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KERNEL CONVERGENCE AND BIHOLOMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES

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We deal with kernel convergence of domains in $\mathbb{C}^n$ which are biholomorphically equivalent to the unit ball B. We also prove that there is an equivalence between the convergence on compact sets of biholomorphic mappings on B, which satisfy a growth theorem, and the kernel convergence. Moreover, we obtain certain consequences of this equivalence in the study of Loewner chains and of starlike and convex mappings on B.

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1. Introduction and preliminaries. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1,\dots,z_n)$ with the usual inner product $\langle z,w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the Euclidean norm $\|z\| = \langle z,z\rangle^{1/2}$, $z \in \mathbb{C}^n$. Let $B(a,r)$ be the open ball of radius r centered at $a \in \mathbb{C}^n$. The ball $B(0,r)$ will be denoted by B_r and the unit ball B_1 will be denoted by B. Also the closed ball of center a and radius r will be denoted by $\overline{B}(a,r)$. In the case of one variable, $B(a,r)$ is denoted by $U(a,r)$, B_r is denoted by U_r, and the unit disc U_1 by U. If G is an open set in $\mathbb{C}^n$, let $H(G)$ be the set of holomorphic mappings from G into $\mathbb{C}^n$. If $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of holomorphic mappings from a domain $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$, we will write $g_k \to g$ to mean that $\{g_k\}_{k\in\mathbb{N}}$ converges (simply or locally uniformly on Ω) to g as $k \to \infty$.

By $L(\mathbb{C}^n,\mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I be the identity in $L(\mathbb{C}^n,\mathbb{C}^n)$. If $f \in H(B)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. We also say that $f \in H(B)$ is locally biholomorphic on B if f has a local holomorphic inverse at each $z \in B$. This is equivalent to $J_f(z) \neq 0$ for $z \in B$, where $J_f(z) = \det Df(z)$ is the complex Jacobian determinant of f at $z \in B$. A biholomorphic mapping of B will also be called a univalent mapping. Let $S(B)$ be the subset of $H(B)$ consisting of normalized univalent mappings on B. In the case of one variable, $S(B)$ is denoted by S. Let $S^*(B)$ and $K(B)$ be the subsets of $S(B)$, consisting, respectively, of starlike and convex mappings on B.

In this paper, we discuss the connection between an unusual notion of convergence of domains in $\mathbb{C}^n$ and biholomorphic mappings which satisfy a growth result. In the case of one variable, the notion of kernel convergence

was introduced and studied by Carathéodory [2] (see also [5, 8, 15, 24, 27]). He proved a fundamental result of independent interest, which was later used to prove certain important results in the theory of univalent functions, especially in the study of Loewner chains and the Loewner differential equation. His result is a complete geometric characterization of the convergence of univalent functions in terms of the convergence of their image domains. Gehring [7] defined the notions of kernel and kernel convergence in the case of domains in $\mathbb{R}^3$, and obtained an analogue of the Carathéodory kernel convergence result in the case of K-quasiconformal mappings in $\mathbb{R}^3$. Other results in this direction were obtained by Reshetnyak [25] in the case of quasiconformal mappings in $\mathbb{R}^n$ (see also [30, pages 72–75]). We mention that a metric space analogue of the Carathéodory kernel convergence result was obtained in [19].

We begin with the following definitions.

Definition 1.1. Let $\{G_k\}_{k\in\mathbb{N}}$ be a sequence of domains in $\mathbb{C}^n$ such that $0 \in G_k$ for $k \in \mathbb{N}$. If 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the *kernel* G of $\{G_k\}_{k\in\mathbb{N}}$ to be the largest domain which contains 0 such that if K is a compact subset of G, then there is a positive integer k_0 such that $K \subset G_k$ for $k \geq k_0$ (in other words, K is contained in all but finitely many of the sets G_k). If 0 is not an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the kernel to be $\{0\}$.

Let $\mathcal{G}$ be the set of all domains Ω in $\mathbb{C}^n$ such that $0 \in \Omega$ and each compact K of Ω is contained in all but finitely many of the sets G_k. We assume that 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$. An application of the Heine-Borel theorem shows that if $D = \bigcup_{\Omega\in\mathcal{G}} \Omega$, then $D \in \mathcal{G}$, and it is clear that no larger domain can belong to $\mathcal{G}$. This yields the existence of the kernel of any domains $G_1, \dots, G_k, \dots$, such that 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$.

Definition 1.2. We say that the $\{G_k\}_{k\in\mathbb{N}}$ kernel converges to G and write $G_k \to G$, if each subsequence of $\{G_k\}_{k\in\mathbb{N}}$ has the same kernel G.

It is not difficult to see that if $\{G_k\}_{k\in\mathbb{N}}$ is an increasing sequence of domains in $\mathbb{C}^n$, that is, $G_k \subseteq G_{k+1}$, $k \in \mathbb{N}$, such that $0 \in G_k$, $k \in \mathbb{N}$, then $G = \bigcup_{k\in\mathbb{N}} G_k$ is the kernel of $\{G_k\}_{k\in\mathbb{N}}$ and $\{G_k\}_{k\in\mathbb{N}}$ converges to G in the sense of kernel convergence.

Let $S^c(B)$ be a compact subset of $S(B)$. Then it is clear that for each $r \in [0,1)$, there exists some $M = M(r) \geq 0$ such that $\|f(z)\| \leq M(r)$ for $\|z\| = r$ for $f \in S^c(B)$. On the other hand, if $z_0 \in B \setminus \{0\}$ is fixed, then the functional $\|f(z_0)\|$ is continuous on $S^c(B)$ with respect to the topology of locally uniform convergence, and hence attains its infimum for some $f_0 \in S^c(B)$. Since f_0 is biholomorphic on B, this infimum cannot be zero. Therefore, there exists a function $m(r)$ which is positive for $r \in (0,1)$ such that $m(r) \leq \|f(z)\|$ for $\|z\| = r < 1$ and $f \in S^c(B)$. (It is also easy to see that $m(r)$ is a strictly increasing function by the maximum principle for holomorphic mappings and $\lim_{r\to 0^+} m(r) = 0$.) Consequently, we have proved that

$$m(r) \leq \|f(z)\| \leq M(r), \quad \|z\| = r, \ \forall f \in S^c(B). \tag{1.1}$$

In [10] it is shown that the set $S^0(B)$, consisting of all mappings in $S(B)$ which have parametric representation, is also a compact subset of $S(B)$ since any mapping in the class $S^0(B)$ satisfies the $1/4$-growth result. Moreover, $S^0(B)$ contains the set $S^*(B)$ as a proper subset (see also [1, 9]). On the other hand, the set $K(B)$ is also a compact subset of $S(B)$ since any mapping in $K(B)$ satisfies the $1/2$-growth result (see [6, 26, 29]).

It is known that in the case of one variable, the class S is compact; however, in several variables, the class $S(B)$ is not compact, and there exist mappings f in $S(B)$ which do not satisfy the above growth result, that is, $S^c(B) \subsetneqq S(B)$ in dimension $n \geq 2$ (see [9, 10]).

In the next section, we will prove that there is an equivalence between the kernel convergence and the convergence on compact sets of biholomorphic mappings on the unit ball B which satisfy the growth result (1.1). In the last section, we will obtain some consequences of this result in the case of the kernel convergence and the convergence on compact sets of normalized starlike and normalized convex mappings on B. Also, we will prove that there is an equivalence between the notions of a Loewner chain, which satisfies a certain normality condition, and kernel convergence.

2. Kernel convergence and biholomorphic mappings. In this section, we prove the main result of this paper, which is an analogue of the Carathéodory kernel convergence theorem [2], on the convergence of conformal functions of one variable for biholomorphic mappings which satisfy the growth result (1.1).

Theorem 2.1. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of biholomorphic mappings on B such that $f_k(0) = 0$ and $Df_k(0) = \alpha_k I$, where $\alpha_k > 0$, $k \in \mathbb{N}$. Assume that $f_k/\alpha_k \in S^c(B)$, $k \in \mathbb{N}$. Also let $G_k = f_k(B)$, $k \in \mathbb{N}$, and let G be the kernel of $\{G_k\}_{k\in\mathbb{N}}$. Then $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B to a mapping f if and only if $G_k \to G \neq \mathbb{C}^n$. In the case of convergence, either $f \equiv 0$ and $G = \{0\}$, or else f is biholomorphic on B, $f/\alpha \in S^c(B)$, where $\alpha = \lim_{k\to\infty} \alpha_k$, and $f(B) = G$. In the latter case, $f_k^{-1} \to f^{-1}$ locally uniformly on G as $k \to \infty$.*

Proof

Necessity. First, assume that $f_k \to f$ locally uniformly on B as $k \to \infty$. In view of a version of Hurwitz's theorem in higher dimensions, we deduce that either $J_f \equiv 0$, or else f is biholomorphic on B.

Case 1. First, assume that $J_f \equiv 0$. Since $f_k \to f$ locally uniformly on B, it follows that $\lim_{k\to\infty} J_{f_k}(0) = J_f(0) = 0$, that is,

$$\lim_{k\to\infty} \alpha_k = 0. \tag{2.1}$$

Since $f_k/\alpha_k \in S^c(B)$, we deduce in view of relations (1.1) and (2.1) that $f_k \to 0$ locally uniformly on B as $k \to \infty$.

Next, we show that $G = \{0\}$ and $G_k \to \{0\}$ in the sense of kernel convergence. Let $g_k = f_k^{-1}$ for $k \in \mathbb{N}$. Suppose that $G \neq \{0\}$. Then there is $\varepsilon > 0$ such that

$B_\varepsilon \subseteq G_k$ for $k \in \mathbb{N}$. Then g_k is a biholomorphic mapping on G_k, and thus in B_ε such that $g_k(0) = 0$ and $\|g_k(w)\| < 1$, $w \in B_\varepsilon$. By the Schwarz lemma for holomorphic mappings, we deduce that $\|g_k(w)\| \le (1/\varepsilon)\|w\|$ for $\|w\| < \varepsilon$ and $\|Dg_k(0)\| \le 1/\varepsilon$ for $k \in \mathbb{N}$. Consequently, we deduce that $\alpha_k \ge \varepsilon$ for $k \in \mathbb{N}$. However, this is a contradiction to (2.1), and thus we must have $G = \{0\}$. Further, since each subsequence of $\{G_k\}_{k\in\mathbb{N}}$ has the same kernel $\{0\}$ by a similar argument as above, we deduce that $G_k \to \{0\}$.

Case 2. We next assume that $J_f \not\equiv 0$, and thus f is biholomorphic on B. Then $\alpha = \lim_{k\to\infty} \alpha_k > 0$ and, taking into account the fact that $f_k/\alpha_k \in S^c(B)$ and $S^c(B)$ is compact, we easily deduce that $f/\alpha \in S^c(B)$ too.

Let $\Omega = f(B)$. We prove that $G = \Omega$ and $G_k \to G$ in the sense of kernel convergence.

First step. We prove that $\Omega \subseteq G$. To this end, it suffices to prove that if K is a compact subset of Ω, then $K \subset G_k$ for sufficiently large k. Indeed, if K is such a compact subset of Ω, $f^{-1}(K)$ is a compact subset of B, and thus there is some $r \in (0,1)$ such that $f^{-1}(K) \subset B_r$. Let $\gamma = \partial B_r$ and $\Gamma = f(\gamma)$. It is obvious that $K \cap \Gamma = \emptyset$ since f is biholomorphic. Further, let η be the Euclidean distance between Γ and K. Then $\eta > 0$ and clearly

$$\eta = \min\{\|f(z) - w\| : w \in K,\ \|z\| = r\}. \tag{2.2}$$

If $v_0 \in K$, then $\|f(z) - v_0\| \ge \eta$ for $z \in \gamma$. On the other hand, since $f_k \to f$ uniformly on γ as $k \to \infty$, there is some $k_0 = k_0(\gamma) \in \mathbb{N}$ such that

$$\|f_k(z) - f(z)\| < \eta, \quad z \in \gamma,\ k \ge k_0. \tag{2.3}$$

Hence, if $k \ge k_0$ and $z \in \gamma$, we obtain

$$\|f_k(z) - f(z)\| < \|f(z) - v_0\|, \tag{2.4}$$

and in view of Rouché's theorem (see [18, Theorem 3] and also [3, 17]), we deduce that both equations

$$f_k(z) - v_0 = 0, \qquad f(z) - v_0 = 0 \tag{2.5}$$

have the same number of solutions inside γ, that is, on B_r, for $k \ge k_0$. But the equation $f(z) - v_0 = 0$ has only one solution on B_r since f is biholomorphic on B, and thus for each $k \ge k_0$, there is a unique point $z_k \in B_r$ such that $v_0 = f_k(z_k)$. Hence, $v_0 \in f_k(B)$ for $k \ge k_0$. Also since k_0 does not depend on v_0 (k_0 depends only on K) and $G_k = f_k(B)$, we deduce that $K \subseteq G_k$ for sufficiently large k. We have therefore proved that $\Omega \subseteq G$.

Second step. We prove that there is a subsequence $\{k_p\}_{p\in\mathbb{N}}$ such that $f_{k_p}^{-1} \to f^{-1}$ locally uniformly on Ω. Indeed, the inverse functions $g_k = f_k^{-1}$ are well defined on any fixed compact subset of Ω for k sufficiently large, since $\Omega \subseteq G$, and moreover $\|g_k(w)\| < 1$ for k large. By Montel's theorem, there is

a subsequence $\{g_{k_p}\}_{p\in\mathbb{N}}$ such that $g_{k_p}\to g$ locally uniformly on Ω. Then g is a holomorphic mapping on Ω, $g(0)=0$, and

$$Dg(0)=\lim_{p\to\infty}Dg_{k_p}(0)=\lim_{p\to\infty}\left[Df_{k_p}(0)\right]^{-1}=\lim_{p\to\infty}\frac{1}{\alpha_{k_p}}I. \tag{2.6}$$

Since f is biholomorphic on B, we must have $\lim_{p\to\infty}\alpha_{k_p}>0$. Hence, $J_g(0)\neq 0$, and thus g is biholomorphic on Ω.

Next, we can prove that $g=f^{-1}$ by an argument based again on the Rouché theorem.

Third step. We next prove that $f_k^{-1}\to f^{-1}$ locally uniformly on Ω as $k\to\infty$ and $\Omega=G$.

The argument in the second step implies that each subsequence of $\{g_k\}_{k\in\mathbb{N}}$ contains a further subsequence which converges locally uniformly on Ω to f^{-1}. Since the sequence $\{g_k\}_{k\in\mathbb{N}}$ is locally uniformly bounded, a further application of Montel's theorem yields that the whole sequence $\{g_k\}_{k\in\mathbb{N}}$ converges locally uniformly on Ω to f^{-1}. In fact, the same argument combined with Vitali's theorem (see, e.g., [20]) yields that $\{g_k\}_{k\in\mathbb{N}}$ converges locally uniformly on G to a biholomorphic mapping ϕ of G onto B. Since $\phi|_\Omega=g$ and g is a biholomorphic mapping of Ω onto G, we must have $\Omega=G$.

We have therefore proved that the kernel of $\{G_k\}_{k\in\mathbb{N}}$ is $f(B)$, and since each subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$ of $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B to f, the corresponding subsequence $\{G_{k_p}\}_{p\in\mathbb{N}}$ of $\{G_k\}_{k\in\mathbb{N}}$ has the same kernel $f(B)$. Hence $G_k\to G$ and $G=f(B)$.

Sufficiency. We now assume that $G_k\to G\neq\mathbb{C}^n$ in the sense of kernel convergence and prove that $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B.

Case 1. First, assume that $G=\{0\}$. We show that $\alpha_k\to 0$ as $k\to\infty$, that is, $J_{f_k}(0)\to 0$ as $k\to\infty$. Otherwise, if $\{\alpha_k\}_{k\in\mathbb{N}}$ does not converge to zero, then there exist some $\varepsilon>0$ and a subsequence $\{\alpha_{k_p}\}_{p\in\mathbb{N}}$ of $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $\alpha_{k_p}\geq\varepsilon$ for $p\in\mathbb{N}$.

Since $\{f_{k_p}/\alpha_{k_p}\}_{p\in\mathbb{N}}\subset S^c(B)$, it follows in view of (1.1) that

$$\alpha_{k_p}m(\|z\|)\leq\|f_{k_p}(z)\|,\quad z\in B,\ p\in\mathbb{N}, \tag{2.7}$$

and thus $f_{k_p}(B)\supseteq B_{\varepsilon\mu}$ for $p\in\mathbb{N}$, where $0<\mu=\lim_{r\to 1^-}m(r)$. (Clearly, $\mu<\infty$ since $f_{k_p}\in S^c(B)$.) However, this is a contradiction to the fact that $G_{k_p}\to\{0\}$. Hence, we must have $\alpha_k\to 0$ as $k\to\infty$. Using an argument similar to that in Case 1 of the proof of necessity, we deduce that $f_k\to 0$ locally uniformly on B as $k\to\infty$.

Case 2. We now assume that $G\neq\{0\}$ and $G\neq\mathbb{C}^n$. We first prove that the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ is bounded. Otherwise, there is a subsequence $\{k_p\}_{p\in\mathbb{N}}$ such that $\alpha_{k_p}\geq p$ for $p\in\mathbb{N}$. Using again an argument similar to that in the previous case, we deduce that $G_{k_p}=f_{k_p}(B)\supseteq B_{p\mu}$, $p\in\mathbb{N}$, and thus the sequence $\{G_{k_p}\}_{p\in\mathbb{N}}$ has the kernel $\mathbb{C}^n$. This contradiction shows that there is $L>0$ such that $\alpha_k\leq L$ for $k\in\mathbb{N}$. Taking into account the relation (1.1), we can easily

obtain

$$\|f_k(z)\| \le \alpha_k M(\|z\|) \le LM(\|z\|), \quad z \in B,\ k \in \mathbb{N}, \tag{2.8}$$

and thus $\{f_k\}_{k\in\mathbb{N}}$ is a locally uniformly bounded sequence on B. In view of Montel's theorem, there is a subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$ of $\{f_k\}_{k\in\mathbb{N}}$ which converges locally uniformly to a holomorphic mapping f. If $J_f \equiv 0$, then using a similar argument as in Case 1 of the proof of necessity, applied to the subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$, we deduce that $f \equiv 0$ and hence $G = \{0\}$. However, this is impossible, and thus f is a biholomorphic mapping of B onto the kernel of $\{G_{k_p}\}_{p\in\mathbb{N}}$ by the necessary part of the proof applied to the sequences $\{f_{k_p}\}_{p\in\mathbb{N}}$ and $\{G_{k_p}\}_{p\in\mathbb{N}}$. But the kernel of $\{G_{k_p}\}_{p\in\mathbb{N}}$ is the same as the kernel of $\{G_k\}_{k\in\mathbb{N}}$, that is G, since $G_k \to G$. Therefore, $f(B) = G$. Further, since $\{f_{k_p}/\alpha_{k_p}\}_{p\in\mathbb{N}} \subset S^c(B)$ and $S^c(B)$ is compact, it follows that $f/\alpha \in S^c(B)$ too and $f_{k_p}^{-1} \to f^{-1}$ locally uniformly on G by the necessary part of the proof.

We next prove that $f_k \to f$ locally uniformly on B as $k \to \infty$. To this end, it suffices to prove that $f_k(z) \to f(z)$ as $k \to \infty$, for all $z \in B$, in view of Vitali's theorem and the fact that $\{f_k\}_{k\in\mathbb{N}}$ is a locally uniformly bounded family on B.

Suppose that there is some $z_0 \in B$ such that $\{f_k(z_0)\}_{k\in\mathbb{N}}$ is not convergent. Since $\{f_k(z_0)\}_{k\in\mathbb{N}}$ is a bounded sequence, there exist two subsequences $\{f_{k'_p}(z_0)\}_{p\in\mathbb{N}}$ and $\{f_{k''_p}(z_0)\}_{p\in\mathbb{N}}$ of $\{f_k(z_0)\}_{k\in\mathbb{N}}$, which converge to some distinct limits denoted by w_0 and w'_0. Since $\{f_{k'_p}\}_{p\in\mathbb{N}}$ and $\{f_{k''_p}\}_{p\in\mathbb{N}}$ are locally uniformly bounded families, we may extract two subsequences of these sequences, again denoted by $\{f_{k'_p}\}_{p\in\mathbb{N}}$ and $\{f_{k''_p}\}_{p\in\mathbb{N}}$, which converge locally uniformly on B to h_1 and h_2, respectively. It is easy to see that h_1 and h_2 are biholomorphic mappings on B, $h_1(0) = h_2(0) = 0$, $Dh_1(0) = \beta I$, and $Dh_2(0) = \gamma I$, where $0 < \beta = \lim_{p\to\infty} \alpha_{k'_p}$ and $0 < \gamma = \lim_{p\to\infty} \alpha_{k''_p}$. It is also obvious that $w_0 = h_1(z_0)$ and $w'_0 = h_2(z_0)$. Moreover, since $G_{k'_p} \to G$ and $G_{k''_p} \to G$ and by the necessary part of the proof, $h_1(B) = h_2(B) = G$. Next, let $q = h_2^{-1} \circ h_1 : B \to B$. Then q is a biholomorphic mapping of B onto B, $q(0) = 0$, and since B is a circular domain, we deduce that q is the restriction of a unitary linear operator (see [28, Theorem 2.1.3]). This yields $\beta = \gamma$. Consequently, $q(0) = 0$, $Dq(0) = I$, and in view of a uniqueness result due to Cartan (see, e.g., [28, Theorem 2.1.1]), we conclude that $q(z) = z$ for $z \in B$, that is, $h_1 \equiv h_2$. However, this is a contradiction to $h_1(z_0) \ne h_2(z_0)$. Thus, we must have $f_k(z) \to f(z)$ as $k \to \infty$, for all $z \in B$. This completes the proof. □

3. Applications. We first apply the result of Theorem 2.1 to obtain the following connections between the kernel convergence and locally uniform convergence of normalized starlike and convex mappings.

Theorem 3.1. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of mappings in $S^*(B)$ and let $G_k = f_k(B)$. Also let G be the kernel of $\{G_k\}_{k\in\mathbb{N}}$. Then $\{f_k\}_{k\in\mathbb{N}}$ converges locally*

uniformly on B to f if and only if $G_k \to G \neq \mathbb{C}^n$. Moreover, $f \in S^*(B)$, $G = f(B)$ (thus G is a starlike domain with respect to the origin), and $f_k^{-1} \to f^{-1}$ locally uniformly on G as $k \to \infty$.

Theorem 3.2. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of mappings in $K(B)$ and let $G_k = f_k(B)$. Also let G be the kernel of $\{G_k\}_{k\in\mathbb{N}}$. Then $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B to f if and only if $G_k \to G \neq \mathbb{C}^n$. Moreover, $f \in K(B)$, $G = f(B)$ (thus G is a convex domain), and $f_k^{-1} \to f^{-1}$ locally uniformly on G as $k \to \infty$.*

Next we use Theorem 2.1 to prove that there is an equivalence between the notions of a Loewner chain $f(z,t)$, such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family, and the kernel convergence of the family $\{f(B,t)\}_{t\geq 0}$. To this end, we recall some notions and results which are useful in the proof of Theorem 3.5.

If $f, g \in H(B)$, we say that f is subordinate to g if there is a Schwarz mapping v (i.e., $v \in H(B)$, $v(0) = 0$, and $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$. We will write $f \prec g$ to mean that f is subordinate to g.

A mapping $f : B \times [0,\infty) \to \mathbb{C}^n$ is called a Loewner chain if the following conditions hold:

(i) $f(\cdot,t)$ is univalent on B, $f(0,t) = 0$, and $Df(0,t) = e^t I$, for each $t \geq 0$;

(ii) $f(\cdot,s) \prec f(\cdot,t)$ whenever $0 \leq s \leq t < \infty$.

Condition (ii) is equivalent to the fact that there is a unique univalent Schwarz mapping $v = v(z,s,t)$ called the transition mapping associated to $f(z,t)$ such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in B,\ 0 \leq s \leq t < \infty. \tag{3.1}$$

Note that $Dv(0,s,t) = e^{s-t}I$, $0 \leq s \leq t < \infty$, in view of the normalization of $f(z,t)$.

Recently, in [10, 14], the authors have proved the following growth result for Loewner chains $f(z,t)$ such that $\{e^{-t}f(z,t)\}_{\geq 0}$ is a normal family. Still this result does not hold for an arbitrary Loewner chain (see [10]).

Lemma 3.3. *Let $f(z,t)$ be a Loewner chain such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on B. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|e^{-t}f(z,t)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ t \geq 0. \tag{3.2}$$

On the other hand, in [12] (see also [13, 14]), Graham and Kohr proved the following absolute continuity result for Loewner chains.

Lemma 3.4. *Let $f(z,t)$ be a Loewner chain. Then, for each $r \in (0,1)$ and $T > 0$, there is $M = M(r,T) > 0$ such that*

$$\|f(z,t_1) - f(z,t_2)\| \leq M(r,T)\,|t_1 - t_2|, \quad \|z\| \leq r,\ t_1, t_2 \in [0,T]. \tag{3.3}$$

Let $S^{\tilde{c}}(B)$ be the subclass of $S(B)$ consisting of all mappings in $S(B)$ which satisfy the $1/4$-growth result. That is, $f \in S^{\tilde{c}}(B)$ if and only if $f \in S(B)$ and

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B. \tag{3.4}$$

Also let $g_t(z) = g(z,t)$ be a biholomorphic mapping of B onto a domain $G(t)$ such that $g_t(0) = 0$, $Dg_t(0) = \alpha(t)I$, where $\alpha(t) > 0$ for $t \ge 0$, and $g_t/\alpha(t) \in S^{\tilde{c}}(B)$, $t \ge 0$. Also let $\alpha_0 = \alpha(0)$. Further, assume that the family $\{G(t)\}_{t\ge 0}$ satisfies the following conditions:

$$G(s) \subsetneqq G(t), \quad 0 \le s < t < \infty, \tag{3.5}$$

$$\begin{aligned} &G(t_k) \longrightarrow G(t_0) \quad \text{if } t_k \longrightarrow t_0 < \infty, \\ &G(t_k) \longrightarrow \mathbb{C}^n \quad \text{if } t_k \longrightarrow \infty. \end{aligned} \tag{3.6}$$

The convergence in question is the kernel convergence. Then we obtain the following result (cf. [24, Chapter 6] and [4]). Theorem 3.5(i) provides an example of a Loewner chain associated to a given family of domains which are biholomorphically equivalent to the unit ball and converge in the sense of kernel convergence. On the other hand, Theorem 3.5(ii) shows that given a Loewner chain $f(z,t)$ such that $\{e^{-t}f(z,t)\}_{t\ge 0}$ is a normal family, the associated family of domains satisfies conditions (3.5) and (3.6). For further applications of Loewner chains in several complex variables, see [10, 11, 12, 13, 14, 16, 21, 22, 23].

Theorem 3.5. (i) *Let g_t and $G(t)$ satisfy the conditions in the previous paragraph.*

(a) *Then α is a strictly increasing continuous function, and $\alpha(t) \to \infty$ as $t \to \infty$.*

(b) *If $\beta(t) = \log[\alpha(t)/\alpha_0]$, then $f(z,t) = \alpha_0^{-1} g(z,\beta^{-1}(t))$ is a Loewner chain and $f(B,t) = \alpha_0^{-1} G(\beta^{-1}(t))$. Further, $\{e^{-t}f(z,t)\}_{t\ge 0}$ is a normal family on B.*

(ii) *Conversely, let $f(z,t)$ be a Loewner chain such that $\{e^{-t}f(z,t)\}_{t\ge 0}$ is a normal family on B. Also let $G(t) = f(B,t)$, $t \ge 0$. Then the family of domains $\{G(t)\}_{t\ge 0}$ satisfies conditions (3.5) and (3.6).*

Proof. First we prove part (i). Using the relation (3.5), we have

$$g(z,s) \prec g(z,t), \quad 0 \le s \le t < \infty, \tag{3.7}$$

and therefore there is a Schwarz mapping $v = v(z,s,t)$ such that

$$g(z,s) = g\big(v(z,s,t),t\big), \quad z \in B,\ 0 \le s \le t < \infty. \tag{3.8}$$

Differentiating both sides of the above relation with respect to z, we obtain

$$\alpha(s)I = Dg(0,s) = Dg(0,t)Dv(0,s,t) = \alpha(t)Dv(0,s,t), \tag{3.9}$$

and thus $\alpha(s)/\alpha(t) = \|Dv(0,s,t)\| \le 1$, that is, $\alpha(s) \le \alpha(t)$. Since $g(B,s) \subsetneqq g(B,t)$, $s < t$, by (3.5), we deduce that $\alpha(s) \ne \alpha(t)$ for $s < t$. Otherwise, if $\alpha(s) = \alpha(t)$ for some $s < t$, then $Dv(0,s,t) = I$. Since $v(B,s,t) \subseteq B$, $v(0,s,t) = 0$, and $Dv(0,s,t) = I$, we deduce in view of a uniqueness result due to Cartan (see [28, Theorem 2.1.1]) that $v(z,s,t) \equiv z$. Hence, $g(z,s) = g(z,t)$, $z \in B$. However, this is a contradiction to (3.5). Thus, $\alpha(s) \ne \alpha(t)$ for $s \ne t$, and, consequently, α is a strictly increasing function from $[0,\infty)$ into $(0,\infty)$. Moreover, since $G(t_k) \to \mathbb{C}^n$ as $t_k \to \infty$, we must have $\alpha(t) \to \infty$ as $t \to \infty$. On the other hand, from Theorem 2.1, we know that $g_{t_k} \to g_t$ locally uniformly on B as $t_k \to t < \infty$, so that the function α is continuous. These arguments prove (a).

We next prove assertion (b). To this end, it suffices to observe that $\alpha : [0,\infty) \to [\alpha_0,\infty)$ is strictly increasing and continuous, hence one-to-one. Consequently, β is also a strictly increasing function from $[0,\infty)$ onto $[0,\infty)$. Using relation (3.7) and the above argument, we obtain

$$f(z,s) \prec f(z,t), \quad z \in B,\ 0 \le s \le t < \infty, \tag{3.10}$$

and since $g(\cdot,t)$ is univalent, we deduce that $f(\cdot,t)$ is also univalent for $t \ge 0$. Moreover, if $\tau = \beta^{-1}(t)$, then $t = \beta(\tau)$ and $e^t = \alpha(\tau)/\alpha_0$. Consequently, we deduce that

$$Df(0,t) = \alpha_0^{-1} Dg(0,\tau) = \alpha_0^{-1}\alpha(\tau) I = e^t I, \quad t \ge 0. \tag{3.11}$$

We conclude that $f(z,t)$ is a Loewner chain. Clearly, $f(B,t) = \alpha_0^{-1} G(\beta^{-1}(t))$, $t \ge 0$. Further, $\{e^{-t} f(z,t)\}_{t\ge 0}$ is a normal family since $g_t/\alpha(t) \in S^{\tilde{c}}(B)$ for $t \in [0,\infty)$.

We now prove part (ii). To this end, let $f_t(z) = f(z,t)$ for $z \in B$ and $t \ge 0$. Obviously, $G(s) \subseteq G(t)$ for $0 \le s \le t < \infty$. Suppose $G(s) = G(t)$ for some $s < t$. Then $q_{s,t} = f_t^{-1} \circ f_s$ is a biholomorphic mapping of B onto B such that $q_{s,t}(0) = 0$. Since B is a circular domain, it follows that $q_{s,t}$ is the restriction of a unitary linear operator. On the other hand, since $Dq_{s,t}(0) = e^{s-t} I$, we must have $s = t$. However, this is a contradiction. The claimed conclusion now follows. This implies (3.5). Further, since $\{e^{-t} f(z,t)\}_{t\ge 0}$ is a normal family, we deduce in view of Lemma 3.3 that $f(B,t) \supseteq B_{e^t/4}$ for $t \ge 0$. Hence, $G(t) = f(B,t) \to \mathbb{C}^n$ as $t \to \infty$. This proves the second condition in (3.6). The first part in (3.6) follows from Theorem 2.1 and Lemma 3.4. This completes the proof. □

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