DISTRIBUTION OF SPECIAL SEQUENCES MODULO A LARGE PRIME

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We study the sets $\{g^x - g^y \pmod{p} : 1 \leq x, y \leq N\}$ and $\{xy : 1 \leq x, y \leq N\}$ where p is a large prime number, g is a primitive root, and $p^{2/3} < N < p$.

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1. Introduction. Let p be a large prime number, g a primitive root $\pmod{p}$, and N a given positive integer, $N < p$. In a series of papers, the distribution of powers $g^n \pmod{p}$ has been investigated by [1, 2, 4, 5]. Vâjăitu and Zaharescu [5] considered the question of A. Odlyzko concerning the set of differences

$$A := \{g^x - g^y \pmod{p} : 1 \leq x, y \leq N\}. \quad (1.1)$$

As it was indicated in [5], A. Odlyzko asks for which values of N the set A contains all residue classes $\pmod{p}$. The conjecture is that one can take N to be as small as $p^{1/2+\varepsilon}$, for any positive ε and $p > c$ with some $c = c(\varepsilon)$. From the result of Rudnick and Zaharescu [4] it follows that in Odlyzko's problem one can take $N = c_0 p^{3/4} \log p$ for some absolute constant c_0 .

One of the main results of [5] is that for the exceptional set of Odlyzko's problem we have

$$\#\{h \pmod{p} : h \notin A\} \ll \frac{p^3 \log p}{N^3}. \quad (1.2)$$

It then follows that for $N > p^{2/3+\varepsilon}$ almost all the residues $\pmod{p}$ belong to A .

Denote

$$B = \{xy \pmod{p} : 1 \leq x, y \leq N\}. \quad (1.3)$$

Vâjăitu and Zaharescu [5] put another problem similar to that of Odlyzko: for which values of N can we be sure that the set B contains all residue classes $\pmod{p}$? They conjectured that N can be taken to be as small as $p^{1/2+\varepsilon}$ and

observed that one can take $N = c_1 p^{3/4} \log p$. This problem is also related to the pair correlation problem for sequences of the form $\alpha n^2 \pmod{1}$. For this account, see Rudnick et al. [3].

In this paper, using an elementary approach we slightly improve by a factor of $\log p$ estimate (1.2) and the estimate for N in Odlyzko's problem and obtain estimate (1.2) with the set B instead of A , see Theorems 1.1, 1.2, and 1.3.

THEOREM 1.1. *For any prime number p , any primitive root $g \pmod{p}$, and $N = 10p^{3/4}$, the set A contains the complete residue system $\pmod{p}$.*

THEOREM 1.2. *For any prime number p , any primitive root $g \pmod{p}$, and any positive integer $N < p$,*

$$\#\{h \pmod{p} : h \notin A\} \ll \frac{p^3}{N^3}. \quad (1.4)$$

THEOREM 1.3. *For any prime number p and any positive integer $N < p$,*

$$\#\{h \pmod{p} : h \notin B\} \ll \frac{p^3 \log p}{N^3}. \quad (1.5)$$

We require the following lemma (see [6, Exercise 14, page 92] and the solution in [6, page 142]) which will be used in the proof of Theorems 1.1 and 1.2.

LEMMA 1.4. *Let $m > 1$, $(a, m) = 1$. Then*

$$\left| \sum_{x=0}^{m-1} \sum_{y=0}^{m-1} v(x) \varrho(y) e^{2\pi i(ax y/m)} \right| \leq \sqrt{mXY}, \quad (1.6)$$

where $v(x)$, $\varrho(y)$ are complex numbers and

$$\sum_{x=0}^{m-1} |v(x)|^2 = X, \quad \sum_{y=0}^{m-1} |\varrho(y)|^2 = Y. \quad (1.7)$$

2. Proof of Theorem 1.1. Note that $0 \in A$. Let h be any integer, $h \not\equiv 0 \pmod{p}$, $N = 10p^{3/4}$, and denote $N_1 = [N/4]$. Our aim is to prove that $J > 0$, where J is the number of solutions in integers x , y , z , and t of the congruence equation

$$g^{x+z} - g^y - hg^t \equiv 0 \pmod{p} \quad (2.1)$$

subject to the condition

$$N_1 + 1 \leq x, y, z \leq 2N_1, \quad 1 \leq t \leq N_1. \quad (2.2)$$

In order to prove it we write J in terms of rational trigonometric sums:

$$pJ = \sum_{a=0}^{p-1} \sum_{x=N_1+1}^{2N_1} \sum_{y=N_1+1}^{2N_1} \sum_{z=N_1+1}^{2N_1} \sum_{t=1}^{N_1} e^{2\pi i(a(g^{x+z} - g^y - hg^t)/p)}. \quad (2.3)$$

Picking up the term with $a = 0$ and estimating other terms by their absolute values, we obtain

$$pJ \geq N_1^4 - \sum_{a=1}^{p-1} \left| \sum_{x=N_1+1}^{2N_1} \sum_{z=N_1+1}^{2N_1} e^{2\pi i(ag^x g^z/p)} \right| \times \left| \sum_{y=N_1+1}^{2N_1} e^{2\pi i(ag^y/p)} \right| \left| \sum_{t=1}^{N_1} e^{2\pi i(ahg^t/p)} \right|. \quad (2.4)$$

We will apply [Lemma 1.4](#) to the double inner sum. To do that, we define $v(u) = \varrho(u) = 1$ if $u \equiv g^x \pmod{p}$ for some $N_1 + 1 \leq x \leq 2N_1$. For all other u , we put $v(u) = \varrho(u) = 0$. Then [Lemma 1.4](#) gives

$$\left| \sum_{x=N_1+1}^{2N_1} \sum_{z=N_1+1}^{2N_1} e^{2\pi i(ag^x g^z/p)} \right| \leq \sqrt{pN_1^2}. \quad (2.5)$$

Hence,

$$pJ \geq N_1^4 - \sqrt{pN_1^2} \sum_{a=0}^{p-1} \left| \sum_{y=N_1+1}^{2N_1} e^{2\pi i(ag^y/p)} \right| \left| \sum_{t=1}^{N_1} e^{2\pi i(ahg^t/p)} \right|. \quad (2.6)$$

For the sum over a , we apply Cauchy inequality. Since g is a primitive root, then

$$\sum_{a=0}^{p-1} \left| \sum_{y=N_1+1}^{2N_1} e^{2\pi i(ag^y/p)} \right|^2 = pN_1, \quad \sum_{a=0}^{p-1} \left| \sum_{t=1}^{N_1} e^{2\pi i(ahg^t/p)} \right|^2 = pN_1. \quad (2.7)$$

Therefore, for each integer h ,

$$pJ > N_1^4 - p^{3/2} N_1^2 \quad (2.8)$$

and [Theorem 1.1](#) follows in view of $N_1 = \lfloor N/4 \rfloor$.

3. Proof of [Theorem 1.2](#). Denote $\overline{A} = \{h \pmod{p} : h \notin A\}$, $N_1 = \lfloor N/2 \rfloor$, and let $|\overline{A}|$ denote the cardinality of $\overline{A}$. Then

$$\sum_{h \in \overline{A}} \sum_{a=0}^{p-1} \sum_{x=1}^{N_1} \sum_{z=1}^{N_1} \sum_{y=1}^N e^{2\pi i(a(g^{x+z} - g^y - h)/p)} = 0. \quad (3.1)$$

Picking up the term with $a = 0$, we obtain

$$N_1^2 N |\overline{A}| \leq \sum_{a=1}^{p-1} \left| \sum_{x=1}^{N_1} \sum_{z=1}^{N_1} e^{2\pi i(ag^x g^z/p)} \right| \left| \sum_{y=1}^N e^{2\pi i(ag^y/p)} \right| \left| \sum_{h \in \overline{A}} e^{2\pi i(ah/p)} \right|. \quad (3.2)$$

We will apply [Lemma 1.4](#) to the double inner sum in the same way as we did in the proof of [Theorem 1.1](#). We obtain

$$\left| \sum_{x=1}^{N_1} \sum_{z=1}^{N_1} e^{2\pi i(ag^x g^z/p)} \right| \leq \sqrt{pN_1^2}. \quad (3.3)$$

Hence,

$$N_1^2 N |\bar{A}| \leq \sqrt{pN_1^2} \sum_{a=0}^{p-1} \left| \sum_{y=1}^N e^{2\pi i(ag^y/p)} \right| \left| \sum_{h \in \bar{A}} e^{2\pi i(ah/p)} \right|. \quad (3.4)$$

In analogy with [Section 2](#), we apply Cauchy inequality to the sum over a . Since

$$\begin{aligned} \sum_{a=0}^{p-1} \left| \sum_{y=1}^N e^{2\pi i(ag^y/p)} \right|^2 &= pN, \\ \sum_{a=0}^{p-1} \left| \sum_{h \in \bar{A}} e^{2\pi i(ah/p)} \right|^2 &= p|\bar{A}|, \end{aligned} \quad (3.5)$$

then

$$N_1^2 N |\bar{A}| \leq \sqrt{pN_1^2 pNp|\bar{A}|}. \quad (3.6)$$

Hence, from $N_1 = [N/2]$, we obtain

$$|\bar{A}| \leq \frac{10p^3}{N^3}. \quad (3.7)$$

This proves [Theorem 1.2](#).

4. Proof of Theorem 1.3. Using Gauss method of estimation of trigonometric sums, one can prove the validity of the following lemma.

LEMMA 4.1. *Let $1 \leq N \leq p$, $(a, p) = 1$. Then*

$$\left| \sum_{x=1}^N e^{2\pi i(ax^2/p)} \right| \ll \sqrt{p \log p}. \quad (4.1)$$

Indeed, if we denote by $|S|$ the value of the left-hand side, then

$$|S|^2 = \sum_{x=1}^N \sum_{y=1}^N e^{2\pi i(a(y^2-x^2)/p)} \leq N + 2 \left| \sum_{1 \leq x < y \leq N} e^{2\pi i(a(y^2-x^2)/p)} \right|. \quad (4.2)$$

Substituting $y = x + t$ gives

$$|S|^2 \ll N + \left| \sum_{x=1}^{N-1} \sum_{t=1}^{N-x} e^{2\pi i(at^2+2atx/p)} \right|. \quad (4.3)$$

Changing the order of summation, we obtain

$$|S|^2 \ll N + \sum_{t=1}^{N-1} \left| \sum_{x=1}^{N-t} e^{2\pi i(2atx/p)} \right| \ll N + \sum_{t=1}^{p-1} \frac{1}{|\sin(\pi 2at/p)|}. \quad (4.4)$$

When t runs through reduced residue system $(\bmod p)$ so does $2at$. Hence,

$$|S|^2 \ll N + \sum_{t=1}^{p-1} \frac{1}{|\sin(\pi t/p)|} \ll N + \sum_{t=1}^{(p-1)/2} \frac{1}{t/p} \ll p \log p. \quad (4.5)$$

We now proceed to prove [Theorem 1.3](#). Put $N_1 = [N/4]$ and denote by B_1 the set

$$B_1 = \{x^2 - y^2 (\bmod p), N_1 \leq x \leq 2N_1, 1 \leq y < N_1\}. \quad (4.6)$$

Since $B_1 \subset B$, then $|\overline{B}| \leq |\overline{B}_1|$ where $\overline{B}$ and $\overline{B}_1$ denote the complement of B and B_1 in the complete residue system $(\bmod p)$, accordingly. Now, as in the proof of [Theorem 1.2](#), we have

$$\sum_{h \in \overline{B}_1} \sum_{a=0}^{p-1} \sum_{x=N_1}^{2N_1} \sum_{y=1}^{N_1-1} e^{2\pi i(a(x^2-y^2-h)/p)} = 0. \quad (4.7)$$

Then it follows that

$$N^2 |\overline{B}_1| \ll \sum_{a=1}^{p-1} \left| \sum_{x=N_1}^{2N_1} e^{2\pi i(ax^2/p)} \right| \left| \sum_{y=1}^{N_1-1} e^{2\pi i(ay^2/p)} \right| \left| \sum_{h \in \overline{B}_1} e^{2\pi i(ah/p)} \right|. \quad (4.8)$$

Now, apply [Lemma 4.1](#) for the sum over x and then use Cauchy inequality as we did in the proof of [Theorems 1.1](#) and [1.2](#). Then, we obtain

$$N^2 |\overline{B}_1| \ll \sqrt{p \log p} \sqrt{p N p |\overline{B}_1|} \quad (4.9)$$

whence, we get

$$|\overline{B}_1| \ll \frac{p^3 \log p}{N^3}. \quad (4.10)$$

Now, [Theorem 1.3](#) follows from $|\overline{B}| \leq |\overline{B}_1|$.

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