SYNGE-BEIL AND RIEMANN-JACOBI JET STRUCTURES WITH APPLICATIONS TO PHYSICS

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In the framework of geometrized first-order jet approach, we study the Synge-Beil generalized Lagrange jet structure, derive the canonic nonlinear and Cartan connections, and infer the Einstein-Maxwell equations with sources; the classical ansatz is emphasized as a particular case. The Lorentz-type equations are described and the attached Riemann-Jacobi structures for two certain uniparametric cases are presented.

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1. Preliminaries. Let $(T, h_{\alpha\beta})$ and (M, γ_{ij}) be two $\mathcal{C}^\infty$ pseudo-Riemannian manifolds of dimensions m and n , respectively. We denote by $\zeta = (E = j^1(T, M), \pi, T \times M)$ the first-order jet bundle of mappings $\varphi : T \rightarrow M$, with the local coordinates

$$(t^\alpha, x^i, y^A)_{(\alpha, i, A) \in I_*} \equiv (y^\mu)_{\mu \in I}. \quad (1.1)$$

Throughout the paper, we consider the sets

$$\begin{aligned} I &= I_h \cup I_v, & I_h &= I_{h_1} \cup I_{h_2}, & I_{h_1} &= \overline{1, m}, & I_{h_2} &= \overline{m+1, m+n}, \\ I_v &= \overline{m+n+1, N}, & I_* &= I_{h_1} \times I_{h_2} \times I_v, & N &= m+n+mn; \end{aligned} \quad (1.2)$$

the indices will implicitly take the values

$$\alpha, \beta, \dots \in I_{h_1}, \quad i, j, \dots \in I_{h_2}, \quad A, B, \dots \in I_v, \quad \lambda, \mu, \dots \in I. \quad (1.3)$$

For $A = m+n+n(i-m-1)+\alpha$, we will denote $A \equiv \binom{i}{\alpha}$ and $y^A \equiv x^{\binom{i}{\alpha}} = \partial x^i / \partial t^\alpha$.

We endow E with the *sub-Riemannian Synge-Beil metric* (see [9])

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha} \binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(t, x, y), \quad (1.4)$$

where

$$g_{ij}(t, x, y) = \gamma_{ij}(x) + \varepsilon U_i(t, x, y) U_j(t, x, y), \quad \forall i, j \in I_{h_2}, \quad \varepsilon \in \{\pm 1\}, \quad (1.5)$$

and $U_i(t, x, y)$ is a distinguished 1-form on E (see [1]). We call $(E, \tilde{g})$ the *Synge-Beil (SB) jet model*. The inverse of g_{ij} is $g^{ij} = y^{ij} - \varepsilon \Theta U^i U^j$, where $U^i = y^{ij} U_j$, $\Theta = (1 + U_*)^{-1}$, and the star index denotes transvection with U^i .

We remark the important particular Synge-Beil uniparametric (SBU) *autonomous normalized case*, where $m = 1$, $s = t^1 = t$, and $h_{11} = 1$, for which we can use the *Finsler-Lagrange tangent space notations* from [5]. Shifting the indices left by one unit (hence, $I_{h_2} = \overline{1, n}$, $I_v = \overline{n+1, 2n}$), we have $y^A \equiv y^{(i)} \stackrel{\text{not}}{=} y^i$. In this case, considering

$$U_i = [k(1 - n^{-2}(x, y))]^{1/2} y_i, \quad k > 0, \quad (1.6)$$

we encounter three important extensively studied cases.

(I) The *Synge classical framework* (see [10]), obtained for $\varepsilon = 1$ and $k = 1$, where $y_i = y_{ij} y^j$, $n(x, y)$ is the refraction index of relativistic optics (see [7, 9]), and the direction $y = X(x)$ is provided by a vector field $X \in \mathcal{X}(M)$.

(II) If the potentials U_i in (1.5) are 0-homogeneous relative to y , in the limit case $n \rightarrow \infty$ with $\varepsilon = 1$, $k \in \mathcal{F}(M)$, we have

$$g_{ij}(x, y) = y_{ij}(x) + k \cdot U_i(x, y) U_j(x, y), \quad (1.7)$$

and we may consider the Finsler fundamental function $F = \sqrt{L}$, where

$$L = g_{ij}(x, y) y^i y^j. \quad (1.8)$$

This is the relativistic Beil-type metric (see [3, 4]) with the two intensively studied subcases

$$U_i \in \{(y_{jk} v^j v^k)^{-1/2} v_i, (s_j(x) v^j)^{-1} v_i\}, \quad (1.9)$$

where $s \in \mathcal{X}^*(M)$, and $v \in X^*(TM)$ is 0-homogeneous in y .

(III) The *generalized Lagrange model* of relativistic optics studied by Miron and Kawaguchi (see [6, 7]) is obtained as limit case $n \rightarrow \infty$ with $\varepsilon = 1$, $k = 1/c^2$ (c = speed of light), where the metric is

$$g_{ij}(x, y) = y_{ij}(x) + c^{-2} \cdot y_i y_j, \quad \forall i, j \in I_{h_2}. \quad (1.10)$$

Considering the general SB-jet case (1.5), we can fix a priori on E a *nonlinear connection* $N = \{N_\mu^A\}_{\mu \in I_h, A \in I_v}$ of coefficients

$$N_\beta^{(i)} = - \left| \frac{y}{\alpha \beta} \right| y^{(i)}_{(y)}, \quad N_j^{(\alpha)} = \left| \frac{i}{jk} \right| y^{(\alpha)}_{(k)}. \quad (1.11)$$

However, an open question (see [9]) addresses the physical significance of choosing an alternative target-nonlinear connection coefficients provided by the spray attached to the Lagrangian

$$L = \tilde{g}_{AB} \mathcal{Y}^A \mathcal{Y}^B, \quad (1.12)$$

given by $\tilde{N}_j^{(i)} = N_j^{(i)} + (\varepsilon/2) g^{ik} \partial_\alpha (U_k U_j)$.

The fixed nonlinear connection leads to a splitting $TE = HE \oplus VE$, where $VE = \text{Ker } \pi_*$, and to the associated local adapted basis of vector fields [1, 9]

$$\mathcal{B} = \left\{ \delta_\alpha \equiv \partial_\alpha - N_\alpha^A \delta_A, \delta_i \equiv \partial_i - N_i^A \delta_A, \delta_A \equiv \partial_A = \frac{\partial}{\partial \mathcal{Y}^A} \right\}_{(\alpha, i, A) \in I_*} \equiv \{\delta_\mu\}_{\mu \in I}, \quad (1.13)$$

where $\partial_\alpha \equiv \partial/\partial t^\alpha$, $\partial_i \equiv \partial/\partial x^i$, of dual basis

$$\begin{aligned} \mathcal{B}^* &= \left\{ \delta^\alpha \equiv dt^\alpha, \delta^i \equiv dx^i, \right. \\ &\quad \left. \delta^A \delta \mathcal{Y}^A = d\mathcal{Y}^A + N_\alpha^A dt^\alpha + N_i^A dx^i \right\}_{(\alpha, i, A) \in I_*} \equiv \{\delta^\mu\}_{\mu \in I}. \end{aligned} \quad (1.14)$$

For N fixed, a linear connection $\nabla = \{L_{\mu\nu}^\lambda\}_{\lambda, \mu, \nu \in I}$ in E has the adapted coefficients provided by $\delta^\lambda (\nabla_{\delta_\nu} \delta_\mu) = L_{\mu\nu}^\lambda$ for all $\lambda, \mu, \nu \in I$; these split into $3^3 = 27$ distinct subsets according to the three index subsets I_{h_1} , I_{h_2} , and I_ν .

We endow E with the metric

$$G = \underbrace{h_{\alpha\beta}(t) dt^\alpha \otimes dt^\beta}_h + \underbrace{g_{ij}(t, x, \mathcal{Y}) dx^i \otimes dx^j}_g + \underbrace{\tilde{g}_{AB}(t, x, \mathcal{Y}) \delta \mathcal{Y}^A \otimes \delta \mathcal{Y}^B}_{\tilde{g}} \quad (1.15)$$

with g_{ij} given in (1.5). The *Cartan linear connection* has the four essential sets of coefficients

$$\begin{aligned} L_{\beta\gamma}^\alpha &= \left| \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right|, & L_{jk}^i &= \left| \begin{array}{c} i \\ jk \end{array} \right| + \tilde{U}_{jk}^i, \\ L_{j\alpha}^i &= \frac{\left[\varepsilon(U^i U_j)_{;\alpha} - \Theta U^i U^m (U_m U_j)_{;\alpha} \right]}{2}, \\ L_{jA}^i &\equiv L_{j(\frac{j}{\alpha})}^i = \mathcal{Y}^{im} U_{(\frac{j}{\alpha})km} - U_{(\frac{j}{\alpha})km} \Theta U^i U^m \end{aligned} \quad (1.16)$$

with

$$\begin{aligned} \tilde{U}_{jk}^i &= \mathcal{Y}^{im} U_{jkm} - \varepsilon \Theta (\mathcal{Y}_{jk*} - U_{jk*}) U^i, \\ U_{ijm} &= \frac{\varepsilon [\delta_{\{j} (U_m U_{i\}) - \delta_m (U_i U_j)]}{2}, \\ U_{(\frac{i}{\alpha})jm} &= \frac{\varepsilon [\delta_{(\frac{j}{\alpha})} (U_m U_{i\}) - \delta_{(\frac{m}{\alpha})} (U_i U_j)]}{2}, \\ \mathcal{Y}_{ijm} &= |ij; m| = \frac{(\partial_{\{j} \mathcal{Y}_{mi\}} - \partial_m \mathcal{Y}_{ij})}{2}, \end{aligned} \quad (1.17)$$

where we denote by α and k the natural covariant derivatives on $(T, h_{\alpha\beta})$ and (M, γ_{ij}) , respectively, by $|\alpha_{\beta\gamma}|$ and $|\frac{i}{jk}|$ the Christoffel symbols of the metrics h and γ , respectively, and $\tau_{[i\dots j]} = \tau_{i\dots j} - \tau_{j\dots i}$, $\tau_{\{i\dots j\}} = \tau_{i\dots j} + \tau_{j\dots i}$.

The torsion and the curvature of ∇ adapted coefficients are given by

$$\delta^\lambda(\mathcal{T}(\delta_\nu, \delta_\mu)) = T_{\mu\nu}^\lambda, \quad \delta^\lambda(\mathcal{R}(\delta_\nu, \delta_\mu)\delta_\rho) = R_{\rho\mu\nu}^\lambda, \quad \forall \lambda, \mu, \nu, \rho \in I. \quad (1.18)$$

In the Cartan connection case, the essential associated *torsion coefficients* are (see [9])

$$\begin{aligned} T_{\alpha j}^i &= -L_{j\alpha}^i, & T_{\beta(\frac{k}{\gamma})}^{(\frac{i}{\alpha})} &= -\delta_\alpha^\gamma L_{j\beta}^i, & T_{j(\frac{k}{\gamma})}^{(\frac{i}{\alpha})} &= -\delta_\alpha^\gamma \tilde{U}_{jk}^i, & T_{jA}^i &= L_{jA}^i, \\ T_{(\frac{j}{\beta})(\frac{k}{\gamma})}^{(\frac{i}{\alpha})} &= \delta_\alpha^{[\beta} L_{j(\frac{k}{\gamma})}^{i]}, & T_{\beta\gamma}^{(\frac{i}{\alpha})} &= -\rho_{\alpha\beta\gamma}^\delta \mathcal{Y}^{(\frac{i}{\delta})}, \\ T_{jk}^{(\frac{i}{\alpha})} &= \rho_{jkl}^i \mathcal{Y}^{(\frac{l}{\alpha})}, & T_{\beta j}^{(\frac{i}{\alpha})} &= 0, \end{aligned} \quad (1.19)$$

where $\rho_{\alpha\beta\gamma}^\delta$ and ρ_{jkl}^i are the curvature components of the metrics h and γ respectively. The *nonholonomy coefficients* $\omega_{\mu\nu}^\lambda$ given by $[\delta_\mu, \delta_\nu] = \omega_{\mu\nu}^\lambda \delta_\lambda$, for all $\mu, \nu \in I$, are related to torsion via $T_{\mu\nu}^\lambda = L_{[\mu\nu]}^\lambda + \omega_{\mu\nu}^\lambda$, for all $\lambda, \mu, \nu \in I$, and the essential curvature N -tensor fields (for explicit expressions, see [9]) are

$$R_{\mu\nu\pi}^\lambda = \delta_{[\pi} L_{\mu]}^\lambda{}_\nu + L^\sigma{}_{\mu[\nu} L_{\sigma\pi]}^\lambda + L_{\mu\sigma}^\lambda \omega_{\nu\pi}^\sigma. \quad (1.20)$$

Denoting by $|\alpha$, $|i$, $|A$, and $|\lambda$ the covariant derivations given by ∇_{δ_μ} , for $\mu \in I_{h_1}, I_{h_2}, I_v$, and I , respectively, the *Ricci identities* for $X \in \mathcal{X}(E)$ and $\theta \in \mathcal{X}^*(E)$ are

$$\begin{aligned} X_{|[\mu|\nu]}^\lambda &= R_{\sigma\mu\nu}^\lambda X^\sigma - T_{\mu\nu}^\sigma X_{|\sigma]}^\lambda, \\ \theta_{\lambda|[\mu|\nu]} &= R_{\lambda\mu\nu}^\sigma \theta_\sigma + T_{\mu\nu}^\sigma \theta_{\lambda|\sigma]}, \quad \forall \lambda, \mu, \nu \in I. \end{aligned} \quad (1.21)$$

The adapted components of the Ricci tensor field are given by $R_{\lambda\mu} = R_{\lambda\mu\nu}^\nu$ and the scalar of curvature is $R \equiv G^{\mu\nu} R_{\lambda\mu\nu}^\lambda = R_h + R_g + R_v$, where

$$\begin{aligned} R_h &= h^{\alpha\beta} \rho_{\alpha\beta\gamma}^\gamma, & R_g &= (\gamma^{ij} - \varepsilon \Theta U^i U^j) (\rho_{ijk}^k + U_{ijk}^k), \\ R_v &= \tilde{g}^{AB} R_{AB}, \end{aligned} \quad (1.22)$$

and $U_{jkl}^i = \tilde{U}_{j[k|l]}^i + \tilde{U}^m{}_{j[k} \tilde{U}_{m|l]}^i + L_{j(\frac{m}{\alpha})}^i \rho_{pkl}^m \mathcal{Y}^{(\frac{p}{\alpha})}$.

2. Einstein-Maxwell equations. Denoting by $E_{\mu\nu} = R_{\mu\nu} + (1/2)RG_{\mu\nu}$ the Einstein N -tensor field, the *Einstein equations with sources*

$$E_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}, \quad \mu, \nu \in I, \quad (2.1)$$

split

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} &= \kappa \mathcal{T}_{\alpha\beta}, \\ R_{ij} - \frac{1}{2} R g_{ij} &= \kappa \mathcal{T}_{ij}, \\ R_{AB} - \frac{1}{2} R g_{AB} &= \kappa \mathcal{T}_{AB}, \\ 0 &= \mathcal{T}_{\alpha i}, \quad 0 = \mathcal{T}_{\alpha A}, \\ R_{i\alpha} &= \kappa \mathcal{T}_{i\alpha}, \quad R_{A\alpha} = \kappa \mathcal{T}_{A\alpha}, \\ R_{iA} &= \kappa \mathcal{T}_{iA}, \quad R_{Ai} = \kappa \mathcal{T}_{Ai}, \end{aligned} \quad (2.2)$$

where $\mathcal{T} = \mathcal{T}_{\mu\nu} \delta^\mu \otimes \delta^\nu \in \mathcal{T}_2^0(E)$ is the *energy-momentum tensor field* and κ is the cosmological constant. They satisfy the *conservation laws*

$$E_{\nu|\mu}^\mu = \kappa \mathcal{T}_{\nu|\mu}^\mu, \quad \forall \mu \in I = I_{h_1} \cup I_{h_2} \cup I_v, \quad (2.3)$$

where the indices are raised by means of the metric G in (1.15).

We note that for $U_i \equiv 0$, the Einstein equations reduce to the classical ones on $(T \times M, h + g)$. Also, if one considers the *extended electromagnetic 2-form*

$$F = F_{A\mu} \delta y^A \wedge \delta y^\mu, \quad (2.4)$$

then the Lagrangian density $\mathcal{L} = (L + F^{\lambda\mu} F_{\lambda\mu}) \sqrt{\det h}$ with L given in (1.12) provides by the Hilbert-Palatini variation the Einstein equations of the form (2.1) with the energy-momentum tensor field

$$T_{\mu\nu} = F_{\mu\rho} F_\nu^\rho - \frac{1}{4} G_{\mu\nu} F^{\rho\pi} F_{\rho\pi}, \quad \mu, \nu \in I. \quad (2.5)$$

In the extended potential Miron-Tatoiu approach [8], the electromagnetic tensor field F has the essential components

$$\begin{aligned} F_{A\beta} &\equiv F_{(\alpha)\beta} = \frac{1}{2} \left(h^{\alpha\gamma} g_{ik} \mathcal{Y}^{(k)}_{[\gamma]} \right)_{|\beta]}, \\ F_{Aj} &\equiv F_{(\alpha)j} = \frac{1}{2} d_{(\alpha)j]} = \left(\mathcal{Y}_{[ik} \tilde{U}_{m]j}^k + \varepsilon U_{[i} \tilde{U}_{m]j}^k U_k \right) h^{\alpha\beta} \mathcal{Y}^{(m)}_{(\beta)}, \\ F_{AB} &\equiv F_{(\alpha)(\beta)} = \frac{1}{2} \tilde{g}_{(\alpha)C} \mathcal{Y}^C_{|(\beta)} = \frac{1}{2} U_{(\beta)[ji]} h^{\alpha\gamma} \mathcal{Y}^{(m)}_{(\gamma)}, \end{aligned} \quad (2.6)$$

derived from the *deflection tensor fields* (detailed in [9])

$$d_\mu^A = \delta^A \nabla_{\delta_\mu} \mathcal{C}, \quad \mu \in I, \quad A \in I_v, \quad (2.7)$$

where $\mathcal{C} = \mathcal{Y}^A \delta_A$ is the Liouville field, by lowering the indices and antisymmetrization; where the raising/lowering of the indices are assumed to be

performed via the metric G . The extended electromagnetic tensor field (2.6) satisfies the following theorem.

THEOREM 2.1. *The 2-form F is subject to the two sets of the Maxwell extended equations with sources*

$$\begin{aligned} S_{\alpha\beta\gamma} F_{(\alpha)}^{\quad(i)}{}_{\beta| \gamma} &= \frac{1}{2} S_{\alpha\beta\gamma} h^{\alpha\varepsilon} g_{ik} \mathcal{Y}_{|\beta|}^{(k)}{}_{|\gamma}, \\ F_{(\alpha)}^{\quad(i)}{}_{j|\beta} &= \frac{1}{2} \left[d_{(\alpha)}^{[i}{}_{\beta|j]} + (\mathcal{Y}_{(\alpha)}^{(m)} L_{[i\beta}^m)_{j]} - d_{(\alpha)}^{\quad(i)}{}_{m} L_{j\beta}^m \right], \\ F_{(\alpha)}^{\quad(i)}{}_{(j)\beta| \gamma} &= \frac{1}{2} \left[d_{(\alpha)}^{[i}{}_{\gamma|} (\mathcal{Y}_{\beta)}^{(j)} + \mathcal{Y}_{(\alpha)}^{(m)} \partial_{(\beta)}^{[j]} L_{i]}^m \right. \\ &\quad \left. - (d_{(\alpha)}^{[i}{}_{\gamma)} (\mathcal{Y}_{\beta)}^{(m)} + L_{[i}^k (\mathcal{Y}_{\beta)}^{(m)} \mathcal{Y}_{\gamma)}^{(k)}) L_{j]}^m \delta_{\beta}^{\gamma} \right], \end{aligned} \quad (2.8)$$

$$\begin{aligned} S_{ijk} F_{(\alpha)}^{\quad(i)}{}_{j|k} &= -\frac{1}{2} S_{ijk} \left[d_{(\alpha)}^{\quad(i)}{}_{(m)}^{\beta} + L_{i(m)}^p \mathcal{Y}_{(\alpha)}^{(p)} \right] \rho_{jkl}^m \mathcal{Y}_{\beta}^{(l)}, \\ S_{ijk} \left[F_{(\alpha)}^{\quad(i)}{}_{j|} (\mathcal{Y}_{\beta)}^{(k)} + F_{(\alpha)}^{\quad(i)}{}_{(j)\beta|} k \right] &= 0, \quad F_{(\alpha)}^{\quad(i)}{}_{(j)\beta|} (\mathcal{Y}_{\gamma)}^{(k)} \equiv 0, \\ \tilde{g}^{BC} F_{B\alpha|C} &= -4\pi J_{\alpha}, \quad \tilde{g}^{BC} F_{Bi|C} = -4\pi J_i, \quad G^{\lambda\mu} F_{A\lambda|\mu} = 4\pi J_A, \end{aligned} \quad (2.9)$$

where we denoted by $J = J_{\mu} \delta^{\mu} \in \mathcal{X}^*(E)$ the adapted dual electric current and by S the cyclic summation of the corresponding indices below.

The last Maxwell equations in (2.8) were first derived in [9]. Note that in the SBU case with $U_i \equiv 0$, the equations above provide in particular the classical Maxwell equations with sources

$$S_{ijk} F_{ij;k} = 0, \quad \mathcal{Y}^{ij} F_{ik;j} = -4\pi J_k. \quad (2.10)$$

3. Extended Lorentz equations. The extended Lorentz equations associated to the SB model are $G_{\nu\rho} (\nabla^{\nu} \mathcal{V}^{\rho} / ds) = F_{A\nu} \mathcal{V}^A$ [2]; denoting $F_A^{\mu} = G^{\mu\nu} F_{A\nu}$, for all $\mu \in I$; they can be rewritten as

$$\frac{\nabla^{\mu} \mathcal{V}^{\mu}}{ds} = F_A^{\mu} \mathcal{V}^A, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{V} &= \mathcal{V}^{\mu} \delta_{\mu}, \\ \{\mathcal{V}^{\mu}\}_{\mu \in I} &\equiv \left(\frac{dt^{\alpha}}{ds}, \frac{dx^i}{ds}, \frac{\delta y^A}{ds} = \frac{dy^A}{ds} + N_{\beta}^A \frac{dt^{\beta}}{ds} + N_j^A \frac{dx^j}{ds} \right)_{(\alpha, i, A) \in I_*} \end{aligned} \quad (3.2)$$

is the covariant velocity along the trajectory of the moving test-particle

$$c : J \subset \mathbb{R} \longrightarrow E, \quad c(s) = (t(s), x(s), y(s)), \quad \forall s \in J. \quad (3.3)$$

We have denoted $\nabla^{\mu} \mathcal{V}^{\mu} / ds = \delta^{\mu} \mathcal{V}^{\mu} / ds + L_{\nu\rho}^{\mu} \mathcal{V}^{\nu} \mathcal{V}^{\rho}$, for all $\mu \in I$, and will further use the dot notation for expressing the s -derivation. The explicit Lorentz

equations were described in [2]. In the SBU case with U_i dependent on x only, the electromagnetic tensors (2.6) have the components

$$F_A^\alpha = 0, \quad F_A^i \equiv g^{ij} F_{(1)j} = g^{ij} g_{[kl} \tilde{U}_{m]j}^l \mathcal{Y}^{(m)}_{(1)}, \quad F_A^B = 0, \quad (3.4)$$

and the Lorentz equations (3.1) reduce to [2]

$$\begin{aligned} \dot{t}^\alpha + \left| \frac{\alpha}{\beta \mathcal{Y}} \right| \dot{t}^\beta \dot{t}^\gamma &= 0, \\ \dot{x}^i + \left| \frac{i}{jk} \right| \dot{x}^j \dot{x}^k &= g^{ij} g_{[kl} \tilde{U}_{m]j}^l \mathcal{Y}^{(m)}_{(1)} \mathcal{V}^{(k)}_{(1)}, \quad \dot{\mathcal{V}}^A = 0, \end{aligned} \quad (3.5)$$

and hence are characterized by constant vertical adapted velocity vector field.

4. Riemann-Jacobi structure and energy-dependent Lagrangians. We further consider in the SBU framework two particular cases.

(I) *The Miron-Kawaguchi generalized Lagrange case*, for g given by

$$g_{ij}(x, \mathcal{Y}) = \gamma_{ij}(x) + k \cdot \mathcal{Y}_i \mathcal{Y}_j \quad (4.1)$$

with $k \in \mathcal{F}(M)$, where the Lagrangian (1.12) becomes $\tilde{L} = \mathcal{Y}_0(1 + k\mathcal{Y}_0)$ and the null index denotes transvection with $\mathcal{Y}$. In this case, the Legendre transformation is given locally by $(x, \mathcal{Y}) \in TM \rightarrow (x, p) \in T^*M$, $p_i = \partial L / \partial \mathcal{Y}^i = 2\mathcal{Y}_i(1 + 2k\mathcal{Y}_0)$, $i = \overline{1, n}$, and is a local diffeomorphism on the set

$$D = \{(x, \mathcal{Y}) \mid \mathcal{Y} \neq 0, 1 + 2k\mathcal{Y}_0 \neq 0\} \subset TM. \quad (4.2)$$

The associated Hamiltonian is

$$H \equiv \mathcal{Y}^i \frac{\partial L}{\partial \mathcal{Y}^i} - L = \mathcal{Y}_0(1 + 3k\mathcal{Y}_0) \quad (4.3)$$

and the local *Riemann-Jacobi structure* $(TU, \hat{g})$ provided by the directional variables $U^i = \mathcal{Y}^i$ is defined by the scaled metric

$$\hat{g}_{ij} \equiv \left(H + \frac{1}{2} U_* \right) \delta_{ij} = \frac{3}{2} (\mathcal{Y}_0 + 2k\mathcal{Y}_0^2) \delta_{ij}, \quad (4.4)$$

where $U_* = \delta_{ij} U^i U^j$.

(II) *The flat local Lagrange space with potential energy-dependence* endowed with Riemann-Jacobi generalized Lagrange metric, with the square-type Lagrangian (see [11])

$$\hat{L} = \frac{1}{2} \delta^{ij} (\mathcal{Y}_i - U_i) (\mathcal{Y}_j - U_j) = \frac{1}{2} \mathcal{Y}_0 + U_0 + f, \quad (4.5)$$

where $f = (1/2)U_*$ and the indices are raised/lowered by means of the Kronecker flat Euclidean metric. Here, the Hamiltonian is $H = (1/2)\mathcal{Y}_0 - f$ and

the Legendre transformation provides momenta as potential shifts of direction $p_i = y_i - U_i$. For obtaining the h -paths associated to the Kern nonlinear connection [5, Theorem 7.4.1, page 113], we apply for $g_{ij} = \delta_{ij}$, $a = c = 1/2$, and $b = 1$ the following lemma.

LEMMA 4.1. *Let (M, γ_{ij}) be a (pseudo-)Riemannian space. Then*

(a) *the spray and the Kern nonlinear connection of the Lagrangian*

$$L'' = a\gamma_0 + bU_0 + cU_*, \quad a, b, c \in \mathbb{R}, \quad (4.6)$$

with $U_i \in \mathcal{X}^(M)$, for raising/lowering performed using the metric γ_{ij} and $U_* = U_i U^i$, are, respectively, given by*

$$\begin{aligned} G^i(x, y) &\equiv \frac{\gamma^{ij}(\partial_j \partial_k L'' \cdot y^k - \partial_j L'')}{4} \\ &= \frac{a}{2} \gamma_{00}^i + \frac{1}{4} \{ b [\gamma^{ia} (y^k U^j \partial_{[k} \gamma_{a]j} - y^k \partial_a U^k) + y^k \partial_k U^i] \\ &\quad - c \gamma^{ia} (U^j U^k \partial_a \gamma_{jk} + 2 U_j \partial_a U^j) \}, \end{aligned} \quad (4.7)$$

where γ_{jk}^i are the Christoffel symbols of γ_{ij} and

$$N_j^i(x, y) \equiv \frac{\partial G^i}{\partial y^j} = a \gamma_{j0}^i + \frac{b}{4} [\gamma^{ia} (\partial_{[j} \gamma_{a]k} \cdot y_{jk} \cdot \partial_a U^k) + \partial_j U^i]; \quad (4.8)$$

(b) *the Euler-Lagrange equations associated to L'' are the spray equations*

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \quad y^i = \dot{x}^i, \quad i = \overline{1, n}, \quad (4.9)$$

and the h -paths are given by

$$\ddot{x}^i + N_j^i(x, \dot{x}) \dot{x}^j = 0, \quad y^i = \dot{x}^i, \quad i = \overline{1, n}, \quad (4.10)$$

with the spray and nonlinear connection determined above.

Then the Kern spray for $g_{ij} = (1/2) \cdot (\text{Hess}_y L)_{ij} = \delta_{ij}$ is

$$G^i = \frac{\delta^{ij} (\partial_{[j} U_{k]}^k + U_k \partial_j U^k)}{4}, \quad (4.11)$$

and denoting $\Omega_{jk} = \partial_{[j} U_{k]}$, this is rewritten as $G^i = \delta^{ij} (\Omega_{j0} + \partial_j f)$. The associated nonlinear connection is then $N_j^i = (1/4) \Omega_j^i$, and its autoparallel curves (the h -paths) satisfy

$$\ddot{x}^i = \frac{1}{4} \Omega_j^i \dot{x}^j. \quad (4.12)$$

Then we have the following theorem.

THEOREM 4.2. *The h -paths described by (4.12) are as well:*

- (a) *the extended Lorentz curves (see [2]) particularized to the almost Riemann Lagrange special (ARLS) jet case associated to the flat metric δ_{ij} and to the potential $U_i/2$;*
- (b) *the solutions of the Lorentz-Udriste force law (see [11]) of the Riemann-Jacobi-Lagrange structure $(M = \mathbb{R}^n, \hat{g}_{ij}, 4N_j^i)$, where $\hat{g}$ is the Riemann-Jacobi metric*

$$\hat{g}_{ij} = (H + f)\delta_{ij} = \frac{\gamma_0}{2}\delta_{ij}; \quad (4.13)$$

- (c) *the stationary curves (see [5]) of the reduced Lagrangian $\hat{L}' = (1/2)\gamma_0 + U_0$.*

PROOF. For (a) and (b), the Riemann-Jacobi and the flat metrics have null Christoffel symbols; for (c), we apply the lemma for $g_{ij} = \delta_{ij}$, $a = 1/2$, $b = 1$, and $c = 0$. $\square$

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