CS-MODULES AND ANNIHILATOR CONDITIONS

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We study S - R -bimodules ${}_S M_R$ with the annihilator condition $S = l_S(A) + l_S(B)$ for any closed submodule A , and a complement B of A , in M_R . Such annihilator condition has a direct connection with the CS-condition for M_R . We make use of this to give a new characterization of CS-modules. Bimodules ${}_S M_R$ for which $r_M l_S(A) = A$ (for every closed submodule A of M_R) are also dealt with. Such modules are called W^* -modules. We give the extra added annihilator conditions to W^* -modules to be equivalent to the continuous (quasicontinuous) modules.

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1. Introduction. Let R and S be rings and let ${}_S M_R$ be a bimodule. For any $X \leq M$ and $T \leq S$, write $l_S(X) = \{s \in S : sX = 0\}$ and $r_M(T) = \{m \in M : Tm = 0\}$. Let $\lambda : S \rightarrow \text{End}(M_R)$ be the canonical ring homomorphism. For each $s \in S$, we identify $\lambda(s)$ with s . A submodule A is essential in M (denoted by $A \leq^e M$) if $A \cap B \neq 0$ for every nonzero submodule B of M . A submodule A is closed in M if it has no proper essential extensions in M . $A \leq^\oplus M$ signifies that A is a direct summand of M (or simply a summand). A module M is called a CS-module if every closed submodule of M is a summand. The module M is continuous if it is a CS-module and satisfies condition (C_2) : if $A \cong B \leq M$ with $A \leq^\oplus M$, then $B \leq^\oplus M$. A generalization of condition (C_2) is (GC_2) (see [4]): if A is a submodule of M with $A \cong M$, then $A \leq^\oplus M$. The module M is quasicontinuous if it is a CS-module and satisfies condition (C_3) : if $A, B \leq^\oplus M$ with $A \cap B = 0$, then $A \oplus B \leq^\oplus M$. It is known that M is quasicontinuous if and only if $M = A \oplus B$ whenever A and B are complements of each other in M (see [3, Theorem 2.8]).

Camillo et al. [1] have dealt with Ikeda-Nakayama rings that are related to continuous and quasicontinuous rings.

For a bimodule ${}_S M_R$, Wisbauer et al. [4] have studied the annihilator condition $l_S(A \cap B) = l_S(A) + l_S(B)$ for any submodules A and B of M_R , and the condition $S = l_S(A) + l_S(B)$ for any submodules A and B of M_R with $A \cap B = 0$. Consequently, they obtained new characterizations of quasicontinuous modules. We adapt their ideas here to study a variation of the above annihilator condition which is connected to CS-modules, and obtain a new characterization of CS-modules in Section 2.

In [Section 3](#), we study the bimodules ${}_S M_R$ which satisfy the following condition:

$$S = l_S(A) + l_S(B) \quad (1.1)$$

for any two relative complements A and B in M_R . Such modules are clearly quasicontinuous modules, while there are quasicontinuous modules which do not satisfy condition (1.1). For example, consider R as a commutative integral domain with field of quotients Q and let $M = Q \oplus Q$. In [Lemma 3.2](#), we give a necessary and sufficient condition for quasicontinuous modules to satisfy condition (1.1). In the case of $S = \text{End}(M_R)$, every quasicontinuous module must have condition (1.1). As a generalization of this condition, we introduce the concept of W^* -modules (bimodules ${}_S M_R$ for which $A = r_M l_S(A)$ for every closed submodule A of M_R). It is clear that any bimodule with condition (1.1) is a W^* -module, while in general the converse is not true. [Proposition 3.8](#) indicates when a W^* -module satisfies condition (1.1).

In [Section 4](#), we discuss the equivalence between W^* -modules and continuous (quasicontinuous) modules over an arbitrary ring S . Then we draw the consequences when S is the endomorphism ring of M_R .

2. CS-modules and annihilator conditions. The proofs of the lemmas and propositions, presented in this section, are adaptations of the arguments in [\[4\]](#).

LEMMA 2.1. *Let ${}_S M_R$ be a bimodule. If for every closed submodule A of M_R there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$, then M_R is a CS-module.*

PROOF. Let A be a closed submodule of M_R . Then by assumption there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$. Write $1_S = u + v$, where $u \in l_S(A)$ and $v \in l_S(B)$. It follows that $a = va$ for all $a \in A$, $b = ub$ for all $b \in B$, and $vB = uA = 0$. Thus $B \subseteq r_M(v) \subseteq r_M(v^2)$ and $r_M(v^2) \cap A = 0$. Since B is a complement of A in M_R , we have $B = r_M(v) = r_M(v^2)$. Similarly, $A = r_M(u) = r_M(u^2)$. Now we show that $(vu)M = 0$. Let $vum = a + b$, where $m \in M$, $a \in A$, and $b \in B$. Noting that $vu = uv$, we have that $(v^2 u^2)m = (vu)(a + b) = 0$. Hence $u^2 m \in r_M(v^2) = r_M(v)$, and this gives that $u^2 v m = v u^2 m = 0$. Then $vm \in r_M(u^2) = r_M(u)$; and thus $vum = uv m = 0$. So $(vu)M \cap (A + B) = 0$. Since $A + B$ is essential in M_R , $(vu)M = 0$. So $uM \subseteq r_M(v) = B$ and $vM \subseteq r_M(u) = A$ and hence $M = vM + uM = A + B = A \oplus B$. Therefore A is a summand of M_R . $\square$

REMARK 2.2. The converse of [Lemma 2.1](#) is not true. For example, there are torsion-free CS-modules over commutative integral domains, which do not satisfy the given condition in [Lemma 2.1](#).

The next lemma follows from [\[4, Lemma 3\]](#).

LEMMA 2.3. Let ${}_S M_R$ be a bimodule, where ${}_S M$ is faithful, and let $M_R = A \oplus B$. If the projection f of M onto A along B is given by $f(m) = sm$ for some $s \in S$, and all $m \in M$, then $S = l_S(A) + l_S(B)$.

For any submodules A and B of M_R and any $t \in S$, define $\alpha_t : A + B \rightarrow M$, $a + b \rightarrow ta$ (see [4]).

PROPOSITION 2.4. Let ${}_S M_R$ be a bimodule such that ${}_S M$ is faithful. The following are equivalent:

- (1) M_R is CS and for any $f^2 = f \in \text{End}(M_R)$, there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$;
- (2) for every closed submodule A of M_R , there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;
- (3) for every closed submodule A of M_R , there exists a complement B of A in M_R such that $S = l_S(A) \oplus l_S(B)$;
- (4) for every closed submodule A of M_R , there exists a complement B of A in M_R such that for every $t \in S$, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A + B & \longrightarrow & M \\ & & \downarrow \alpha_t & & \\ & & M & & \end{array} \quad (2.1)$$

can be extended by $\lambda(s)$, for some $s \in S$.

PROOF. (1) $\Rightarrow$ (2). Let A be a closed submodule of M_R . Since M_R is a CS-module, there exists $f^2 = f \in \text{End}(M_R)$ such that $A = fM$. By (1), there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$. Hence $(s^2 - s)M = (f^2 - f)M = 0$. Since ${}_S M$ is faithful, it follows that s is an idempotent in S . Now we have

$$l_S(A) = l_S(fM) = l_S(sM) = l_S(s) = S(1 - S). \quad (2.2)$$

Similarly, $l_S(B) = S_S$, where $B = (1 - f)M$. Thus $S = l_S(A) + l_S(B)$.

(2) $\Rightarrow$ (1). It is clear by Lemma 2.1 that M_R is CS. Now let $f^2 = f \in \text{End}(M_R)$, and denote $A = f(M)$. By (2), there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$. The argument of the proof of Lemma 2.1 shows that $M = A \oplus B$. Let π be the projection of M onto A along B . Then

$$l_S(A) = l_S(\pi M) = \{s \in S : s\pi = 0\} \quad (2.3)$$

(by considering s the homomorphism given by left multiplication by s) and

$$l_S(B) = l_S((1 - \pi)M) = \{s \in S : s(1 - \pi) = 0\}. \quad (2.4)$$

Let $1 = s' + s$, where $s' \in l_S(A)$ and $s \in l_S(B)$. Thus $s'\pi = 0$ and $s(1 - \pi) = 0$. It follows that $0 = s(1 - \pi) = (1 - s')(1 - \pi) = 1 - \pi - s'$. Therefore $f(m) = \pi(m) = sm$ for all $m \in M$.

(2) $\Rightarrow$ (3). From the argument in the proof of [Lemma 2.1](#), we have $M = A \oplus B$. Since ${}_S M$ is faithful, we have $0 = l_S(M) = l_S(A + B) = l_S(A) \dot{+} l_S(B)$ and hence $S = l_S(A) \oplus l_S(B)$.

(3) $\Rightarrow$ (4). Let A be a closed submodule of M_R . By (3), there exists a complement B of A such that $S = l_S(A) \oplus l_S(B)$. Write $t = u + v$, where $u \in l_S(A)$ and $v \in l_S(B)$. Then $\alpha_t(a + b) = ta = (u + v)a = va = v(a + b) = \lambda(v)(a + b)$.

(4) $\Rightarrow$ (2). Let A be a closed submodule of M_R . By (4), there exists a complement B of A in M_R satisfying diagram (2.1). By (4), there exists $s \in S$ such that $\lambda(s)$ extends α_t . Thus, for all $a \in A$ and $b \in B$, $ta = \alpha_t(a + b) = \lambda(s)(a + b) = s(a + b)$. It follows that $(1 - s)a + (-s)b = 0$, for all $a \in A$ and $b \in B$. So $1 - s \in l_S(A)$ and $-s \in l_S(B)$ and hence $1 = (1 - s) - (-s) \in l_S(A) + l_S(B)$. Therefore $S = l_S(A) + l_S(B)$. $\square$

COROLLARY 2.5. *The following are equivalent for a bimodule ${}_S M_R$ with $S = \text{End}(M_R)$:*

- (1) M_R is a CS-module;
- (2) for every closed submodule A of M_R , there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;
- (3) for every closed submodule A of M_R , there exists a complement B of A in M_R such that $S = l_S(A) \oplus l_S(B)$;
- (4) for every closed submodule A of M_R , there exists a complement B of A in M_R such that for every $t \in S$, diagram (2.1) can be extended by some $g: M \rightarrow M$.

PROPOSITION 2.6. *Let S be the center of $\text{End}(M_R)$. The following are equivalent:*

- (1) for every closed submodule A of M_R , there exists a complement B of A in M_R such that $S = l_S(A) + l_S(B)$;
- (2) M_R is CS and every idempotent of $\text{End}(M_R)$ is central;
- (3) M_R is CS and every closed submodule of M_R is fully invariant.

PROOF. (1) $\Leftrightarrow$ (2) by [Proposition 2.4](#).

(2) $\Rightarrow$ (3). Let A be a closed submodule of M . By CS, A is a direct summand of M_R . Then $A = f(M)$ for some $f^2 = f \in \text{End}(M_R)$. For any $g \in \text{End}_R(M)$, since f is central by (2), $g(A) = g(f(M)) = f(g(M)) \subseteq f(M) = A$. This shows that A is a fully invariant submodule of M .

(3) $\Rightarrow$ (2). Let $f, g \in \text{End}_R(M)$ with $f^2 = f$. Therefore $f(M)$ is a closed submodule of M_R . By (3), $g(f(M)) \subseteq f(M)$ and $g((1 - f)(M)) \subseteq (1 - f)(M)$. It follows that $f g f = g f$ and $(1 - f)g(1 - f) = g(1 - f)$. Thus, $g - g f = g(1 - f) = (1 - f)g(1 - f) = g - g f - f g + f g f = g - g f - f g + g f = g - f g$. This shows that $f g = g f$. $\square$

3. Condition (1.1) and its generalizations. The next lemma is clear.

LEMMA 3.1. *The following are equivalent for a bimodule ${}_S M_R$:*

- (1) $S = l_S(A) + l_S(B)$ for any two relative complements A and B of M_R ;
- (2) for any submodules A and B of M_R with $A \cap B = 0$, $S = l_S(A) + l_S(B)$.

We say that a bimodule ${}_S M_R$ has condition (1.1) if it satisfies one of the equivalent conditions of Lemma 3.1.

The next lemma follows from [4, Lemma 3].

LEMMA 3.2. *Let ${}_S M_R$ be a bimodule such that ${}_S M$ is faithful. Then the following are equivalent:*

- (1) M has condition (1.1);
- (2) M is quasicontinuous and every idempotent in $\text{End}(M_R)$ is a left multiplication by an element of S .

REMARK 3.3 [4, Theorem 8]. In the case of $S = \text{End}(M_R)$, it is clear from Lemma 3.2 that an R -module M is quasicontinuous if and only if M has condition (1.1).

PROPOSITION 3.4. *Let ${}_S M_R$ be a bimodule which satisfies condition (1.1). Then $A = r_M l_S(A)$ for all closed submodules A of M_R .*

PROOF. Let A be a closed submodule of M_R and B a submodule of $r_M l_S(A)$ such that $A \cap B = 0$. By Zorn's lemma, there exists a complement C of A in M_R with $B \subseteq C$. By condition (1.1), we have $S = l_S(A) + l_S(C) \subseteq l_S(A) + l_S(B)$, so $S = l_S(A) + l_S(B)$. Since $l_S(A) = l_S r_M l_S(A) \leq l_S(B)$, it follows that $S = l_S(B)$ and hence $B = 0$. This shows that $A \leq^e r_M l_S(A)$. Since A is a closed submodule of M_R , we have $A = r_M l_S(A)$. $\square$

A bimodule ${}_S M_R$ is called a W^* -module if $A = r_M l_S(A)$ for every closed submodule A of M_R . It is clear by Proposition 3.4 that every bimodule ${}_S M_R$ with condition (1.1) is a W^* -module. But there are bimodules which are W^* -modules and do not satisfy condition (1.1). For example, let $S = R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is any field and let $M = {}_R R_R$. It is clear that M is W^* -module. But M_R is not quasicontinuous, and hence M does not satisfy condition (1.1).

LEMMA 3.5. *The following are equivalent for a bimodule ${}_S M_R$:*

- (1) $A \leq^e r_M l_S(A)$ for all submodules A of M_R ;
- (2) ${}_S M_R$ is a W^* -module.

PROOF. (1) $\Rightarrow$ (2). This implication is obvious.

(2) $\Rightarrow$ (1). Let A be a submodule of M_R and C a maximal essential extension of A in M_R . We have by (2) that $A \leq^e C = r_M l_S(C)$. Since $r_M l_S(A) \leq r_M l_S(C)$, we have $A \leq^e r_M l_S(A)$. $\square$

PROPOSITION 3.6. *If ${}_S M_R$ is a W^* -module, then $r_M(T) = 0$, or $r_M(T)$ is uniform for every maximal left ideal T of S .*

PROOF. Let T be a maximal left ideal of S . Since $T \subseteq l_S r_M(T)$, we have either $l_S r_M(T) = T$ or $l_S r_M(T) = S$. If $l_S r_M(T) = S$, then $r_M(T) = 0$. If $l_S r_M(T) = T$, let N be a nonzero submodule of $r_M(T)$. Then $T = l_S r_M(T) \subseteq l_S(N) \subseteq S$, and the maximality of T yields $T = l_S(N)$. It follows that $r_M(T) = r_M l_S(N)$. Since M is W^* -module, we have by Lemma 3.5 that $N \leq^e r_M(T)$. Therefore $r_M(T)$ is uniform. $\square$

COROLLARY 3.7. *Let ${}_S M_R$ be a W^* -module, where every maximal left ideal of S is a left annihilator. Then $r_M(T)$ is uniform for every maximal left ideal T of S .*

PROOF. Let T be a maximal left ideal of S . From Proposition 3.6, it is enough to show that $r_M(T) \neq 0$. Let $r_M(T) = 0$. By assumption, $T = l_S r_M(T) = l_S(0) = S$, which contradicts the maximality of T . $\square$

PROPOSITION 3.8. *The following are equivalent for a bimodule ${}_S M_R$:*

- (1) ${}_S M_R$ is a W^* -module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements A and B in M_R ;
- (2) ${}_S M_R$ has condition (1.1).

PROOF. (1) $\Rightarrow$ (2). Let A and B be two relative complements in M_R . Then by (1), $S = l_S(0) = l_S(A \cap B) = l_S(r_M l_S(A) \cap r_M l_S(B)) = l_S r_M(l_S(A) + l_S(B)) = l_S(A) + l_S(B)$. Therefore M has condition (1.1).

(2) $\Rightarrow$ (1). This implication is obvious. $\square$

4. The relation between W^* -modules and (quasi-) continuous modules.
The following is an immediate consequence of Proposition 3.8.

PROPOSITION 4.1. *Let ${}_S M_R$ be a bimodule with $S = \text{End}(M_R)$. Then the following are equivalent:*

- (1) ${}_S M_R$ is a W^* -module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements A and B of M_R ;
- (2) M_R is quasicontinuous.

PROPOSITION 4.2. *Let ${}_S M_R$ be a bimodule, where ${}_S M$ is faithful. Then the following are equivalent:*

- (1) ${}_S M_R$ is a W^* -module, $l_S(A) + l_S(B)$ is an annihilator for any two relative complements A and B of M_R , and M_R has GC_2 ;
- (2) M_R is a continuous module and every idempotent in $\text{End}(M_R)$ is a left multiplication by an element of S .

PROOF. (1) $\Rightarrow$ (2). We have by Proposition 3.8 that M_R has condition (1.1). Therefore, by Lemma 3.2, M_R is a quasicontinuous module. Let $s \in \text{End}(M_R)$ be a monomorphism, with $sM \leq^e M$. By GC_2 it follows that $sM = M$. Then by [3, Lemma 3.14], M_R is a continuous module. The rest of the proof of (2) follows from Lemma 3.2.

(2) $\Rightarrow$ (1). This implication is obvious. $\square$

COROLLARY 4.3. *Let ${}_S M_R$ be a bimodule with $S = \text{End}(M_R)$. Then the following are equivalent:*

- (1) *${}_S M_R$ is a W^* -module, $l_S(A) + l_S(B)$ is an annihilator for any two relative complements A and B of M_R , and M_R has GC_2 ;*
- (2) *M_R is a continuous module.*

In particular, if M_R is of finite uniform dimension, then S is semiperfect.

PROOF. It is clear that every monomorphism $f \in \text{End}(M_R)$ is an isomorphism (due to GC_2 and M of finite uniform dimension). Hence, M satisfies the assumptions in Camps and Dicks [2, Theorem 5], and so $\text{End}(M_R)$ is semilocal. Therefore by using [3, Proposition 3.5 and Lemma 3.7], idempotents of $S/J(S)$ lift to idempotents of S , and thus S is semiperfect. $\square$

LEMMA 4.4. *Let ${}_S M_R$ be a bimodule such that every finitely generated left ideal of S is a left annihilator of a subset of M_R , and every closed submodule of M_R is a right annihilator of a finite subset of S . Then M has condition (1.1).*

PROOF. Let A_1 and A_2 be complements of each other in M_R . Then by assumption, we have $A_i = r_M(Y_i)$ for some finite subsets Y_i of S . Again by assumption, $SY_i = l_S(K_i)$ for some subsets K_i in M_R , where $i = 1, 2$. Now $S = l_S(A_1 \cap A_2) = l_S(r_M(Y_1) \cap r_M(Y_2)) = l_S r_M(SY_1 + SY_2) = SY_1 + SY_2$ (due to the assumption and since $SY_1 + SY_2$ is finitely generated). Hence $S = l_S(K_1) + l_S(K_2) = l_S r_M l_S(K_1) + l_S r_M l_S(K_2) = l_S r_M(Y_1) + l_S r_M(Y_2) = l_S(A_1) + l_S(A_2)$. Therefore M satisfies condition (1.1). $\square$

LEMMA 4.5. *Let ${}_S M_R$ be a bimodule and let every idempotent in $\text{End}(M_R)$ be a left multiplication by an element of S . If M_R is a CS-module, then every closed submodule of M_R is a right annihilator of a finite subset of S .*

PROOF. Let A be a closed submodule of M_R . Then by CS, there exists $f^2 = f \in \text{End}(M_R)$ such that $A = r_M(1 - f) = \{m \in M : (1 - s)m = 0\} = r_M(1 - s)$, where $(1 - s) \in S$. $\square$

The following corollary is an immediate consequence of Lemmas 4.4 and 4.5.

COROLLARY 4.6. *Let ${}_S M_R$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of S be a left annihilator of a subset of M . Then the following are equivalent:*

- (1) *every closed submodule of M is a right annihilator of a finite subset of S ;*
- (2) *M is a CS-module.*

THEOREM 4.7. *Let ${}_S M_R$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of S be a left annihilator of a subset of M . Then the following are equivalent:*

- (1) *M is a CS-module;*
- (2) *M is continuous.*

PROOF. By Lemmas 4.4 and 4.5, we have that M has condition (1.1). By Remark 3.3, M is quasicontinuous. To show that M is continuous, by [3, Lemma 3.14], it is enough to show that every essential monomorphism $s \in S$ is an isomorphism. Let $s \in S$ be a monomorphism, with $sM \leq^e M$. By assumption, $S_S = l_S(X)$ for some subset X of M . It follows that $X = 0$ and hence $S_S = s$. Then s is a split monomorphism, and therefore $sM = M$. $\square$

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