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Retraction

Retracted: On the Interlacing Property and the Routh-Hurwitz Criterion

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International Journal of Mathematics and Mathematical Sciences has retracted the article titled "On the Interlacing Property and the Routh-Hurwitz Criterion" [1], this article has been retracted as it is found to contain a substantial amount of material from the paper "An Elementary Derivation of the Routh-Hurwitz Criterion," authored by M. T. Ho, A. Datta, and S. P. Bhathacharyya, and it is published in "IEEE Transactions on Automatic Control" in 1998.

References

[1] Z. Zahreddine, "On the Interlacing Property and the Routh-Hurwitz Criterion," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, Article ID 624382, 11 pages, 2003.

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ON THE INTERLACING PROPERTY AND THE ROUTH-HURWITZ CRITERION

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Unlike the Nyquist criterion, root locus, and many other stability criteria, the well-known Routh-Hurwitz criterion is usually introduced as a mechanical algorithm and no attempt is made whatsoever to explain why or how such an algorithm works. It is widely believed that simple derivations of this important criterion are highly requested by the mathematical community. In this paper, we address this problem and provide a simple proof of the Routh-Hurwitz criterion based on two generalizations of an interesting property known in stability theory as the interlacing property. Within the same context, the singularities that may arise in the Routh-Hurwitz criterion are also dealt with.

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1. Introduction. The problem of root distribution of a polynomial has been long treated, and it is of virtual importance in diverse mathematical and engineering applications: spectral analysis, numerical computations, control theory, and digital signal processing, to name a few. The first systematic approach to investigate root distribution of a real polynomial was presented by Sturm [23]. Then, the stability of a linear continuous-time systems of differential equations with real coefficients was studied by many authors, and the number of roots of the characteristic polynomial in the open right-half plane was obtained by Hermite [10], Routh [21], Hurwitz [12], Marden [19], and Liënard and Chipart [17]. More recently, Kreĭn and Naĭmark [14], Levinson and Redheffer [16], Lipatov and Sokolov [18], and others had further contributions, which were still mainly restricted to the real case. Complex systems of differential equations arise in the study of multidimensional systems [8]. The complex counterpart of the Routh array was considered in [25], where necessary and sufficient conditions were given for the asymptotic stability of a system of differential equations with complex coefficients. In [25], the extended Routh array (ERA) was introduced and proved to be the natural extension of the Routh array to the complex case. In [26], a generalized version of the ERA was proposed to handle the singularities that may arise in the ERA, and it generalized the results of [6] restricted to the real case. For the stability of a discrete-time system of difference equations, the number of roots outside the unit circle was determined by Cohn and Schur [5, 22]. For further work on the stability of

discrete systems, see, for example, [13, 15, 20]. Explicit relationships between Routh-Hurwitz and Schur-Cohn types of stability were established in [24]. The concept of Routh-Hurwitz stability was extended to the convex hull of $n \times n$ matrices in [28] with applications to the stability of interval dynamical systems.

The general problem of root distribution of a polynomial in some subregions of the complex plane, for example, half-planes, circles, sectors, and ellipses, has been investigated by many authors [2, 3, 4, 7, 9, 11, 27]. The historic Routh stability criterion remains the backbone of stability analysis in linear systems, and it has been used to solve a wide range of problems. By far, the most authoritative reference for the Routh-Hurwitz test is Gantmacher [7] where the proofs depend on Cauchy indices and Sturm chains. However, much research efforts are still made towards, and many new results are continuously derived on this subject, not only for the further theoretical development but also for the establishment of simpler and more easily realizable criteria in practice.

In Section 2 of this paper, we offer two generalizations of the interlacing property based on the net-accumulated phase of the frequency-response of a real polynomial. The new results are then applied in Section 3 to derive a very simple proof of the Routh-Hurwitz stability criterion, something desperately required in standard literature on stability analysis and control theory. In Section 4, we look at the singularities that may arise in the Routh-Hurwitz criterion, and we offer appropriate remedies to each case. We end with some concluding remarks.

2. Generalizations of the interlacing property. In this section, we derive two generalizations of the interlacing property by first stating a fundamental relationship between the net-accumulated phase of the frequency-response of a real polynomial and the difference between the numbers of roots of the polynomial in the open left and open right half-planes and, second, by developing a procedure for systematically determining the net-accumulated phase. Consider a real polynomial $f(z)$ of degree n with no zeros on the imaginary axis

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n. \tag{2.1}$$

Definition 2.1. Let l and r denote the numbers of roots of $f(z)$ in the open left and open right half-planes, respectively. Then, the signature of $f(z)$, denoted as $\sigma(f)$, is defined by $\sigma(f) = l - r$.

Since $n = l + r$, it follows that $\sigma(f)$ and n uniquely determine l and r and hence the root distribution of $f(z)$. Now, for every frequency $s \in \mathbb{R}$, $f(js)$ is a point in the complex plane. Let $g(s)$ and $h(s)$ be two functions defined pointwise by $g(s) = \mathrm{Re}[f(js)]$ and $h(s) = \mathrm{Im}[f(js)]$. It follows that, for all $s \in \mathbb{R}$,

$$f(js) = g(s) + jh(s). \tag{2.2}$$

Furthermore, we recall that, at any given frequency s, the phase angle of $f(js)$ is given by $\theta(s) = \tan^{-1}[h(s)/g(s)]$. If $\Delta_0^\infty \theta$ represents the net change in argument $\theta(s)$, as s increases from 0 to ∞, then we can state the following lemma [7].

Lemma 2.2. *Let $f(z)$ be a real polynomial with no imaginary axis roots. Then, $\Delta_0^\infty \theta = (\pi/2)\sigma(f)$.*

Since $\theta(s) = \tan^{-1}[h(s)/g(s)]$, then the rate of change of phase, with respect to frequency, is given by

$$\begin{aligned}\frac{d\theta(s)}{ds} &= \frac{1}{1+h^2(s)/g^2(s)}\frac{(dh(s)/ds)g(s)-(dg(s)/ds)h(s)}{g^2(s)}\\ &= \frac{(dh(s)/ds)g(s)-(dg(s)/ds)h(s)}{g^2(s)+h^2(s)}.\end{aligned} \tag{2.3}$$

If $g(s)$ and $h(s)$ are known for all values of s, then we can integrate (2.3) to obtain the net phase accumulation. Since we know that every time the polar plot $f(js)$ makes a transition from the real axis to the imaginary axis or vice versa, there can be at most a net phase change of $\pm(\pi/2)$ radians. Therefore, to calculate the net accumulation of a phase over all frequencies, it is not necessary to know the precise rate of change of a phase at each and every frequency. The precise sign of the phase change can be determined by checking (2.3) at the real or imaginary axis crossing of the $f(js)$ plot. Since at a real or imaginary axis crossing one of the two terms in the numerator of (2.3) vanishes and the denominator is always positive, the actual determination of the sign of the phase change becomes even simpler.

For a given polynomial $f(z)$ of a degree greater than or equal to one, either the real part or the imaginary part or both of $f(js)$ become infinitely large as $s \to \pm\infty$. However, if we want to count the total phase accumulation in integral multiples of real to imaginary axis crossings or imaginary to real axis crossings, it is important that the frequency-response plot used approaches either the real or imaginary axis as $s \to \pm\infty$. To achieve this, we normalize the plot of $f(js)$ by scaling it with $1/k(s)$, where $k(s) = (1+s^2)^{n/2}$. Since $k(s)$ does not have any real roots, this scaling will ensure that the normalized frequency-response plot $f_k(js) = g_k(s) + jh_k(s)$ intersects either with the real axis or the imaginary axis at $\pm\infty$ while, at the same time, keeping unchanged the finite frequencies at which the $f(js)$ plot intersects the real and imaginary axes.

The subsequent development of the paper makes extensive use of standard signum function $\operatorname{sgn}: \mathbb{R} \to \{-1, 0, 1\}$ defined by

$$\operatorname{sgn}[x] = \begin{cases} -1, & \text{if } x < 0,\\ 0, & \text{if } x = 0,\\ 1, & \text{if } x > 0.\end{cases} \tag{2.4}$$

Now, with $f(z)$, $g(s)$, $h(s)$, $g_k(s)$, and $h_k(s)$ as defined above, let

$$0 = s_0 < s_1 < s_2 < \cdots < s_{m-1} \tag{2.5}$$

be the real, nonnegative distinct finite zeros of $h_k(s)$ with odd multiplicities. Clearly, the zeros of $h_k(s)$ of even multiplicities can be skipped while counting the net phase accumulation because the function $h_k(s)$ does not change sign while passing through a real zero of even multiplicity. Also, define $s_m = \infty$.

The following simple facts can now be stated:

(i) if s_i, s_{i+1} are both zeros of $h_k(s)$, then

$$\Delta_{s_i}^{s_{i+1}}\theta = \frac{\pi}{2}\{\operatorname{sgn}[g_k(s_i)] - \operatorname{sgn}[g_k(s_{i+1})]\}\cdot\operatorname{sgn}[h_k(s_i^+)]; \tag{2.6}$$

(ii) if s_i is a zero of $h_k(s)$ while s_{i+1} is not a zero of $h_k(s)$, a situation possible only when $s_{i+1} = \infty$ is a zero of $g_k(s)$ (n odd), then

$$\Delta_{s_i}^{s_{i+1}}\theta = \frac{\pi}{2}\operatorname{sgn}[g_k(s_i)]\cdot\operatorname{sgn}[h_k(s_i^+)]; \tag{2.7}$$

(iii) and

$$\operatorname{sgn}[h_k(s_{i+1}^+)] = -\operatorname{sgn}[h_k(s_i^+)], \quad i = 0,1,2,\ldots,m-2. \tag{2.8}$$

Equation (2.6) is straightforward, while (2.8) simply states that $h_k(s)$ changes sign as it passes through a zero of odd multiplicity. Equation (2.7) follows directly from (2.3).

The repetitive use of (2.8) leads to

$$\operatorname{sgn}[h_k(s_i^+)] = (-)^{m-1-i}\cdot\operatorname{sgn}[h_k(s_{m-1}^+)], \quad i = 0,1,2,\ldots,m-1. \tag{2.9}$$

When (2.9) is substituted into (2.6), we find that if s_i and s_{i+1} are both zeros of $h_k(s)$, then

$$\Delta_{s_i}^{s_{i+1}}\theta = \frac{\pi}{2}\{\operatorname{sgn}[g_k(s_i)] - \operatorname{sgn}[g_k(s_{i+1})]\}\cdot(-1)^{m-1-i}\operatorname{sgn}[h_k(s_{m-1}^+)]. \tag{2.10}$$

Based on the above facts, the following theorem concerning $\sigma(f)$ can now be given.

Theorem 2.3. *Let $f(z)$ be a given real polynomial of degree n with no roots on the imaginary axis, that is, the normalized plot $f_k(js)$ does not pass through the origin. Let $0 = s_0 < s_1 < s_2 < \cdots < s_{m-1}$ be the real nonnegative distinct*

finite zeros of $h_k(s)$ with odd multiplicities. Let $s_m = \infty$. Then,

$$\sigma(f) = \begin{cases} \{\operatorname{sgn}[g_k(s_0)] - 2\operatorname{sgn}[g_k(s_1)] + 2\operatorname{sgn}[g_k(s_2)] + \cdots \\ \quad + (-1)^{m-1} 2\operatorname{sgn}[g_k(s_{m-1})] + (-1)^m \operatorname{sgn}[g_k(s_m)]\} \\ \cdot (-1)^{m-1} \operatorname{sgn}[h(\infty)], & \text{if } n \text{ is even,} \\ \{\operatorname{sgn}[g_k(s_0)] - 2\operatorname{sgn}[g_k(s_1)] + 2\operatorname{sgn}[g_k(s_2)] + \cdots \\ \quad + (-1)^{m-1} 2\operatorname{sgn}[g_k(s_{m-1})]\} \cdot (-1)^{m-1} \operatorname{sgn}[h(\infty)], & \text{if } n \text{ is odd.} \end{cases} \tag{2.11}$$

Proof. Suppose first that n is even. Then, $s_m = \infty$ is a zero of $h_k(s)$. Since $\operatorname{sgn}[h_k(s_{m-1}^+)] = \operatorname{sgn}[h(\infty)]$, the first expression in (2.11) is obtained by repeatedly using (2.10) to determine $\Delta_0^\infty \theta$ and then applying Lemma 2.2.

When n is odd, $s_m = \infty$ is not a zero of $h_k(s)$. Therefore, using (2.7) and (2.10), we get

$$\begin{aligned} \Delta_0^\infty \theta &= \sum_{i=0}^{m-2} \Delta_{s_i}^{s_{i+1}} \theta + \Delta_{m-1}^\infty \theta \\ &= \sum_{i=0}^{m-2} \frac{\pi}{2} \{\operatorname{sgn}[g_k(s_i)] - \operatorname{sgn}[g_k(s_{i+1})]\} \cdot (-1)^{m-1-i} \operatorname{sgn}[h_k(s_{m-1}^+)] \\ &\quad + \frac{\pi}{2} \operatorname{sgn}[g_k(s_{m-1})] \cdot \operatorname{sgn}[h_k(s_{m-1}^+)]. \end{aligned} \tag{2.12}$$

Since $\operatorname{sgn}[h_k(s_{m-1}^+)] = \operatorname{sgn}[h(\infty)]$, the desired expression follows by applying Lemma 2.2. □

Now, we state a result similar to Theorem 2.3 where the signature $\sigma(f)$ of a real polynomial $f(z)$ is to be determined using the values of the frequencies such that $f_k(js)$ crosses the imaginary axis.

Theorem 2.4. *Let $f(z)$ be a given real polynomial of degree n with no roots on the imaginary axis. Let $0 < s_1 < s_2 < \cdots < s_{m-1}$ be the real nonnegative distinct finite zeros of $g_k(s)$ with odd multiplicities. Let $s_m = \infty$. Then,*

$$\sigma(f) = \begin{cases} -\{2\operatorname{sgn}[h_k(s_1)] - 2\operatorname{sgn}[h_k(s_2)] + \cdots \\ \quad + (-1)^{m-2} 2\operatorname{sgn}[h_k(s_{m-1})]\} \cdot (-1)^m \operatorname{sgn}[g(\infty)], & \text{if } n \text{ is even,} \\ -\{2\operatorname{sgn}[h_k(s_1)] - 2\operatorname{sgn}[h_k(s_2)] + \cdots \\ \quad + (-1)^{m-2} 2\operatorname{sgn}[h_k(s_{m-1})] + (-1)^{m-1} \operatorname{sgn}[h_k(s_m)]\} \\ \cdot (-1)^m \operatorname{sgn}[g(\infty)], & \text{if } n \text{ is odd.} \end{cases} \tag{2.13}$$

The proof follows along the same lines as that of Theorem 2.3.

3. The Routh-Hurwitz stability criterion. In this section, we offer a very simple proof of the Routh-Hurwitz stability criterion based on Theorems 2.3 and 2.4. Consider a real polynomial $f(z)$ of degree n,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0, \tag{3.1}$$

and let

$$f(z) = f_e(z) + f_o(z), \tag{3.2}$$

where $f_e(z)$ and $f_o(z)$ are the polynomials made up of the terms in $f(z)$ consisting of the even and odd powers of z, respectively. To avoid singularities of the first or the second kind [7] in the Routh array, we make the following assumptions:

(1) $a_{n-1} \neq 0$;

(2) $f_e(z)$ and $f_o(z)$ are coprime.

In order to generate the Routh algorithm, we start with the polynomial $f(z)$ and construct a polynomial $f_1(z)$ of order $n-1$ in the following way.

If n is even, then

$$f_1(z) = \left[f_e(z) - \frac{a_n}{a_{n-1}} \cdot z \cdot f_o(z) \right] + f_o(z), \tag{3.3}$$

and if n is odd, then

$$f_1(z) = \left[f_o(z) - \frac{a_n}{a_{n-1}} \cdot z \cdot f_e(z) \right] + f_e(z). \tag{3.4}$$

The next theorem expresses a relationship between the signatures of $f(z)$ and $f_1(z)$, respectively.

Theorem 3.1. *If $f(z)$ and $f_1(z)$ are as already defined, then*

$$\sigma(f) - \sigma(f_1) = \begin{cases} 1, & \text{if } a_n a_{n-1} > 0, \\ -1, & \text{if } a_n a_{n-1} < 0. \end{cases} \tag{3.5}$$

Proof. Suppose that

$$f(js) = g(s) + jh(s). \tag{3.6}$$

Consider first the case when n is even. Then, from (3.3),

$$f_1(js) = \left[g(s) + \frac{a_n}{a_{n-1}} s h(s) \right] + jh(s). \tag{3.7}$$

From (3.6) and (3.7), it follows that the finite zeros of $h_k(s)$ are the same for both $f(js)$ and $f_1(js)$. Also, at these frequencies, both $f(js)$ and $f_1(js)$ have

the same real part so that $\operatorname{sgn}[g_k(s)]$ is also identical for both these polynomials at these frequencies. Therefore, when we subtract the second expression on the right-hand side of (2.11) from the first one, we get

$$\sigma(f)-\sigma(f_1) = -\operatorname{sgn}[g_k(\infty)]\cdot\operatorname{sgn}[h(\infty)]. \tag{3.8}$$

Now, when s is positive and large, we get

$$g(s) \simeq (-1)^{n/2} a_n s^n, \tag{3.9}$$

while

$$h(s) \simeq (-1)^{(n-2)/2} a_{n-1} s^{n-1} \tag{3.10}$$

so that

$$\operatorname{sgn}[g_k(\infty)]\cdot\operatorname{sgn}[h(\infty)] = -\operatorname{sgn}[a_n a_{n-1}]. \tag{3.11}$$

Hence,

$$\sigma(f)-\sigma(f_1) = \begin{cases} 1, & \text{if } a_n a_{n-1} > 0, \\ -1, & \text{if } a_n a_{n-1} < 0. \end{cases} \tag{3.12}$$

When n is odd, we conclude from (3.4) that

$$f_1(js) = g(s) + j\left[h(s) - \frac{a_n}{a_{n-1}} s g(s)\right]. \tag{3.13}$$

From (3.6) and (3.13), it follows that the finite zeros of $g_k(s)$ are the same for both $f(js)$ and $f_1(js)$. Also, at these frequencies, both $f(js)$ and $f_1(js)$ have the same imaginary part so that $\operatorname{sgn}[h_k(s)]$ is also identical for both these polynomials at these frequencies. Therefore, from (2.13), we get

$$\begin{aligned} \sigma(f)-\sigma(f_1) &= -(-1)^{m-1}(-1)^m \operatorname{sgn}[h_k(\infty)]\cdot\operatorname{sgn}[g(\infty)] \\ &= \operatorname{sgn}[g(\infty)]\cdot\operatorname{sgn}[h_k(\infty)]. \end{aligned} \tag{3.14}$$

Now, when s is positive and large, we get

$$\begin{aligned} g(s) &\simeq (-1)^{(n-1)/2} a_{n-1} s^{n-1}, \\ h(s) &\simeq (-1)^{(n-1)/2} a_n s^n, \end{aligned} \tag{3.15}$$

so that

$$\operatorname{sgn}[g(\infty)]\cdot\operatorname{sgn}[h_k(\infty)] = -\operatorname{sgn}[a_n a_{n-1}]. \tag{3.16}$$

Hence,

$$\sigma(f)-\sigma(f_1) = \begin{cases} 1, & \text{if } a_n a_{n-1} > 0, \\ -1, & \text{if } a_n a_{n-1} < 0, \end{cases} \tag{3.17}$$

and the proof is complete. □

The following is a corollary of Theorem 3.1.

Corollary 3.2. *Let $f(z)$ be a given real polynomial and let $f_1(z)$ be defined by either (3.3) or (3.4), depending on the parity of n. Let l, l_1 denote the numbers of open left half-plane zeros of $f(z)$, $f_1(z)$, and let r, r_1 denote the numbers of open right half-plane zeros of $f(z)$, $f_1(z)$. Then,*

$$\begin{aligned} l_1 &= l-1, \quad r_1 = r, \qquad \text{if } a_n a_{n-1} > 0, \\ l_1 &= l, \qquad r_1 = r-1, \quad \text{if } a_n a_{n-1} < 0. \end{aligned} \tag{3.18}$$

Proof. We know that $\sigma(f) = l - r$ and $\sigma(f_1) = l_1 - r_1$. Then, by Theorem 3.1, we have

$$l - r - l_1 + r_1 = \begin{cases} 1, & \text{if } a_n a_{n-1} > 0, \\ -1, & \text{if } a_n a_{n-1} < 0. \end{cases} \tag{3.19}$$

Also,

$$l + r - (l_1 + r_1) = 1. \tag{3.20}$$

Adding (3.19) and (3.20) leads to

$$l - l_1 = \begin{cases} 1, & \text{if } a_n a_{n-1} > 0, \\ 0, & \text{if } a_n a_{n-1} < 0. \end{cases} \tag{3.21}$$

Subtracting (3.19) and (3.20), we get

$$r - r_1 = \begin{cases} 0, & \text{if } a_n a_{n-1} > 0, \\ 1, & \text{if } a_n a_{n-1} < 0. \end{cases} \tag{3.22}$$

The desired result follows from (3.21) and (3.22). □

For a given real polynomial $f(z)$, Routh's algorithm is equivalent to reducing the degree of $f(z)$ by one at a time using (3.3) and (3.4) alternately. This has been clearly articulated in [1, 2, 16, 25]. The Sturm sequence calculation in [7] is equivalent to the alternate application of (3.3) and (3.4). Therefore, Corollary 3.2 leads to the immediate conclusion that $f(z)$ is Hurwitz if and only if the leading coefficients of all the polynomials that result from alternately applying (3.3) and (3.4) to $f(z)$ are of the same sign. Moreover, it is also evident that the number of open right half-plane zeros of $f(z)$ is equal to the number of sign changes in the leading coefficients of the successive polynomials. Clearly, this is the Routh-Hurwitz criterion.

4. Singularities in the Routh-Hurwitz criterion. The generation of the Routh-Hurwitz criterion in the last section dealt only with the regular case, that is, the case in which the degree of $f(z)$ can be successively reduced by

the alternate application of (3.3) and (3.4) until we finally reach a zero-order polynomial. However, this process would terminate prematurely if, while applying (3.3) or (3.4), we encounter $a_{n-1} = 0$. This is what we call singular cases in the Routh-Hurwitz criterion, which we deal with in this section.

We start with a given real polynomial $f_0(z)$ of degree n,

$$f_0(z) = a_0^0 + a_1^0 z + a_2^0 z^2 + \cdots + a_n^0 z^n. \tag{4.1}$$

Suppose that, by using (3.3) and (3.4) alternately, we obtain a sequence of polynomials $\{f_0(z), f_1(z), f_2(z), \ldots, f_m(z)\}$, where the leading coefficient of each $f_i(z)$, $i = 0, 1, 2, \ldots, m$ is nonzero. Let

$$f_m(z) = a_0^m + a_1^m z + a_2^m z^2 + \cdots + a_{n-m-1}^m z^{n-m-1} + a_{n-m}^m z^{n-m}, \tag{4.2}$$

where $a_{n-m}^m \neq 0$. So, if $a_{n-m-1}^m = 0$, then, clearly, the Routh's algorithm comes to a halt because, in the next step of the Routh's algorithm, we need to divide by a_{n-m-1}^m, which is now equals zero. To deal with this singularity, we consider three different cases that may occur.

Case 4.1. We have $a_{n-m-1}^m = 0$, but there exists at least one k, $k = 3, 5, 7, \ldots$ such that $a_{n-m-k}^m \neq 0$, that is, if the first element in any row of the Routh table vanishes, then there is at least one nonzero element in that row. If $f_0(z)$ has no imaginary zeros, then we can proceed as follows. Replace $a_{n-m-1}^m = 0$ by a small nonzero number ϵ of arbitrary sign and then proceed with the Routh's algorithm. If another singularity is encountered later, then introduce another parameter to replace the new zero element, and so on.

The replacement of $a_{n-m-1}^m = 0$ by ϵ leads to the modification of the original polynomial $f_0(z)$. Using (3.3) and (3.4) for $a_{n-m-1}^m = \epsilon$, we can work our way backward to obtain the modified polynomial $f_0(z, \epsilon)$ whose coefficients are rational functions of ϵ. Since $f_0(z)$ has no roots on the imaginary axis, it follows by continuity that, when ϵ is small enough, $\sigma(f_0(z)) = \sigma(f_0(z, \epsilon))$. It is for this specific reason that the above modification can be used to handle a singularity of this type and still allow to count the number of open right half-plane zeros.

Case 4.2. Suppose $a_{n-m-k}^m = 0$, for $k = 1, 3, 5, 7, \ldots$, that is, all the elements in one row of the Routh array are zeros. It follows that $f_0(z)$ must have at least one pair of complex conjugate zeros symmetrically distributed about the origin. This includes the case of purely imaginary zeros and the case of purely real zeros having opposite signs.

To deal with this kind of singularity, we can simply replace $f_0(z)$ by $f_0(z - \epsilon)$, where ϵ is a sufficiently small positive number, and then continue with Routh's algorithm. The net result is that the number of closed right half-plane zeros of $f_0(z)$ equals the number of sign changes in the leading coefficients of the successive polynomials.

Case 4.3. It is possible that Cases 4.1 and 4.2 occur at different stages in the same problem when proceeding with Routh's algorithm. Again, we can replace $f_0(z)$ by $f_0(z-\epsilon)$, where ϵ is a sufficiently small positive number, and then continue with Routh's algorithm. Alternatively, we can factor out the imaginary axis zeros as in [7] and then apply Routh's algorithm to the new polynomial.

Remark 4.4. The derivation of the Routh-Hurwitz criterion in [7] is carried out using the Cauchy index which ignores the imaginary axis roots. Therefore, in [7], it is possible to deal with the singular cases and obtain a count of the number of open right half-plane zeros by conveniently modifying Routh's algorithm. However, the modifications proposed here allow us to count the number of closed right half-plane roots when the original polynomial has roots on the imaginary axis.

5. Conclusion. In this paper, we provided generalized versions of the interlacing property, leading to a simple proof of the Routh-Hurwitz criterion and recovering the unstable zero-counting capability of Routh's algorithm. As mentioned earlier, such simple derivations are highly needed to make the proof of the Routh-Hurwitz criterion accessible to as many audience as possible on the mathematical stage. It is also expected that the generalizations of the interlacing property presented here are likely to have far reaching implications on some long standing stability problems. Such concerns are currently under investigation and will be addressed in a future work.

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