TWO-POINT DISTORTION THEOREMS FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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We give two-point distortion theorems for various subfamilies of analytic univalent functions. We also find the necessary and sufficient condition for these subclasses of analytic functions.

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1. Introduction. Let \( \Omega \) be the family of functions \( \omega(z) \) regular in the unit disc \( D = \{ z \mid |z| < 1 \} \) and satisfying the conditions \( \omega(0) = 0, |\omega(z)| < 1 \) for \( z \in D \).

For arbitrary fixed numbers \( A, B, -1 \leq B < A \leq 1 \), denote by \( P(A,B) \) the family of functions

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\]

regular in \( D \), such that \( p(z) \in P(A,B) \) if and only if

\[
p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]

for some functions \( \omega(z) \in \Omega \) and every \( z \in D \). This class was introduced by Janowski [6].

Moreover, let \( C(A,B,b) \) denote the family of functions

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots
\]

regular in \( D \), such that \( f(z) \in C(A,B,b) \) if and only if

\[
1 + \frac{1}{b^2} f''(z) f'(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]

where \( b \neq 0 \), \( b \) is a complex number, for some functions \( p(z) \in P(A,B) \) and all \( z \in D \).

Next we consider the following class of functions defined in \( D \).
Let \( S^*(A,B,b) \) denote the family of functions

\[
f(z) = z + b_1 z + b_2 z^2 + b_3 z^3 + \cdots
\]
regular in \( D \), such that \( f(z) \in S^*(A,B,b) \) if and only if

\[
1 + \frac{1}{b} \left( z \frac{f'(z)}{f(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]

where \( b \neq 0 \), \( b \) is a complex number, for some functions \( \omega(z) \in \Omega \) and all \( z \in D \).

We obtain the following subclasses of \( C(A,B,b) \) by giving specific values to \( A \), \( B \), and \( b \). For \( A = 1 \), \( B = -1 \), and \( b = 1 \), \( C(1,-1,1) \) is the well-known class of convex functions [3, 4]. For \( A = 1 \), \( B = -1 \), and \( b = 1 - \alpha \) (\( 0 \leq \alpha < 1 \)), \( C(1,-1,1-\alpha) \) is the class of convex functions of order \( \alpha \) introduced by Robertson [9].

For \( A = 1 \), \( B = -1 \), \( C(1,-1,b) \) is the class of convex functions of complex order; this class was introduced by Wiatrowski [12]. For \( A = 1 \), \( B = -1 \), and \( b = e^{-i\lambda}\cos \lambda \), \( |\lambda| < \pi/2 \), \( C(1,-1,\alpha) \) is the class of functions for which \( zf''(z) \) is \( \lambda \)-spirallike; this class was introduced by Robertson [10].

If we write \( 1 + \left( \frac{1}{b} \right) z \left( f''(z) / f'(z) \right) = C(f'(z),f''(z),b) \), then we obtain the following classes:

1. the class \( C(1,0,b) \) defined by \( |C(f'(z),f''(z),b) - 1| < 1 \),
2. the class \( C(\beta,0,b) \) defined by \( |C(f'(z),f''(z),b) - 1| < \beta \), \( 0 \leq \beta < 1 \),
3. the class \( C(\beta,-\beta,b) \) defined by

\[
\left\| \frac{C(f'(z),f''(z),b) - 1}{C(f'(z),f''(z),b) + 1} \right\| < 1, \quad 0 < \beta,
\]

(1.7)

4. the class \( C(1,(1-1/M),b) \) defined by \( |C(f'(z),f''(z),b) - M| < M \), \( M > 1 \).

Similarly, the subclasses of \( S^*(A,B,b) \) are obtained by giving specific values to \( A \), \( B \), and \( b \). These subclasses are obtained in [1, 2, 7, 8, 11].

2. Preliminary lemmas. For the purpose of this paper, we give the following lemmas.

**Lemma 2.1.** The necessary and sufficient condition for \( f(z) \in C(A,B,b) \) is

\[
f(z) = \begin{cases} 
\int_0^z (1 + B\omega(\zeta))^{b(A-B)/B} d\zeta, & B \neq 0 \\
\int_0^z e^{bA\omega(\zeta)} d\zeta, & B = 0,
\end{cases}
\]

(2.1)

where \( \omega(z) \in \Omega \).
**Proof.** Let $B \neq 0$ and let $f(z) \in C(A,B,b)$. From the definition of the class $C(A,B,b)$, we can write

$$1 + \frac{1}{B} z \frac{f''(z)}{f'(z)} = p(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}. \quad (2.2)$$

Equality (2.2) can be written in the form

$$\frac{f''(z)}{f'(z)} = b(A - B) \frac{\omega'(z)}{1 + B \omega(z)} \quad (2.3)$$

by using Jack’s lemma [5]. Integrating both sides of equality (2.3), we obtain

$$f(z) = \int_0^z (1 + B \omega(\zeta))^{b(A - B)/B} d\zeta. \quad (2.4)$$

Equality (2.4) shows that $f(z) \in C(A,B,b)$. Conversely, if we take differentiation from equality (2.3), we obtain

$$f''(z) = (1 + B w(z))^{b(A - B)/B}. \quad (2.5)$$

Differentiating both sides of equality (2.5), we obtain

$$z \frac{f''(z)}{f'(z)} = b(A - B) \frac{z \omega'(z)}{1 + B \omega(z)}. \quad (2.6)$$

Using Jack’s lemma [5] and after the simple calculations from (2.6), we obtain

$$1 + \frac{1}{B} z \frac{f''(z)}{f'(z)} = p(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}. \quad (2.7)$$

This equality shows that $f(z) \in C(A,B,b)$. Similarly, we obtain

$$f(z) = \int_0^z e^{bAw(\zeta)} d\zeta \iff f(z) \in C(A,B,b), \quad B = 0. \quad (2.8)$$

**Lemma 2.2.** Let $f(z) \in C(A,B,b) \Rightarrow zf'(z) \in S^*(A,B,b)$.

**Proof.** Let

$$g(z) = zf'(z). \quad (2.9)$$

Taking a logarithmic derivative of (2.9), and after simple calculations, we get

$$1 + \frac{1}{B} \left( z \frac{g'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{B} z \frac{f''(z)}{f'(z)}. \quad (2.10)$$

This shows that the lemma is true.

**Lemma 2.3.** The class $C(A,B,b)$ is invariant under the rotation so that $f(e^{i\alpha}z) \in C(A,B,b)$, $|\alpha| \leq 1$, whenever $f(z) \in C(A,B,b)$. 

\[ \square \]
Proof. Let \( g(z) = f(e^{i\alpha}z) \). After the simple calculations from this equality we get
\[
1 + \frac{1}{b} z g''(z) g'(z) = 1 + \frac{1}{b} (e^{i\alpha}z) f''(e^{i\alpha}z) f'(e^{i\alpha}z), \quad |\zeta| = |e^{i\alpha}z| < 1.
\] (2.11)

This shows that the lemma is true. \(\blacksquare\)

We note that the class \( S^*(A,B,b) \) is invariant under the rotation so that \( f(e^{i\alpha}z) \in S^*(A,B,b) \), |\( \alpha \)| ≤ 1, whenever \( f(z) \in S^*(A,B,b) \).

Lemma 2.4. Let \( f(z) \) be regular and analytic in \( D \) and normalized so that \( f(0) = 0 \) and \( f'(0) = 1 \). A necessary and sufficient condition for \( f(z) \in C(A,B,b) \) is that for each member \( g(z) \),
\[
g(z) = z \left( \frac{f(z) - f(\zeta)}{z - \zeta} \right)^2, \quad z, \zeta \in D, \quad z \neq \zeta, \quad \zeta = \eta z, \quad |\eta| \leq 1,
\] (2.12)

must be satisfied.

Proof. Let \( f(z) \in C(A,B,b) \), then this function is analytic, regular, and continuous in the unit disc \( D \) and by using Lemmas 2.2 and 2.3, equality (2.12) can be written in the form
\[
g(z) = z (f'(z))^2.
\] (2.13)

Taking the logarithmic derivative from equality (2.13) and after simple calculations, we get
\[
1 + \frac{1}{b} z f''(z) f'(z) = 1 + \frac{1}{2b} \left( z g'(z) - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}.
\] (2.14)

If we consider equality (2.14), the definition of \( C(A,B,b) \), and the definition of \( S^*(A,B,b) \), we obtain that \( g(z) \in S^*(A,B,b) \).

Conversely, let \( g(z) \in S^*(A,B,b) \), then on simple calculations from equality (2.12), we get
\[
1 + \frac{1}{b} \left( z g'(z) - 1 \right) = \frac{1}{b} \left[ \frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b}.
\] (2.15)

If we write
\[
F(z,\zeta) = \frac{1}{b} \left[ \frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b},
\] (2.16)
equality (2.15) can be written in the form
\[
F(z,\zeta) = 1 + \frac{1}{b} \left( z g'(z) - 1 \right).
\] (2.17)
On the other hand,

\[
\lim_{\zeta \to z} F(z, \zeta) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = 1 + \frac{1}{b} \left( \frac{z g'(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.18}
\]

Equality (2.18) shows that \( f(z) \in C(A, B, b) \).

**Corollary 2.5.** If \( f(z) \in C(A, B, b) \), then

\[
2 \left[ 1 + \frac{1}{b} \left( \frac{z f'(z)}{f(z)} - 1 \right) \right] - 1 = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.19}
\]

**Proof.** If we take \( \zeta = 0 \) in \( F(z, \zeta) \), we obtain

\[
F(z, 0) = \frac{1}{b} \left( 2z \frac{f'(z)}{f(z)} - 1 \right) + 1 - \frac{1}{b} = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.20}
\]

This shows that the corollary is true. \( \square \)

**Corollary 2.6.** If \( f(z) \in C(A, B, b) \), then the set of values of \( (z f'(z) / f(z)) \) is the closed disc with centre \( C(r) \) and radius \( g(r) \), where

\[
C(r) = \frac{2 - [2B^2 + |b|(AB - B^2)]r^2}{2(1 - B^2r^2)}, \quad g(r) = \frac{|b|(A - B)r}{2(1 - B^2r^2)}. \tag{2.21}
\]

The proof of this corollary is obtained by using (2.19) and the inequality

\[
\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}, \quad p(z) \in P(A, B). \tag{2.22}
\]

Inequality (2.22) was proved by Janowski [6].

**Lemma 2.7.** If \( f(z) \in C(A, B, b) \) and \( h_{\rho}(z) \) is defined by

\[
h_{\rho}(z) = \frac{f(\rho((z + a)/(1 + z\overline{a}))) - f(a)}{(1 - |a|^2)f'(a)}, \quad a, z \in D, \quad \rho \in (0, 1), \tag{2.23}
\]

then \( h_{\rho}(z) \in C(A, B, b) \).
**Proof.** Let \( B \neq 0 \). After simple calculations from (2.23), we obtain

\[
1 + \frac{1}{b} z \frac{h''_\rho(z)}{h'_\rho(z)} = \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \left[ 1 + \frac{1}{b} \left( \rho \left( \frac{z + a}{1 + za} \right) \right) \frac{f''(\rho((z + a)/(1 + za)))}{f'(\rho((z + a)/(1 + za)))} \right] + \left[ 1 - \frac{1}{b} \frac{2za}{1 + za} - \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \right].
\] (2.24)

On the other hand, if we use Lemma 2.1, we can write

\[
f(\rho((z + a)/(1 + za))) - f(a) = \int_0^z (1 + B\omega(\zeta))^b(A - B)^d\zeta.
\] (2.25)

After a brief computation from equality (2.25), we get

\[
1 + A\omega(z) = \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \left[ 1 + \frac{1}{b} \left( \rho \left( \frac{z + a}{1 + za} \right) \right) \frac{f''(\rho((z + a)/(1 + za)))}{f'(\rho((z + a)/(1 + za)))} \right] + \left[ 1 - \frac{1}{b} \frac{2za}{1 + za} - \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \right].
\] (2.26)

Let \( B = 0 \). Similarly,

\[
f(\rho((z + a)/(1 + za))) - f(a) = \int_0^z e^{b\omega(\zeta)} d\zeta \Rightarrow 1 + A\omega(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} = 1 + A\omega(z)
\]

\[
= \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \left[ 1 + \frac{1}{b} \left( \rho \left( \frac{z + a}{1 + za} \right) \right) \frac{f''(\rho((z + a)/(1 + za)))}{f'(\rho((z + a)/(1 + za)))} \right] + \left[ 1 - \frac{1}{b} \frac{2za}{1 + za} - \frac{(1 - |a|^2)z}{(z + a)(1 + za)} \right].
\] (2.27)

In (2.26) and (2.27), letting \( z = e^{i\theta} \) and \( \omega = \rho((e^{i\theta} + a)/(1 + e^{i\theta}a)) \) gives

\[
\frac{1 + A\omega(z)}{1 + B\omega(z)} = \frac{(1 - |a|^2)}{1 + ae^{-i\theta}a^2} \left[ 1 + \frac{1}{b} \frac{f''(\omega)}{f'(\omega)} \right] + \left[ 1 - \frac{1}{b} \frac{2e^{i\theta}a}{1 + e^{i\theta}a} - \frac{(1 - |a|^2)e^{i\theta}}{1 + e^{-i\theta}a} \right].
\] (2.28)
and we conclude that $h_\rho(z)$ is in (2.27) for every admissible $\rho$. From the compactness of $C(A,B,b)$ and (2.28), we infer that $h(z) = \lim_{\rho \to 1} h_\rho(z)$ is in $C(A,B,b)$.

3. Two-point distortion for the class $C(A,B,b)$. In this section, we give two-point distortion theorems for the class $C(A,B,b)$.

**Theorem 3.1.** Let $f(z) \in C(A,B,b)$. Then for $|z| = r$, $0 \leq r < 1$,

$$
\frac{(1 + B|z|)^{(B-A)(|b|-\text{Re}b)/2B}}{(1 - B|z|)^{(B-A)(|b|+\text{Re}b)/2B}} \leq |f''(z)| \leq \frac{(1 - B|z|)^{(B-A)(|b|-\text{Re}b)/2B}}{(1 + B|z|)^{(B-A)(|b|+\text{Re}b)/2B}}, \quad B \neq 0,
$$

$$
e^{-A|b||z|} \leq |f'(z)| \leq e^{A|b||z|}, \quad B = 0. \tag{3.1}
$$

**Proof.** If we use the definition of the class $C(A,B,b)$, then we obtain

$$
\text{Re} \left( z \frac{f'''(z)}{f''(z)} \right) \geq \frac{\text{Re} b(B^2 - AB)r^2 - |b|(A - B)r}{1 - B^2r^2}, \quad B \neq 0, \tag{3.2}
$$

since

$$
\text{Re} z \frac{f''(z)}{f'(z)} = r \frac{\partial}{\partial r} \log |f'(z)|, \quad |z| = r; \tag{3.3}
$$

and using (3.2), we obtain

$$
\frac{\partial}{\partial r} \log |f'(z)| \geq \frac{\text{Re} b(B^2 - AB)r - |b|(A - B)}{(1 - B^2r^2)}. \tag{3.4}
$$

Integrating both sides of inequality (3.4) from 0 to $r$, we obtain

$$
|f'(z)| \geq \left( \frac{1 + B|z|}{1 - B|z|} \right)^{(B-A)(|b|-\text{Re}b)/2B} \tag{3.5}
$$

Similarly, we obtain the bounds on the right-hand side of (3.1). If $B = 0$, then we have

$$
-|b|Ar \leq \text{Re} z \frac{f'''(z)}{f''(z)} \leq |b|Ar; \tag{3.6}
$$
and using (3.3), we obtain

$$-|b|A \leq \frac{\partial}{\partial r} \log |f'(z)| \leq |b|A. \quad (3.7)$$

Integrating both sides of inequality (3.7) from 0 to $r$, we obtain the desired result.

**Theorem 3.2.** If $f(z) \in C(A,B,b)$, then, for $|z| = r$, $0 \leq r < 1$,

$$\frac{|z|(1 + B|z|)^{(B-A)(|b|-Re b)/4B}}{(1 - B|z|)^{(B-A)(|b|+Re b)/4B}} \leq |f(z)| \leq \frac{|z|(1 - B|z|)^{(B-A)(|b|-Re b)/4B}}{(1 + B|z|)^{(B-A)(|b|+Re b)/4B}}, \quad B \neq 0,$$

$$|z|e^{-|b||z|/2} \leq |f(z)| \leq |z|e^{|b||z|/2}, \quad B = 0.

(3.8)

**Proof.** If we use Corollaries 2.5 and 2.6 and the definition of the classes $C(A,B,b)$ and $P(A,B)$, we can write

$$\left|2 \left(1 + \frac{1}{b} \left(\frac{z^2 f'(z)}{f(z)} - 1\right) - 1\right) - \frac{1 - ABr^2}{1 - B^2 r^2}\right| \leq \frac{(A - B)r}{1 - B^2 r^2}. \quad (3.9)$$

After the simple calculations from inequality (3.9), we get

$$\text{Re} z \frac{f'(z)}{f(z)} \geq \frac{2 - |b|(A - B)r - (2B^2 - (B^2 - AB)\text{Re} b)r^2}{1 - B^2 r^2} \quad (3.10)$$

since

$$\text{Re} z \frac{f'(z)}{f(z)} = r \frac{\partial}{\partial r} \log |f(z)|; \quad (3.11)$$

and using (3.10), we obtain

$$\frac{\partial}{\partial r} \log |f(z)| \geq \frac{2 - |b|(A - B)r - (2B^2 - (B^2 - AB)\text{Re} b)r^2}{2r(1 - B^2 r^2)}. \quad (3.12)$$

Integrating both sides of this inequality from 0 to $r$, we obtain

$$|f(z)| \geq \frac{|z|(1 + B|z|)^{(B-A)(|b|-Re b)/4B}}{(1 - B|z|)^{(B-A)(|b|+Re b)/4B}}. \quad (3.13)$$

Similarly, we obtain the upper bounds in (3.8). Thus we end the proof. \qed
We note that the bounds in Theorems 3.1 and 3.2 are sharp because the extremal function is

\[
f_* (z) = \begin{cases} 
e^{Abz}, & B \neq 0, \\
\frac{z(1-Bz)}{(1+Bz)} (B-A) \frac{|b|-2Reb}{4B}, & B = 0,
\end{cases}
\]

(3.14)

**Corollary 3.3.** Let \( f(z) \in C(A,B,b) \). Then

\[
\alpha F_1 (u,v) \leq |f(u) - f(v)| \leq \alpha F_2 (u,v), \quad B \neq 0,
\]

\[
\alpha G_1 (u,v) \leq |f(u) - f(v)| \leq \alpha G_2 (u,v), \quad B = 0,
\]

(3.15)

where

\[
\alpha = (1 - |v|^2) \frac{|u - v|}{|1 - v u|},
\]

\[
F_1 (u,v) = \frac{(1+B|z|)}{(1-B|z|)} (B-A) \frac{|b|-2Reb}{4B},
\]

(3.16)

\[
F_2 (u,v) = \frac{(1-B|z|)}{(1+B|z|)} (B-A) \frac{|b|-2Reb}{4B},
\]

\[
G_1 (u,v) = e^{-(3/2)|b|A(|u-v|/|1-u v|)} ,
\]

\[
G_2 (u,v) = e^{(3/2)|b|A(|u-v|/|1-u v|)}.
\]

**Proof.** If we consider Lemmas 2.1 and 2.7 and Theorem 3.2, then we can write

\[
\frac{|z|(1+B|z|)}{(1-B|z|)} (B-A) \frac{|b|-2Reb}{4B} \leq \left| \frac{f((z+a)/(1+z\overline{a})) - f(a)}{(1-|a|^2)f'(a)} \right| \leq \frac{|z|(1-B|z|)}{(1+B|z|)} (B-A) \frac{|b|-2Reb}{4B}, \quad B \neq 0,
\]

(3.17)

\[
|z| e^{-|b|A|z|/2} \leq \left| \frac{f((z+a)/(1+z\overline{a})) - f(a)}{(1-|a|^2)f'(a)} \right| \leq |z| e^{-|b|A|z|/2}, \quad B = 0.
\]
Inequalities (3.17) can be written in the form
\[
(1 - |a|^{2}) |f'(a)| M_{1}(|z|) \leq \left| f\left( \frac{z + a}{1 + \overline{z}a} \right) - f(a) \right|
\leq (1 - |a|^{2}) |f'(a)| M_{2}(|z|), \quad B \neq 0,
\]
\[
(1 - |a|^{2}) |f'(a)| N_{1}(|z|) \leq \left| f\left( \frac{z + a}{1 + \overline{z}a} \right) - f(a) \right|
\leq (1 - |a|^{2}) |f'(a)| N_{2}(|z|), \quad B = 0,
\]
where
\[
M_{1}(|z|) = \frac{|z|(1 + B|z|)^{(B-A)(|b|-\text{Re}b)/4B}}{(1 - B|z|)^{(B-A)(|b|+\text{Re}b)/4B}} \cdot \frac{(1 + B|z|)^{(B-A)(|b|+\text{Re}b)/4B}}{(1 - B|z|)^{(B-A)(|b|-\text{Re}b)/4B}},
\]
\[
M_{2}(|z|) = \frac{|z|(1 - B|z|)^{(B-A)(|b|-\text{Re}b)/4B}}{(1 + B|z|)^{(B-A)(|b|+\text{Re}b)/4B}} \cdot \frac{(1 + B|z|)^{(B-A)(|b|+\text{Re}b)/4B}}{(1 - B|z|)^{(B-A)(|b|-\text{Re}b)/4B}},
\]
\[
N_{1}(|z|) = |z|e^{-|b|A}|z|/2,
\]
\[
N_{2}(|z|) = |z|e^{-|b|A}|z|/2.
\]

If we take \( v = a, u = (z + v)/(1 + z\overline{v}) \), or \( z = (u - v)/(1 - u \cdot \overline{v}) \), and if we use Theorem 3.1 in inequalities (3.18), we obtain the desired result. \( \square \)

We note that these inequalities are sharp because the extremal function is
\[
f_{\ast}(z) = \begin{cases} e^{Abz}, & B \neq 0 \\ \frac{z(1-Bz)^{(B-A)(|b|-2\text{Re}b)/4B}}{(1+Bz)^{(B-A)(|b|+\text{Re}b)/4B}}, & B = 0. \end{cases}
\]

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