COMMUTING IDEMPOTENTS OF AN H^* -ALGEBRA

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Commutative H^* -algebra is characterized in terms of idempotents. Here we offer three characterizations.

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1. Introduction. In the past, the author used commuting idempotents to characterize continuous functions defined on a certain space [3, 4]. For example, it was shown in [3] that a certain Banach algebra is isometrically isomorphic to the space $C(S)$ of all continuous complex-valued functions on a totally disconnected compact space S . In the sequel, we use idempotents to characterize commutative H^* -algebras. An interesting consequence of this results (Theorems 3.1, 3.2, and 3.3 below) is somewhat unusual forms of characterizations of Hilbert spaces.

2. Preliminaries. A proper H^* -algebra is a Hilbert algebra (a Banach algebra with a Hilbert space norm) which has an involution $x \rightarrow x^*$ such that $(xy, z) = (y, x^*z) = (x, zy^*)$ for all $x, y, z \in A$. An idempotent is a *nonzero* member e of A such that $e^2 = e$.

DEFINITION 2.1. An idempotent e in an algebra a is said to be *primary* if $ef = e$ for any idempotent $f \in A$ such that $ef \neq 0$.

Note that the product of any two distinct primary idempotents is zero.

3. Main results. Let A be a complex Banach algebra. Let I be the set of idempotents in A , let P be the set of all primary idempotents, and let A_o be the set of all (complex) finite linear combinations of primary idempotents $A_o = \{\sum_{i=1}^n \lambda_i e_i : e_i \in P, i = 1, \dots, n \text{ and } \lambda_1, \dots, \lambda_n \text{ are complex numbers}\}$.

THEOREM 3.1. *Let A be a complex Banach algebra such that all members of P commute and A_o is dense in A . Assume further that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ for all $x, y \in A_o$ such that $xy = 0$. Then, A is a proper commutative H^* -algebra.*

PROOF. First, note that A is commutative since members of A_o commute. Condition " $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $xy = 0$ " implies that $\|x\|^2 = \sum_{i=1}^n |\lambda_i|^2 \|e_i\|^2$

for any member $x = \sum_{i=1}^n \lambda_i e_i$ of A_o (it can be readily established using induction on n). This fact can be used to show that there is an inner product $(\cdot, \cdot)$ and an involution $x \rightarrow x^*$ such that $(x, x) = \|x\|^2 = \|x^*\|^2$ and $(xy, z) = (y, x^*z)$ for all $x, y, z \in A_o$. In fact, all we have to do is to set $(x, y) = (\sum_i \lambda_i e_i, \sum_j \mu_j e_j) = \sum_{i,j} \lambda_i \bar{\mu}_j \|e_i e_j\|^2$ and $x^* = \sum \bar{\lambda}_i e_i$ for members $x = \sum \lambda_i e_i$ and $y = \sum \mu_j e_j$ of A_o (note that $e_i e_j = 0$ if $i \neq j$).

We leave it to the reader to verify that A is isometrically isomorphic to the space $L^2(P, \mu)$ of all complex-valued functions $x(\cdot)$ on P such that $\sum_{e \in P} |x(e)|^2 \|e\|^2 < \infty$, with pointwise multiplication of members of $L^2(P, \mu)$ ($xy(e) = x(e)y(e)$ for all $x, y \in L^2(P, \mu)$). (Measure μ on P is the set function that associates with each member e of P the positive number $\|e\|$.) (One can interpret the expression " $\sum_{e \in P} |x(e)|^2 \|e\|^2 < \infty$ " to mean "there exists a countable subset $P_x = \{e_1, e_2, \dots, e_n, \dots\}$ of P such that $x(e) = 0$ if $e \notin P_x$ and $\sum_{i=1}^\infty |x(e_i)|^2 \|e_i\|^2$ converges"). Obviously, $L^2(P, \mu)$ is a proper commutative H^* -algebra under the pointwise multiplication. $\square$

THEOREM 3.2. *Let A be a complex Banach algebra such that the members of I commute and that the set A_1 of finite linear combinations of I is dense in A ($A_1 = \{\sum_{i=1}^n \lambda_i e_i, e_1 \cdot \dots \cdot e_n \in I \text{ and } \lambda_1, \dots, \lambda_n \text{ are complex numbers}\}$). Assume further that*

- (i) *for each $e \in I$, there exists $f \in P$ such that $ef \neq 0$,*
- (ii) *if $x, y \in A$ and $xy = 0$, then $\|x + y\|^2 + \|x\|^2 + \|y\|^2$.*

Then, A is a commutative proper H^ -algebra.*

PROOF. First, note that A is commutative. Also, it follows from assumption (i) that, for each $e \in I$, there are primary idempotents $f_1 \cdot \dots \cdot f_n$ (a finite number) such that $e = f_1 + f_2 + \dots + f_n$ and $f_i f_j = 0$ if $i \neq j$. To see this, let $e \in I$ and $f \in P$ be such that $ef \neq 0$. Then, $ef = f$ and $g = e - f$ is also an idempotent such that $fg = 0$. This means that $\|e\|^2 = \|f + g\|^2 = \|f\|^2 + \|g\|^2$, and so $\|g\|^2 = \|e\|^2 - \|f\|^2 < \|e\|^2 - 1$ since $\|f\| > 1$ ($\|f\| = \|f^2\| \leq \|f\|^2$ and $\|f\| \neq 0$). It follows that if n is any natural number such that $\|e\|^2 < n$, then $\|g\|^2 < n - 1$. Now, we can use induction on n to see that each idempotent can be represented as a finite sum of primary idempotents. (Note that if g is a finite sum of mutually annihilating members of P , then so is $e = f + g$ since $f \in P$ and $fg = 0$.)

But this means that A_o (the space of finite linear combinations of the members of P) is dense in A . Theorem 3.1 now implies that A is an H^* -algebra. $\square$

THEOREM 3.3. *Let A be a Banach algebra such that all members of I commute, that the space of all finite linear combinations of members of I is dense in A , and that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $xy = 0$ for any $x, y \in A$.*

Assume further that, for each closed ideal J in A , there is an ideal J^1 such that $J \cap J^1 = (0)$ and $J + J^1 = A$, that is, for any $a \in A$, there are $a_1 \in J$ and $a_2 \in J^1$ such that $a = a_1 + a_2$. Then, A is a commutative H^ -algebra.*

PROOF. We only need to show that, for each $e \in I$, there exists $f \in P$ such that $ef \neq 0$.

Let $e \in I$ and let N be the annihilator of e , $N = \{x \in A : xe = 0\}$. Then, $ex - x \in N$ for each $x \in A$, that is, e is a relative identity modulo N and N is a regular ideal [2, Section 20] (see also [2, Subsection 22D and Subsection 22E]). Let M be the maximal regular ideal such that $M \supset N$ [2, Subsection 20B] and let M^1 be an ideal such that $M + M^1 = A$ and $M \cap M^1 = \{0\}$. Write $e = f + u$ with $f \in M^1$, $u \in M$. Then, f is also a relative identity modulo M ($fx - x = (e - u)x - x = ex - x - ux \in M$). Also, f is an idempotent since $ff - f \in M^1 \cap M = 0$ and $f \neq 0$ (otherwise $e \in M$).

Now, we show that $f \in P$, that is, f is primary. Let $h \in I$ be such that $fh \neq 0$. If $fh \neq f$, then $f - fh \neq 0$, and we have a decomposition $f = f_1 + f_2$ of f as a sum of nonzero idempotents $f_1 = fh$ and $f_2 = f - fh$ such that $f_1 f_2 = 0$. Let $M_1 = f_1 A + M = \{f_1 a + m : a \in A, m \in M\}$. It is a regular ideal including M strictly larger than M ($f_1 \in M_1$, $f_1 \notin M$) (also, $ex - x \in M \subset M_1$). This contradicts the maximality of M . Thus, $fh = f$ for each $h \in I$ with $fh \neq 0$. Theorem 3.2 now implies that A is a commutative H^* -algebra. $\square$

4. Some properties of H^* -algebra. Now, we show that every proper commutative H^* -algebra satisfies assumptions of Theorems 3.1, 3.2, and 3.3. First, note that, in any commutative Banach algebra, an idempotent e is primary if and only if it *cannot* be written as a sum $e = e_1 + e_2$ of two mutually annihilating, $e_1 e_2 = 0$, nonzero idempotents e_1 and e_2 .

Indeed, let e be primary and assume that $e = e_1 + e_2$ for some nonzero idempotents e_1 and e_2 such that $e_1 e_2 = 0$. Then, $e_1 e = e_1$ and $e_1 e = e$ since e is primary. This implies $e_2 = 0$, which is a contradiction.

Conversely, assume that $e \neq ef$ for some idempotent f such that $e_1 = ef \neq 0$. Then, $e_2 = f - ef$ is also nonzero idempotent such that $e_1 e_2 = 0$ ($e_2^2 = (e - f)^2 = e - ef - ef + ef = e_2$ and $e_1 e_2 = ef(e - f) = ef - ef = 0$). This means that if e is not primary, then it has a decomposition $e = e_1 + e_2$ into mutually annihilating nonzero idempotents.

In the case of a proper commutative H^* -algebra, the fact that e is primary would also imply that e is selfadjoint: $e^* e = e$ since e^* is also idempotent ($(e^*)^2 = (ee)^* = e^*$). This means that, in this case, e is primary if and only if e is selfadjoint and primitive in the sense of Ambrose [1, Definition 3.3, page 376].

It follows that [1, Corollary 4.1, page 382] implies that, in each proper commutative H^* -algebra, the set A_o (of finite linear combinations of primary idempotents) is dense in A .

The remark in [1, page 369] of (orthogonal complement of any ideal is an ideal of the same kind (in the paragraph above Definition 1.4)) implies that every commutative H^* -algebra satisfies the condition of Theorem 3.3 about the existence of the ideal J^1 . But it was shown in the proof of Theorem 3.3 that

this fact implies that, for each idempotent e , there is a primary idempotent f such that $ef \neq 0$, which is one of the assumptions of [Theorem 3.2](#).

It remains to show that $xy = 0$ for any x, y in a proper commutative H^* -algebra implies that x is orthogonal to y (which implies that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$). We will use the terminology of [\[1\]](#).

So, let A be a commutative proper H^* -algebra. Let P be the set of all primary idempotent. Then, P is a maximal family of doubly orthogonal primitive idempotents (see [\[1, Definition 3.1\]](#)) (it was remarked above that the product ef of any two distinct primary idempotents e, f is zero, $ef = 0$) and so it follows from [\[1\]](#) that $A = \sum_{\alpha} e_{\alpha} A$ and each $e_{\alpha} A$ is isomorphic to the complex field. This means that each $x \in A$ has the form $x = \sum_{e \in P} x(e)e$ for some complex number $x(e)$ for each $e \in P$ and $\sum_{e \in P} |x(e)|^2 \|e\|^2 < \infty$. It is easy to see that the products xy and (x, y) are expressible in terms of this representation by the formulae $xy = \sum_{e \in P} x(e)y(e)e$ and $(x, y) = \sum_{e \in P} x(e)\overline{y(e)}\|e\|^2$. From this, it is easy to show that $xy = 0$ implies $(x, y) = 0$ (note that if the product of any two complex numbers is zero, then either of the numbers (or both) is zero).

5. Some consequences. One of consequences of [Theorems 3.1, 3.2, and 3.3](#) is that each of the above theorems can be used to characterize Hilbert spaces. The reason for that is the fact that each Hilbert space has also a structure of a proper H^* -algebra. To see this, all we have to do is to take any orthonormal base $\{e_{\alpha}\}_{\alpha \in \Gamma}$ of a Hilbert space H and define multiplication on it by setting $xy = \sum_{\alpha \in \Gamma} x(e_{\alpha})y(e_{\alpha})e_{\alpha}$, where $x = \sum_{\alpha \in \Gamma} x(e_{\alpha})e_{\alpha}$ and $y = \sum_{\alpha \in \Gamma} y(e_{\alpha})e_{\alpha}$ are representations of x and y in terms of the orthonormal base $\{e_{\alpha}\}_{\alpha \in \Gamma}$ ($x(e_{\alpha}) = (x, e_{\alpha})$) [\[5\]](#). It is easy to see that H becomes a proper commutative H^* -algebra with respect to the involution $x \rightarrow x^*$ with $x^* = \sum_{\alpha \in \Gamma} \overline{x(e_{\alpha})}e_{\alpha}$. Thus, we have the following characterization of a Hilbert space (it is somewhat awkward, yet it is a characterization). It is stated as a corollary of the above theorems.

COROLLARY 5.1. *Let B be a Banach space. Assume that it is possible to define a multiplication on B with respect to which it is a Banach algebra that has properties stated in either one of the theorems above ([Theorems 3.1, 3.2, and 3.3](#)). Then, B is a Hilbert space.*

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