We study convergences of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

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1. Introduction and preliminaries. Let $D$ be a nonempty subset of a real Banach space $X$ and $T : D \to D$ a nonlinear mapping. The mapping $T$ is said to be asymptotically quasi-nonexpansive (see [5]) if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - p\| \leq (1 + k_n) \|x - p\| \quad (1.1)$$

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$. The mapping $T$ is said to be asymptotically nonexpansive (see [3]) if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n) \|x - y\| \quad (1.2)$$

for all $x, y \in D$ and $n \in \mathbb{N}$. The mapping $T$ is said to be a mapping of asymptotically nonexpansive type [4] if

$$\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad (1.3)$$

for any $y \in D$.


Recently, Liu [5] extended results of [2, 7] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points of asymptotically quasi-nonexpansive mappings.
First, we introduce the concept of class of mappings of asymptotically quasi-nonexpansive type: the mapping \( T \) is said to be a mapping of asymptotically quasi-nonexpansive type if \( F(T) \neq \emptyset \) and

\[
\limsup_{n \to \infty} \sup_{x \in D} (\| T^nx - p \| - \| x - p \|) \leq 0 \quad \text{for any } p \in F(T).
\]  

\[ \text{(1.4)} \]

**Remark 1.1.** If \( T \) is a mapping of asymptotically nonexpansive type with \( F(T) \neq \emptyset \), then \( T \) is a mapping of asymptotically quasi-nonexpansive type.

**Remark 1.2.** If \( D \) is bounded and \( T \) is an asymptotically quasi-nonexpansive mapping, then \( T \) is a mapping of asymptotically quasi-nonexpansive type. In fact, if \( T \) is an asymptotically quasi-nonexpansive mapping, then there exists a sequence \( \{k_n\} \) in \([0, \infty)\) with \( \lim_{n \to \infty} k_n = 0 \) such that

\[
\| T^nx - p \| \leq (1 + k_n) \| x - p \|
\]

\[ \text{(1.5)} \]

for all \( x \in D, \ p \in F(T), \) and \( n \in \mathbb{N} \), which implies

\[
\sup_{x \in D} \{\| T^nx - T^ny \| - \| x - y \|\} \leq k_n \cdot \text{diam}D
\]

\[ \text{(1.6)} \]

for any \( y \in F(T) \) and \( n \in \mathbb{N} \). Hence

\[
\limsup_{n \to \infty} \sup_{x \in D} (\| T^nx - T^ny \| - \| x - y \|) \leq 0 \quad \text{for any } y \in F(T).
\]

\[ \text{(1.7)} \]

We observe from Remarks 1.1 and 1.2 that the class of mappings of asymptotically nonexpansive type is an intermediate class between the class of mappings of asymptotically quasi-nonexpansive type and that of mappings of asymptotically nonexpansive type with nonempty fixed-point sets. Let

\[ C_1 = \{ T : T : D \to D \text{ is a nonexpansive mapping} \}, \]

\[ C_2 = \{ T : T : D \to D \text{ is a quasi-nonexpansive mapping} \}, \]

\[ C_3 = \{ T : T : D \to D \text{ is an asymptotically nonexpansive mapping} \}, \]

\[ C_4 = \{ T : T : D \to D \text{ is an asymptotically quasi-nonexpansive mapping} \}, \]

\[ C_5 = \{ T : T : D \to D \text{ is a mapping of asymptotically nonexpansive type} \}, \]

\[ C_6 = \{ T : T : D \to D \text{ is a mapping of asymptotically quasi-nonexpansive type} \}. \]

\[ \text{(1.8)} \]

Then we have the following implications:

\[ C_1 \iff C_2 \]

\[ \downarrow \quad \downarrow \]

\[ C_3 \iff C_4 \]

\[ \downarrow \quad \downarrow \]

\[ C_5 \iff C_6. \]
In this paper, we are mainly interested in the problem of approximation of fixed points of the more general class of mappings of asymptotically quasi-nonexpansive type than that of asymptotically quasi-nonexpansive mappings. The purpose of this paper is to continue discussion concerning convergence of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces. We give necessary and sufficient conditions for the Mann and Ishikawa iteration processes to converge to fixed points of mappings of asymptotically quasi-nonexpansive type. Further, we obtain extensions of various results obtained quite recently by Deng [1], Ghosh and Denath [2], Liu [5], and Tan and Xu [9, 10] to more general types of space as well as families of operators.

We say that a Banach space \( X \) satisfies Opial’s condition \([6]\) if, for each sequence \( \{x_n\} \) in \( X \) weakly convergent to a point \( x \) and for all \( y \neq x \),

\[
\liminf_{n \to \infty} ||x_n - x|| < \liminf_{n \to \infty} ||x_n - y||.
\] (1.10)

The examples of Banach spaces which satisfy Opial’s condition are Hilbert spaces, and all \( L^p[0, 2\pi] \) with \( 1 < p \neq 2 \) fail to satisfy Opial’s condition \([6]\).

Let \( D \) be a nonempty closed convex subset of a Banach space \( X \). Then \( I - T \) is demiclosed at zero if, for any sequence \( \{x_n\} \) in \( D \), condition \( x_n \rightharpoonup x \) weakly and \( \lim_{n \to \infty} ||x_n - Tx_n|| = 0 \) implies \( (I - T)x = 0 \).

2. Main results. In this section, we establish some weak and strong convergences for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

**Lemma 2.1.** Let \( D \) be a nonempty subset of a normed space \( X \) and let \( T : D \to E \) be a mapping of asymptotically quasi-nonexpansive type. For two given real sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \([0, 1]\), let a sequence \( \{x_n\} \) in \( D \) be defined by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,
\]

\[
y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n = 1, 2, \ldots.
\] (2.1)

If \( p \) is a fixed point of \( T \), then

(a) \( \|x_{n+1} - p\| \leq \|x_n - p\| + (1 + \beta_n) \sup_{x \in D} (\|T^n x - p\| - \|x - p\|) \), \( n = 1, 2, \ldots \).

(b) \( \lim_{n \to \infty} ||x_n - p|| \) exists.

**Proof.** Let \( p \) be a fixed point of \( T \).

(a) From (2.1), we have

\[
\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\|
\]

\[
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n (\|T^n y_n - p\| - \|y - p\|) + \alpha_n\|y_n - p\|
\]
\[ \alpha_n \left( \frac{1}{1 - \beta_n} \right) \leq \| \mathbf{x}_n - p \| - \| \mathbf{y}_n - p \| \]

\[ + \beta_n \left( \| T^n \mathbf{x}_n - p \| - \| \mathbf{x}_n - p \| \right) \]

\[ \leq \| \mathbf{x}_n - p \| + \left( \| T^n \mathbf{y}_n - p \| - \| \mathbf{y}_n - p \| \right) \]

\[ + \beta_n \left( \| T^n \mathbf{x}_n - p \| - \| \mathbf{x}_n - p \| \right) \]

\[ \leq \| \mathbf{x}_n - p \| + (1 + \beta_n) \sup_{x \in D} \left( \| T^n x - p \| - \| x - p \| \right). \]

\[ (2.2) \]

(b) For \( m, n \in \mathbb{N} \), we have

\[ \| \mathbf{x}_{n+m} - p \| \leq \| \mathbf{x}_{n+m-1} - p \| + 2 \sup_{x \in D} \left( \| T^{n+m-1} x - p \| - \| x - p \| \right) \]

\[ \leq \| \mathbf{x}_{n+m-1} - p \| + 2 \sup_{x \in D} \left( \| T^n x - p \| - \| x - p \| \right) \]

\[ \leq \| \mathbf{x}_{n+m-2} - p \| + 4 \sup_{x \in D} \left( \| T^n x - p \| - \| x - p \| \right) \]

\[ \leq \cdots \leq \| \mathbf{x}_n - p \| + 2n \sup_{x \in D} \left( \| T^n x - p \| - \| x - p \| \right). \]

Hence, for \( n \in \mathbb{N} \),

\[ \limsup_{m \to \infty} \| \mathbf{x}_m - p \| \leq \| \mathbf{x}_n - p \| + 2n \limsup_{m \to \infty} \left( \| T^n x - p \| - \| x - p \| \right) \]

\[ \leq \| \mathbf{x}_n - p \|. \]

\[ (2.4) \]

It follows that

\[ \limsup_{m \to \infty} \| \mathbf{x}_m - p \| \leq \liminf_{n \to \infty} \| \mathbf{x}_n - p \|. \]

\[ (2.5) \]

Thus \( \lim_{n \to \infty} \| \mathbf{x}_n - p \| \) exists.

\[ \square \]

**Lemma 2.2.** Let \( D \) and \( T \) be as in Lemma 2.1. For a given real sequence \( \{ \alpha_n \} \) in \( [0, 1] \), let a sequence \( \{ \mathbf{x}_n \} \) in \( D \) be defined by

\[ \mathbf{x}_{n+1} = (1 - \alpha_n) \mathbf{x}_n + \alpha_n T^n \mathbf{x}_n, \quad n = 1, 2, \ldots \]

\[ (2.6) \]

If \( p \) is a fixed point of \( T \), then

(a) \( \| \mathbf{x}_{n+1} - p \| \leq \| \mathbf{x}_n - p \| + \sup_{x \in D} (\| T^n x - p \| - \| x - p \|), \ n = 1, 2, \ldots \)

(b) \( \lim_{n \to \infty} \| \mathbf{x}_n - p \| \) exists.

**Theorem 2.3.** Let \( X \) be a Banach space which satisfies Opial’s condition and let \( D \) be a weakly compact subset of \( X \). Let \( T \) and \( \{ \mathbf{x}_n \} \) be as in Lemma 2.1. Suppose that \( T \) has a fixed point, \( I - T \) is demiclosed at zero, and \( \{ \mathbf{x}_n \} \) is an approximating fixed-point sequence for \( T \), that is, \( \lim_{n \to \infty} \| \mathbf{x}_n - T \mathbf{x}_n \| = 0 \). Then \( \{ \mathbf{x}_n \} \) converges weakly to a fixed point of \( T \).

**Proof.** First, we show that \( \omega_w(\mathbf{x}_n) \subset F(T) \). Let \( \mathbf{x}_{n_k} \to \mathbf{x} \) weakly. By assumption, we have \( \lim_{n \to \infty} \| \mathbf{x}_n - T \mathbf{x}_n \| = 0 \). Since \( I - T \) is demiclosed at zero,
By Opial's condition, \( \{x_n\} \) possesses only one weak limit point, that is, \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Theorem 2.4.** Let \( X \) be a Banach space which satisfies Opial's condition and let \( D \) be a weakly compact subset of \( X \). Let \( T \) and \( \{x_n\} \) be as in Lemma 2.2. Suppose that \( T \) has a fixed point, \( I - T \) is demiclosed at zero, and \( \{x_n\} \) is an approximating fixed-point sequence for \( T \), that is, \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). Then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Remark 2.5.** Theorem 2.3 improves Theorem 2 of Deng [1] for mappings of asymptotically quasi-nonexpansive type. Theorem 2.4 generalizes Theorem 2.1 of Schu [8].

**Theorem 2.6.** Let \( D \) be a closed subset of Banach space, let \( T : D \to D \) be a mapping of asymptotically quasi-nonexpansive type, and \( F(T) \) be nonempty closed set. For two given real sequences \( \{\alpha_n\} \) and \( \{\beta\} \) in \([0, 1]\), let the Ishikawa iterative sequence \( \{x_n\} \) in \( D \) be defined by (2.1). Then \( \{x_n\} \) converges strongly to a fixed point of \( T \) if and only if \( \liminf_n d(x_n, F(T)) = 0 \).

**Proof.** Let \( \{x_n\} \) converge strongly to a point \( z \in F(T) \). Then \( \lim_n d(x_n, F(T)) = 0 \). Conversely, suppose \( \liminf_n d(x_n, F(T)) = 0 \). From Lemma 2.1(a),

\[
\|x_{n+1} - p\| \leq \|x_n - p\| + 2\sup_{x \in D} (\|T^n x - p\| - \|x - p\|) \tag{2.7}
\]

for any \( n \in \mathbb{N} \) and \( p \in F(T) \). Since \( T \) is a mapping of asymptotically quasi-nonexpansive type, we have

\[
\limsup_{n \to \infty} \sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq 0 \tag{2.8}
\]

Hence, there exists a positive integer \( n_0 \) and a sequence \( \{a_n\} \) of positive real numbers with \( \lim_n a_n = 0 \) such that

\[
\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq a_n \tag{2.9}
\]

for any \( n \geq n_0 \). Without loss of generality, we can assume that \( a_n = 1/2n^2 \). Hence,

\[
\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^k x - p\| - \|x - p\|) \right\} \leq \frac{1}{2n^2} \tag{2.10}
\]

for any \( n \geq n_0 \). It follows from (2.7) that

\[
\|x_{n+1} - p\| \leq \|x_n - p\| + \frac{1}{n^2} \tag{2.11}
\]
for all $n \geq n_0$, that is,

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \frac{1}{n^2}$$

(2.12)

for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have

$$d(x_{n+m}, F(T)) \leq d(x_n, F(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}.$$  

(2.13)

Using [10, Lemma 1, page 303], we obtain that $\lim_n d(x_n, F(T))$ exists, and it follows from $\liminf_n d(x_n, F(T)) = 0$ that $\lim_n d(x_n, F(T)) = 0$. Thus, $\lim_n d(x_n, F(T)) = 0$. For each $\varepsilon > 0$, there exists a natural number $m_0$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{3}$$

(2.14)

for all $n \geq m_0$. Then there exists a $p' \in F(T)$ such that $d(x_n, p') < \varepsilon/2$ for all $n \geq m_0$. If $n, m \geq m_0$, then

$$d(x_n, x_m) \leq d(x_n, p') + d(p', x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$  

(2.15)

This shows that $\{x_n\}$ is a Cauchy sequence in $D$. Let $\lim_n x_n = v \in D$. Since $F(T) \subset D$ is closed and $\lim_n d(x_n, F(T)) = 0$, we conclude that $v \in F(T)$. This completes the proof.

As a consequence of Theorem 2.6, we obtain the following result.

**Theorem 2.7.** Let $D$ be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and let $F(T)$ be a nonempty closed set. For a given sequence $\{\alpha_n\}$ in $[0, 1]$, let the Mann iterative sequence $\{x_n\}$ in $D$ be defined by (2.6). Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_n d(x_n, F(T)) = 0$.

** Remark 2.8.** Theorems 2.6 and 2.7 extend corresponding results of Ghosh and Debnath [2], Liu [5], and Petryshyn and Williamson [7] from quasi-nonexpansive or asymptotically quasi-nonexpansive mapping to large class of non-Lipschitzian mappings.

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