)^{1/3}, \frac{b + ab(ab^2)^{1/3} + (a^2b)^{1/3}}{3}\right\}. \quad (2.3)$$
These integral bases are due to Dedekind [1]. From (2.2) and (2.3), we deduce that the discriminant \( d(K) \) of \( K \) is given by
\[
d(K) = -3f^2, \tag{2.4}
\]
where
\[
f = \begin{cases} 
3ab, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
ab, & \text{if } a^2 \equiv b^2 \pmod{9}.
\end{cases} \tag{2.5}
\]

The relative discriminant \( d(L/k) \) of \( L/k \) is given by
\[
d(L/k) = f^2 = \begin{cases} 
9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
a^2b^2, & \text{if } a^2 \equiv b^2 \pmod{9},
\end{cases} \tag{2.6}
\]
see [1]. We note that if \( \alpha, \beta \in O_L \) are such that
\[
d_{L/k}(1, \alpha, \beta) = d(L/k), \tag{2.7}
\]
then \( \{1, \alpha, \beta\} \) is a RIB for \( L/k \).

3. RIB for \( L/k \). We show that \( \{1, \alpha, \beta\} \) is a RIB for \( L/k \), where \( \alpha \) and \( \beta \) are given in Table 3.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>3 \mid a, 3 \mid b</td>
<td>((ab^2)^{1/3})</td>
<td>((a^2b)^{1/3}) \sqrt{-3}\</td>
</tr>
<tr>
<td>(ii)</td>
<td>3 \mid a, 3 \mid b</td>
<td>((ab^2)^{1/3}) \sqrt{-3}\</td>
<td>((a^2b)^{1/3})</td>
</tr>
<tr>
<td>(iii)</td>
<td>3 \mid a, 3 \mid b, 9 \mid a^2 - b^2</td>
<td>((ab^2)^{1/3})</td>
<td>(b + ab(ab^2)^{1/3} + (a^2b)^{1/3}) \sqrt{-3}\</td>
</tr>
<tr>
<td>(iv)</td>
<td>3 \mid a, 3 \mid b, 9 \mid a^2 - b^2</td>
<td>((ab^2)^{1/3} - a) \sqrt{-3}\</td>
<td>(b + ab(ab^2)^{1/3} + (a^2b)^{1/3}) \sqrt{-3}\</td>
</tr>
</tbody>
</table>

An easy calculation making use of (2.2), (2.3), (2.4), and (2.5) shows that
\[
d_{L/k}(1, \alpha, \beta) = \begin{cases} 
9a^2b^2, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
a^2b^2, & \text{if } a^2 \equiv b^2 \pmod{9},
\end{cases} \tag{3.1}
\]
so that (2.7) holds in view of (2.6). Clearly, \( \alpha \in L \) and \( \beta \in L \). We now show that \( \alpha \in \Omega \) and \( \beta \in \Omega \) so that \( \alpha \in O_L \) and \( \beta \in O_L \), proving that \( \{1, \alpha, \beta\} \) is a RIB for \( L/k \). Clearly, \( \alpha \in \Omega \) in Cases (i) and (iii), and \( \beta \in \Omega \) in Cases (ii) and (iv), see (2.3)
for the latter. In the remaining cases, it suffices to give a monic polynomial $f_\alpha(x) \in \mathbb{Z}[x]$ of which $\alpha$ is a root in Cases (ii) and (iv), and a monic polynomial $f_\beta(x) \in \mathbb{Z}[x]$ of which $\beta$ is a root in Cases (i) and (iii).

**Case (i).** Here,

$$f_\beta(x) = x^6 + 3a_1^2b^2, \quad a_1 = \frac{a}{3} \in \mathbb{Z}. \quad (3.2)$$

**Case (ii).** Here,

$$f_\alpha(x) = x^6 + 3a_2^2b_1^2, \quad b_1 = \frac{b}{3} \in \mathbb{Z}. \quad (3.3)$$

**Case (iii).** We have

$$a^2 \equiv b^2 \equiv 1 \pmod{3}, \quad a^2 - b^2 \equiv 0 \pmod{3}, \quad (3.4)$$

so that

$$a^4b^4 - 3a^2b^2 + a^2 + b^2 = (a^2 - b^2)^2 + (a^2 - 1)(b^2 - 1)(a^2b^2 + a^2 + b^2) \equiv 0 \pmod{9}, \quad (3.5)$$

and we define $m \in \mathbb{Z}$ by

$$m = \frac{(a^4b^4 - 3a^2b^2 + a^2 + b^2)}{9}. \quad (3.6)$$

In this case,

$$f_\beta(x) = x^6 + (2a^2 + 1)b^2x^4 + ((a^2 - 1)^2b^2 - 6m)b^2x^2 + 3b^2m^2. \quad (3.7)$$

**Case (iv).** We have

$$a^2 \equiv b^2 \equiv 1 \pmod{3}, \quad a^2 - b^2 \equiv 0 \pmod{9}, \quad a^2 + 2b^2 \equiv 0 \pmod{9} \quad (3.8)$$

so that we can define $r, s \in \mathbb{Z}$ by

$$r = \frac{(a^2 + 2b^2)}{3}, \quad s = \frac{(a^2 - b^2)}{9}. \quad (3.9)$$

Here,

$$f_\alpha(x) = x^6 + a^2x^4 + a^2rx^2 + 3a^2s^2. \quad (3.10)$$

This completes the proof that $\{1, \alpha, \beta\}$ is a RIB for $L/k$.

We conclude with four examples.

**Example 3.1** (cf. [2, Illustration 5.2]). A RIB for $\mathbb{Q}(\sqrt[3]{2} \sqrt[3]{3}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (i))

$$\left\{1, 213^{1/3}, \frac{213^{2/3}}{\sqrt{-3}} \right\}. \quad (3.11)$$
Example 3.2. A RIB for $\mathbb{Q}(\sqrt[3]{5}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (ii))
\[
\left\{1, \frac{9^{1/3}}{\sqrt[3]{-3}}, 3^{1/3}\right\}.
\]

Example 3.3 (cf. [2, Illustration 5.5]). A RIB for $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iii))
\[
\left\{1, 2^{1/3}, \frac{1 + 2 \cdot 2^{1/3} + 2^{2/3}}{\sqrt{-3}}\right\}.
\]

Example 3.4 (cf. [2, Illustration 5.7]). A RIB for $\mathbb{Q}(\sqrt[3]{10}, \sqrt{-3})$ over $\mathbb{Q}(\sqrt{-3})$ is (see Case (iv))
\[
\left\{1, \frac{10^{1/3} - 10}{\sqrt{-3}}, \frac{1 + 10 \cdot 10^{1/3} + 10^{2/3}}{3}\right\}.
\]

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