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DUALITY IN THE OPTIMAL CONTROL FOR DAMPED HYPERBOLIC SYSTEMS WITH POSITIVE CONTROL

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We study the duality theory for damped hyperbolic equations. These systems have positive controls and convex cost functionals. Our main results lie in the application of duality theorem, that is, $\inf J = \sup K$, on various cost functions.

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1. Introduction. Lions [2] introduced optimal control problems of the variety of distributed parameter systems, for example, elliptic, parabolic, and hyperbolic. Here, we see that adjoint state systems are given cost functional and distributed parameter systems. Duality theory is $\inf J = \sup K$, where J satisfies systems and K satisfies adjoint state systems. The duality theory for the corresponding parabolic systems has been given by Chan [1] and Tanimoto [7]. Park and Lee [5, 6] studied the duality theory for hyperbolic systems. Also, they [4] obtained same results for hyperbolic systems with damping terms. In this paper, we study the duality theory for damped hyperbolic systems with positive controls. These systems have various convex cost functionals. The main objective is to prove the duality theorem for damped hyperbolic systems with positive controls and various cost functions. The main tools are integration by parts and Green's formula.

2. Preliminaries. Let X be a Hilbert space $(\cdot,\cdot)$ and let $\|\cdot\|_X$ denote the inner product and the induced norm on X; X' denotes the dual space of X and $\langle\cdot,\cdot\rangle_{X',X}$ denotes the dual pairing between X' and X. We introduce underlying Hilbert spaces to describe damped second-order evolution equations. Let H be a real pivot Hilbert space; its norm $\|\cdot\|_H$ is simply denoted by $|\cdot|_H$. For $i = 1,2$, let V_i be a real separable Hilbert space. Assume that each pair (V_i,H) is a Gelfand triple space with a notation $V_i \hookrightarrow H \equiv H' \hookrightarrow V_i'$. We suppose that V_1 is continuously embedded in V_2. Then we see that $V_1 \hookrightarrow V_2 \hookrightarrow H \equiv H' \hookrightarrow V_2' \hookrightarrow V'$ and the equalities $\langle\phi,\psi\rangle_{V_1',V_1} = \langle\phi,\psi\rangle_{V_2',V_2}$ for $\phi \in V_2'$, $\psi \in V$, and $\langle\phi,\psi\rangle_{V_1',V_1} = (\phi,\psi)_H$ for $\phi \in H$, $\psi \in V$ hold. Let T be a positive number. We define a function space $W(0,T)$ by

$$W(0,T) = \{y \mid y \in L^2(0,T;V_1),\ y' \in L^2(0,T;V_2),\ y'' \in L^2(0,T;V_1')\} \tag{2.1}$$

with an inner product

$$(y_1,y_2)_{W(0,T)} = \int_0^T \Big\{ (y_1(t),y_2(t))_{V_1} + (y_1'(t),y_2'(t))_{V_2} + (y_1''(t),y_2''(t))_{V_1'} \Big\} dt. \tag{2.2}$$

This becomes a Hilbert space with norm

$$\|y\|_{W(0,T)} = \Big(\|y\|^2_{L^2(0,T;V_1)} + \|y'\|^2_{L^2(0,T;V_2)} + \|y''\|^2_{L^2(0,T;V_1')} \Big)^{1/2}, \tag{2.3}$$

where $' = d/dt$ and $'' = d^2t/dt^2$.

From now on, we set $V = V_2 = H_0^1(\Omega)$ and $H = L^2(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ, and let $Q = \Omega \times (0,T)$. We will give an exact description of damped second-order evolution equations. We consider the bilinear forms defined by

$$\begin{aligned} a_1(t;\phi,\psi) &= \sum_{i,j=1}^n \int_\Omega a_{ij}(t,x) \frac{\partial \phi(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} dx \\ &\quad + \int_\Omega a_0(t,x)\phi(x)\psi(x)dx \quad \forall \phi,\psi \in V, \\ a_2(t;\phi,\psi) &= \sum_{i,j=1}^n \int_\Omega b_{ij}(t,x) \frac{\partial \phi(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} dx \\ &\quad + \int_\Omega b_0(t,x)\phi(x)\psi(x)dx \quad \forall \phi,\psi \in V, \end{aligned} \tag{2.4}$$

where a_{ij}, b_{ij}, a_0, and b_0 are the functions satisfying the following properties:

(i) $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$,

(ii) $a_{ij}, b_{ij}, a_0, b_0 \in C^1([0,T];L^\infty(\Omega))$,

(iii) $\sum_{i,j=1}^n a_{i,j}(t,x)\xi_i\xi_j \ge c_1(\xi_1^2 + \cdots + \xi_n^2)$, $c_1 > 0$, $\xi_i \in \mathbb{R}$,

(iv) $\sum_{i,j=1}^n b_{i,j}(t,x)\xi_i\xi_j \ge c_2(\xi_1^2 + \cdots + \xi_n^2)$, $c_2 > 0$, $\xi_i \in \mathbb{R}$.

Using the above properties, we can show the coercivity condition of a_1 and a_2. Indeed, by (i) and (ii), there exists $K > 0$ such that $|a_0(t,x)| \le K$ a.e., $x \in \Omega$, and for all $t \in [0,T]$. The coercivity condition of a_1 follows from

$$\begin{aligned} a_1(t;\phi,\psi) &\ge c_1 \sum_{i=1}^n \int_\Omega \left| \frac{\partial \phi(x)}{\partial x_i} \right|^2 dx - K \int_\Omega |\phi(x)|^2 dx \\ &\ge c_1 \|\phi\|^2_{H_0^1(\Omega)} - K|\phi|^2_{L^2(\Omega)}. \end{aligned} \tag{2.5}$$

Similarly, we can show the coercivity condition of a_2. Then we can define the operator $A_i(t) \in \mathcal{L}(V_i, V_i')$ for $t \in [0,T]$ deduced by the relation

$$a_i(t;\phi,\psi) = \langle A_i(t)\phi, \psi \rangle_{V_i', V_i} \quad \forall \phi,\psi \in V_i,\ i = 1,2. \tag{2.6}$$

Let $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Then by Nakagiri and Ha [3], there exists a solution $y \in W(0,T)$ of

$$\begin{gathered}\frac{\partial^2 y(u;t)}{\partial t^2} + A_2(t)\frac{\partial y(u;t)}{\partial t} + A_1(t)y(u;t) = u \quad \text{in } (0,T),\\ y(u;0) = y_0, \quad y'(u;0) = y_1 \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Sigma.\end{gathered} \tag{2.7}$$

3. Duality. We choose a control variable space $\mathcal{U} = L^2(Q)$. Let $u \in \mathcal{U}_{ad} = \{u \mid u \geq 0 \text{ a.e. in } Q\} \subset \mathcal{U}$ and y satisfying (2.7). we consider a cost functional given by

$$J(y,u) = \frac{1}{2}\int_0^T |y(u) - z_d|^2 dt + \frac{1}{2}\int_0^T (Nu,u)dt. \tag{3.1}$$

For our purpose, we consider the following systems:

$$\begin{aligned}&y''(u;t) + A_2(t)y'(u;t) + A_1(t)y(u;t) = u \quad \text{in } Q,\\ &y(u;0) = y_0, \quad y'(u;0) = y_1 \quad \text{in } \Omega,\\ &u \geq 0 \quad \text{a.e. in } Q, \quad y(0) \geq 0, \quad y'(0) \geq 0, \quad y = 0 \quad \text{on } \Sigma,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&p''(u;t) - A_2(t)p'(u;t) + \big(A_1(t) - A_2'(t)\big)p(u;t) = y - z_d \quad \text{in } Q,\\ &p = 0 \quad \text{on } \Sigma, \quad p(u;T) = 0, \quad p'(u;T) = 0\\ &p + Nu \geq 0 \quad \text{in } Q, \quad p(0) \geq 0, \quad p'(0) \geq 0,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&u(p + Nu) = 0,\\ &p(0)y(0) = 0 \quad \text{in } \Omega, \quad y'(0)p(0) = 0, \quad y(0)p'(0) = 0.\end{aligned} \tag{3.4}$$

Theorem 3.1. *Let $J = (1/2)\int_0^T |y - z_d|^2 dt + (1/2)\int_0^T (Nu,u)dt$ and $K = -(1/2)\int_0^T |y|^2 dt + (1/2)\int_0^T |z_d|^2 dt - (1/2)\int_0^T (Nu,u)dt$. Assume that y_0, u_0, and p_0 satisfy (3.2), (3.3), and (3.4), respectively, y and u in J satisfy (3.2), and y and u in K satisfy (3.3). Then*

$$\inf_{(3.2)} J = J(y_0,u_0) = K(y_0,u_0) = \sup_{(3.3)} K. \tag{3.5}$$

Proof. (i) We begin by showing that $J = K$ at (y_0,u_0,p_0).

$$\begin{aligned}J(y_0,u_0) &= J(y_0,u_0) - \int_0^T (u_0,p_0)dt - \int_0^T (u_0,Nu_0)dt\\ &= J(y_0,u_0) - \int_0^T (y_0'' + A_2(t)y_0' + A_1(t)y_0, p_0)dt - \int_0^T (u_0,Nu_0)dt\end{aligned}$$

$$
\begin{aligned}
&= J(y_0,u_0) - \int_0^T (y_0, p_0'' - A_2'(t)p_0 - A_2(t)p_0' + A_1(t)p_0)\,dt \\
&\quad - \int_0^T (u_0, Nu_0)\,dt \\
&= J(y_0,u_0) - \int_0^T (y_0, y_0 - z_d)\,dt - \int_0^T (u_0, Nu_0)\,dt.
\end{aligned}
\tag{3.6}
$$

(ii) To show $\inf_{(3.2)} J = J(y_0,u_0)$, we must check that $J(y,u) \geq J(y_0,u_0)$ under (3.2) for (y,u,p) and under (3.2), (3.3), and (3.4) for (y_0,u_0,p_0). Now, we have

$$
\begin{aligned}
J(y,u) - y(y_0,u_0) &\geq \int_0^T (y_0 - z_d, y - y_0)\,dt + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T (p_0'' - A_2(t)p_0' + (A_1(t) - A_2'(t))p_0, y - y_0)\,dt \\
&\quad + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T (p_0, y'' + A_2(t)y' + A_1(t)y)\,dt \\
&\quad - \int_0^T (p_0, y_0'' + A_2(t)y_0' + A_1(t)y_0)\,dt \\
&\quad + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T (p_0, u - u_0)\,dt + \int_0^T (Nu_0, u - u_0)\,dt \\
&\geq 0.
\end{aligned}
\tag{3.7}
$$

(iii) We claim that $K(y,u) \leq K(y_0,u_0)$ under (3.3) for (y,u,p):

$$
\begin{aligned}
&J(y_0,u_0) - J(y,u) \\
&\quad \geq \int_0^T (y - z_d, y - y_0)\,dt + \int_0^T (Nu, u - u_0)\,dt \\
&\quad = \int_0^T (p'' - A_2(t)p' + (A_1(t) - A_2'(t))p, y_0 - y)\,dt + \int_0^T (Nu, u_0 - u)\,dt \\
&\qquad + \int_0^T (y_0'' + A_2(t)y_0' + A_1(t)y_0 - u_0, p_0 - p)\,dt \\
&\quad = -\int_0^T (y - z_d, y)\,dt - \int_0^T (Nu, u)\,dt + \int_0^T (y_0, y_0 - z_d)\,dt \\
&\qquad + \int_0^T (Nu_0, u_0)\,dt + \int_0^T (Nu + p, u_0)\,dt - \int_0^T (u_0, p_0 + Nu_0)\,dt
\end{aligned}
$$

$$\geq -\int_0^T (y-z_d,y)dt - \int_0^T (Nu,u)dt + \int_0^T (y_0,y_0-z_d)dt + \int_0^T (Nu_0,u_0)dt. \tag{3.8}$$

Therefore

$$K(y_0,u_0) \geq K(y,u). \tag{3.9}$$

This completes the proof. □

Now, we observe the terminal value of $y(u;t)$. Since the observation $z(u)$ is given by $y(u;T)$, the cost function is given as

$$J(y,u) = \frac{1}{2}|y(T)-z_d|^2 + \frac{1}{2}\int_0^T (Nu,u)dt. \tag{3.10}$$

We introduce the following systems:

$$\begin{aligned} &y''(u;t)+A_2(t)y'(u;t)+A_1(t)y(u;t)=u \quad \text{in } Q,\\ &y(u;0)=y_0, \quad y'(u;0)=y_1 \quad \text{in } \Omega,\\ &u\geq 0 \quad \text{a.e. in } Q, \quad y(0)\geq 0, \quad y'(0)\geq 0, \quad y=0 \quad \text{on } \Sigma, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &p''(u;t)-A_2(t)p'(u;t)+(A_1(t)-A_2'(t))p(u;t)=0 \quad \text{in } Q,\\ &p=0 \quad \text{on } \Sigma,\\ &p(u;T)=0, \quad p'(u;T)=y(T)-z_d,\\ &-p+Nu\geq 0 \quad \text{in } Q, \quad p(0)\geq 0, \quad p'(0)\geq 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &u(-p+Nu)=0,\\ &p(0)y(0)=0 \quad \text{in } \Omega, \quad y'(0)p(0)=0, \quad y(0)p'(0)=0. \end{aligned} \tag{3.13}$$

Theorem 3.2. *Let $J=(1/2)|y(T)-z_d|^2+(1/2)\int_0^T(Nu,u)dt$ and $K=-(1/2)|y(T)|^2+(1/2)|z_d|-(1/2)\int_0^T(Nu,u)dt$. Assume that y_0, u_0, and p_0 satisfy (3.11), (3.12), and (3.13), respectively, y and u in J satisfy (3.11), and y and u in K satisfy (3.12). Then*

$$\inf_{(3.11)} J = J(y_0,u_0) = K(y_0,u_0) = \sup_{(3.12)} K. \tag{3.14}$$

Proof. (i) We now prove that $J(y_0,u_0)=K(y_0,u_0)$:

$$\begin{aligned}
J(y_0,u_0) &= J(y_0,u_0)+\int_0^T (p_0,u_0)\,dt-\int_0^T (Nu_0,u_0)\,dt\\
&= J(y_0,u_0)+\int_0^T (y_0''+A_2(t)y_0'+A_1(t)y_0,p_0)\,dt-\int_0^T (Nu_0,u_0)\,dt\\
&= \frac{1}{2}|y_0(T)-z_d|^2-(y_0(T)-z_d,y_0(T))-\frac{1}{2}\int_0^T (Nu_0,u_0)\,dt\\
&= -\frac{1}{2}|y_0(T)|^2+\frac{1}{2}|z_d|^2-\frac{1}{2}\int_0^T (Nu_0,u_0)\,dt\\
&= K(y_0,u_0).
\end{aligned} \tag{3.15}$$

(ii) We show that $J(y,u)\geq J(y_0,u_0)$ under (3.11) for (y,u,p) and under (3.11), (3.12), and (3.13) for (y_0,u_0,p_0):

$$\begin{aligned}
J(y,u)-J(y_0-u_0) &\geq (y_0(T)-z_d,y(T)-y_0(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&= (y_0(T)-z_d,y(T)-y_0(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&\quad -\int_0^T (p''-A_2(t)p'+(A_1(t)-A_2'(t))p,y-y_0)\,dt\\
&= (y_0(T)-z_d,y(T)-y_0(T))\\
&\quad +\int_0^T (Nu_0,u-u_0)\,dt-(y_0(T)-z_d,y(T)-y_0(T))\\
&\quad -\int_0^T (p_0,y''-y_0''+A_2(t)y'-A_2(t)y_0'\\
&\qquad +A_1(t)y-A_1(t)y_0)\,dt\\
&= \int_0^T (-p_0+Nu_0,u-u_0)\,dt\\
&\geq 0.
\end{aligned} \tag{3.16}$$

(iii) We have to check that $K(y,u)\leq K(y_0,u_0)$ under (3.12) for (y,u,p):

$$\begin{aligned}
J(y_0,u_0)-J(y,u) &\geq (y(T)-z_d,y_0(T)-y(T))+\int_0^T (Nu,u_0-u)\,dt\\
&= (y(T)-z_d,y_0(T)-y(T))+\int_0^T (Nu,u_0-u)\,dt\\
&\quad -\int_0^T (y_0''+A_2(t)y_0'+A_1(t)y_0-u_0,p_0-p)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= -(y(T)-z_d, y(T)) - \int_0^T (Nu,u) + (y_0(T)-z_d, y_0(T))dt \\
&\quad + \int_0^T (Nu_0,u_0)dt + \int_0^T (-p+Nu,u_0)dt \\
&\quad + \int_0^T (p_0-Nu_0,u_0)dt \\
&\geq -(y(T)-z_d, y(T)) - \int_0^T (Nu,u)dt \\
&\quad + (y_0(T)-z_d, y_0(T)) + \int_0^T (Nu_0,u_0)dt.
\end{aligned}
\tag{3.17}
$$

This shows that

$$K(y_0,u_0) \geq K(y,u). \tag{3.18}$$

Therefore, Theorem 3.2 is proved. □

When the observation $z(u)$ is given by $z(u) = y'(u)$, the cost function is defined as

$$J(y',u) = \frac{1}{2}\int_0^T |y'(u)-z_d|^2 dt + \frac{1}{2}\int_0^T (Nu,u)dt. \tag{3.19}$$

We will consider the following systems:

$$
\begin{aligned}
&y''(u;t) + A_2(t)y'(u;t) + A_1(t)y(u;t) = u \quad \text{in } (0,T), \\
&y(u;0) = y_0, \quad y'(u;0) = y_1 \quad \text{in } \Omega, \\
&u \geq 0 \quad \text{a.e. in } Q, \quad y(0) \geq 0, \quad y'(0) \geq 0, \quad y = 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&p''(u;t) - A_2(t)p'(u;t) + A_1(t)p(u;t) + \int_t^T A_1'(\sigma)p(u;\sigma)d\sigma \\
&\qquad = y'(u;t) - z_d \quad \text{in } Q, \\
&p = 0 \quad \text{on } \Sigma, \quad p(u;T) = 0, \quad p'(u;T) = 0, \\
&-p' + Nu \geq 0 \quad \text{in } Q, \quad p(0) \geq 0, \quad p'(0) \geq 0,
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&u(-p+Nu) = 0 \\
&p(0)y(0) = 0, \quad y'(0)p(0) = 0, \quad y(0)p'(0) = 0, \\
&y(0)\int_0^T A'(\sigma)p(\sigma)d\sigma = 0 \quad \text{in } \Omega.
\end{aligned}
\tag{3.22}
$$

Theorem 3.3. *Let $J = (1/2)\int_0^T |y'-z_d|^2 dt + \int_0^T (1/2)(Nu,u)dt$ and $K = -(1/2)\int_0^T |y'|^2 dt + (1/2)\int_0^T |z_d|^2 dt - (1/2)\int_0^T (Nu,u)dt$. Assume that y_0, u_0, and p_0 satisfy (3.20), (3.21), and (3.22), respectively, y and u in J satisfy (3.20),*

and y and u in K satisfy (3.21). Then

$$\inf_{(3.20)} J = J(y_0', u_0) = K(y_0', u_0) = \sup_{(3.21)} K. \tag{3.23}$$

Proof. (i) First, we claim that $J(y_0', u_0) = K(y_0', u_0)$:

$$\begin{aligned}
J(y_0', u_0) &= J(y_0', u_0) + \int_0^T (p_0', u_0)\,dt - \int_0^T (Nu_0, u_0)\,dt \\
&= J(y_0', u_0) + \int_0^T (p_0', y_0'' + A_2(t)y_0' + A_1(t)y_0)\,dt - \int_0^T (Nu_0, u_0)\,dt \\
&= J(y_0', u_0) + \int_0^T \Bigg(-p_0'' + A_2(t)p_0' - A_1(t)p_0 \\
&\qquad\qquad - \int_t^T A'(\sigma)p(\sigma)\,d\sigma, y' \Bigg)\,dt - \int_0^T (Nu_0, u_0)\,dt \\
&= J(y_0', u_0) - \int_0^T (y_0' - z_d)\,dt - \int_0^T (Nu_0, u_0)\,dt \\
&= -\frac{1}{2}\int_0^T |y_0'|^2\,dt + \frac{1}{2}\int_0^T |z_d|^2\,dt - \frac{1}{2}\int_0^T (Nu_0, u_0)\,dt.
\end{aligned} \tag{3.24}$$

(ii) Second, we must show that $\inf J = J(y_0', u_0)$:

$$\begin{aligned}
J(y', u) - J(y_0', u_0) &\geq \int_0^T (y_0' - z_d, y' - y_0')\,dt + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T \Bigg(p_0'' - A_2(t)p_0' + A_1(t)p_0 \\
&\qquad\qquad + \int_t^T A_1'(\sigma)p_0(\sigma)\,d\sigma, y' - y_0' \Bigg)\,dt \\
&\quad + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T (p_0', -(y'' - y_0'') - A_2(t)(y' - y_0') - A_1(t)(y - y_0))\,dt \\
&\quad + \int_0^T (Nu_0, u - u_0)\,dt \\
&= \int_0^T (-p_0' + Nu_0, u - u_0)\,dt \\
&\geq 0.
\end{aligned} \tag{3.25}$$

(iii) Finally, we check that $\sup K = K(y_0', u_0)$:

$$
\begin{aligned}
&J(y_0', u_0) - J(y', u) \\
&\quad \geq \int_0^T (y' - z_d, y_0', y')\,dt + \int_0^T (Nu, u_0 - u)\,dt \\
&\quad = \int_0^T \left(p'' - A_2(t)p' + A_1(t)p + \int_t^T A_1'(\sigma)p(\sigma)\,d\sigma, y_0' - y' \right) dt \\
&\qquad - \int_0^T (y_0'' + A_2(t)y_0' + A_1(t)y_0 - u_0, p_0' - p')\,dt + \int_0^T (Nu, u_0 - u)\,dt \\
&\quad = -\int_0^T (y' - z_d, y')\,dt + \int_0^T (p', -y_0'' - A_2(t)y_0' - A_1(t)y_0)\,dt \\
&\qquad - \int_0^T \left(y_0', -p_0'' + A_2(t)p_0' - A_1(t)p_0 - \int_t^T A'(\sigma)p_0(\sigma)\,d\sigma \right) dt \\
&\qquad + \int_0^T (Nu, u_0)\,dt - \int_0^T (Nu, u)\,dt + \int_0^T (u_0, p')\,dt \\
&\qquad + \int_0^T (u_0, p_0')\,dt - \int_0^T (u_0, p')\,dt \\
&\quad = -\int_0^T (y' - z_d, y')\,dt - \int_0^T (Nu, u)\,dt + \int_0^T (y_0', y_0' - z_d)\,dt \\
&\qquad + \int_0^T (Nu, u_0)\,dt + \int_0^T (u_0, p_0')\,dt \\
&\qquad - \int_0^T (u_0, p')\,dt + \int_0^T (Nu_0, u_0)\,dt - \int_0^T (Nu_0, u_0)\,dt \\
&\quad \geq -\int_0^T (y' - z_d, y')\,dt - \int_0^T (Nu, u)\,dt \\
&\qquad + \int_0^T (y_0' - z_d, y_0')\,dt + \int_0^T (Nu_0, u_0)\,dt.
\end{aligned}
\tag{3.26}
$$

This implies that

$$
K(y_0', u_0) \geq K(y, u). \tag{3.27}
$$

So, we claimed Theorem 3.3. □

In this case, we observe the terminal value $y'(u;T)$. Since the observation $z(u)$ is given by $y'(u;T)$, the cost functional is given as

$$
J(y', u) = \frac{1}{2}\left| y'(u;T) - z_d \right|^2 + \frac{1}{2}\int_0^T (Nu, u)\,dt. \tag{3.28}
$$

We introduce the following systems:

$$\begin{aligned}
&y''(u;t)+A_2(t)y'(u;t)+A_1(t)y(u;t)=u \quad \text{in } Q,\\
&y(u;0)=y_0, \quad y'(u;0)=y_1 \quad \text{in } \Omega,\\
&u\geq 0 \quad \text{a.e. in } Q, \quad y(0)\geq 0, \quad y'(0)\geq 0 \quad \text{in } \Omega, \quad y=0 \quad \text{on } \Sigma,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&p''(u;t)-A_2(t)p'(u;t)+\big(A_1(t)-A_2'(t)\big)p(u;t)=0 \quad \text{in } Q,\\
&p=0 \quad \text{on } \Sigma,\\
&p(u;T)=y'(T)-z_d, \quad p'(u;T)=A_2(T)y'(T)-z_d,\\
&p+Nu\geq 0 \quad \text{in } Q,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&p(0)\geq 0, \quad p'(0)\geq 0 \quad \text{in } \Omega,\\
&u(p+Nu)=0,\\
&p(0)y(0)=0, \quad y'(0)p(0)=0, \quad y(0)p'(0)=0 \quad \text{in } \Omega.
\end{aligned} \tag{3.31}$$

Theorem 3.4. *Let $J=(1/2)|y'(T)-z_d|^2+(1/2)\int_0^T(Nu,u)dt$ and $K=-(1/2)|y'(T)|^2+(1/2)|z_d|-(1/2)\int_0^T(Nu,u)dt$. Assume that y_0, u_0, and p_0 satisfy (3.29), (3.30), and (3.31), respectively, y and u in J satisfy (3.29), and y and u in K satisfy (3.30). Then*

$$\inf_{(3.29)} J=J(y_0',u_0)=K(y_0',u_0)=\sup_{(3.30)} K. \tag{3.32}$$

Proof. (i) First, we show that $J=K$ at (y_0',u_0,p_0):

$$\begin{aligned}
J(y_0',u_0)&=J(y_0',u_0)-\int_0^T(u_0,p_0)dt-\int_0^T(u_0,Nu_0)dt\\
&=J(y_0',u_0)-\int_0^T(y_0''+A_2(t)y_0'+A_1(t)y_0,p_0)dt-\int_0^T(Nu_0,u_0)dt\\
&=\frac{1}{2}|y_0'(T)-z_d|^2+\frac{1}{2}\int_0^T(Nu_0,u_0)dt-(y_0'(T)-z_d,y_0'(T))\\
&\quad-\int_0^T(Nu_0,u_0)dt\\
&=-\frac{1}{2}|y_0'(T)|^2+\frac{1}{2}|z_d|^2-\frac{1}{2}\int_0^T(Nu_0,u_0)dt\\
&=K(y_0',u_0).
\end{aligned} \tag{3.33}$$

(ii) Second, we check that $J(y',u) \geq J(y_0',u_0)$:

$$\begin{aligned}
&J(y',u)-J(y_0',u_0)\\
&\quad\geq (y_0'(T)-z_d,y'(T)-y_0'(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&\quad= (y_0'(T)-z_d,y'(T)-y_0'(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&\qquad+\int_0^T (p''-A_2(t)p'+(A_1(t)-A_2'(t))p,y-y_0)\,dt\\
&\quad= \int_0^T (p,y''-y_0''+A_2(t)y'-A_2(t)y_0'+A_1(t)y-A_1(t)y_0)\,dt\\
&\qquad+\int_0^T (Nu_0,u-u_0)\,dt\\
&\quad= \int_0^T (p+Nu_0,u-u_0)\,dt\\
&\quad\geq 0.
\end{aligned} \tag{3.34}$$

(iii) Finally, we claim that $K(y',u) \leq K(y_0',u_0)$:

$$\begin{aligned}
&J(y_0',u_0)-J(y',u)\\
&\quad\geq (y_0'(T)-z_d,y'(T)-y_0'(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&\quad= (y_0'(T)-z_d,y'(T)-y_0'(T))+\int_0^T (Nu_0,u-u_0)\,dt\\
&\qquad+\int_0^T (y_0''+A_2(t)y_0'+A_1(t)y_0-u_0,p_0-p)\,dt\\
&\quad= -(y'(T)-z_d,y'(T))-\int_0^T (Nu,u)\,dt+(y_0'(T)-z_d,y'(T))\\
&\qquad+\int_0^T (Nu_0,u_0)\,dt+\int_0^T (Nu+p,u_0)\,dt-\int_0^T (p_0+Nu_0,u_0)\,dt\\
&\quad\geq -(y'(T)-z_d,y'(T))-\int_0^T (Nu,u)\,dt+(y_0'(T)-z_d,y'(T))\\
&\qquad+\int_0^T (Nu_0,u_0)\,dt.
\end{aligned} \tag{3.35}$$

This implies that

$$K(y_0',u_0) \geq K(y',u). \tag{3.36}$$

This completes the proof. □

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