THE CONVERGENCE ESTIMATES FOR GALERKIN-WAVELET SOLUTION OF PERIODIC PSEUDODIFFERENTIAL INITIAL VALUE PROBLEMS

NGUYEN MINH CHUONG and BUI KIEN CUONG

Received 13 March 2002

Using the discrete Fourier transform and Galerkin-Petrov scheme, we get some results on the solutions and the convergence estimates for periodic pseudodifferential initial value problems.


1. Introduction. In recent years, wavelets have been developing intensively and have become a powerful tool to study mathematics and technology, for example, the theory of the singular integral, singular integro-differential equations, the areas such as sound analysis, image compression, and so on (see [9, 10] and references therein). In this paper, we use a scaling function and a multilevel approach to estimate the error of the problem

\[ \frac{\partial u(x,t)}{\partial t} = a \cdot Au(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \ a \in \mathbb{R}, \]

\[ u(x,0) = [u_0](x), \quad x \in \mathbb{R}^n, \] (1.1)

where \( A \) is a pseudodifferential operator (see [1, 2, 3, 4, 6, 8, 9, 12]) with a symbol \( \sigma \in C^\infty(\mathbb{R}^n) \), \( \sigma \) is positively homogeneous of degree \( r > 0 \) such that

\[ |D^\alpha \sigma(\xi)| \leq C_\alpha (1 + |\xi|)^{r-|\alpha|}, \text{ for all multi-index } \alpha \in \mathbb{N}^n, \] (1.2)

\( \mathbb{R}^n = \mathbb{R}^n / \mathbb{Z}^n \), and \([u_0](x) = \sum_{k \in \mathbb{Z}^n} u_0(x+k) \) is a periodic operator.

We discuss only problem (1.1) with the following condition:

\[ a \sigma(\xi) \leq 0, \quad \forall \xi \in \mathbb{Z}^n. \] (1.3)

2. Preliminaries and notations. The continuous Fourier transform of the function \( f \in L_2(\mathbb{R}^n) \) is defined by

\[ \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}^n \] (2.1)
with the inverse Fourier formula
\[ f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \hat{f}(\xi) d\xi, \quad \xi \in \mathbb{R}^n \] (2.2)
(see [4, 8, 11]).

The discrete Fourier transform of the function \( f \in L_2(\mathbb{Z}^n) \) is
\[ \mathcal{F}(f)(\xi) = \tilde{f}(\xi) := \int_{[0,1]^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{Z}^n, \] (2.3)
and the inverse Fourier transform is
\[ f(x) := \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) e^{2\pi i x \xi} \] (2.4)
(see [6]).

Some simple properties of the discrete Fourier transform are
\[ (f,g)_0 = \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) \overline{\tilde{g}(\xi)}, \] (2.5)
where \((\cdot, \cdot)_0\) is the \(L_2(\mathbb{Z}^n)\)-inner product,
\[ \|f\|_0^2 = \sum_{\xi \in \mathbb{Z}^n} |\tilde{f}(\xi)|^2 = \|\tilde{f}\|_{l_2}^2, \] (2.6)
where \(\|\cdot\|_0\) is \(L_2(\mathbb{Z}^n)\)-norm and \(\|\cdot\|_{l_2}\) is \(l_2\)-norm.

Let \(s \in \mathbb{R}\). Denote
\[ H^s(\mathbb{Z}^n) = \{ u \in D'(\mathbb{Z}^n) \mid (D)^s u \in L_2(\mathbb{Z}^n) \}, \] (2.7)
where
\[ \langle \xi \rangle = \begin{cases} 1 & \text{if } \xi = 0, \\ |\xi| & \text{if } \xi \neq 0, \end{cases} \] (2.8)
then \(H^s(\mathbb{Z}^n)\) is the Sobolev space endowed with the norm
\[ \|u\|_s^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \] (2.9)
and the inner product
\[ \langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)}. \] (2.10)

Here, we also define the discrete Sobolev space \(H^s_d(\mathbb{R}^n), s \in \mathbb{R}\), of the functions \(f \in H^s(\mathbb{R}^n)\) such that the following norm is finite:
\[ \|f\|_{s,d}^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2. \] (2.11)
Denote
\[ \mathcal{L}_2 = \left\{ f \in L^2(\mathbb{R}^n) : \sum_{\xi \in \mathbb{Z}^n} |f(\cdot - \xi)| \in L^2([0,1]^n) \right\}. \] (2.12)

It is clear that any function \( f \in L^2(\mathbb{R}^n) \), which has compact support, or any function, for which \( \int_{k+1}^{k+0.1} |f(x)|^2 \, dx \) decays exponentially as \( |k| \) tends to infinity, belongs to \( \mathcal{L}_2 \). The periodic operator \( [u] \) is totally defined if \( u \in \mathcal{L}_2 \).

Here, we assume that \( u_0 \in \mathcal{L}_2 \).

**Remark 2.1.** (1) It follows from (2.1) and (2.3) that if \( u \in \mathcal{L}_2 \), then \( \mathcal{F}([u])(\xi) = \hat{u}(\xi) \), \( \xi \in \mathbb{Z}^n \).

(2) It is clear that if \( t \leq s \), \( s, t \in \mathbb{R} \), then \( H^t(\mathcal{J}^n) \subset H^s(\mathcal{J}^n) \).

Using the variable separate method and the discrete Fourier transform, the solution of problem (1.1) can be represented as
\[ u(x,t) = E(t)[u_0](x) = \sum_{\xi \in \mathbb{Z}^n} \exp(a\sigma(\xi)t)\mathcal{F}([u_0])(\xi)e^{2\pi ix \xi}, \] (2.13)
where \( E(t) \) is a differentiable function and \( E(0) = 1 \).

We recall that a multiresolution approximation (MRA) of \( L^2(\mathbb{R}^n) \) is, as a definition, an increasing sequence \( V_j, j \in \mathbb{Z} \), of closed linear subspaces of \( L^2(\mathbb{R}^n) \) with the following properties:
\[ \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n); \] (2.14)
for all \( f \in L^2(\mathbb{R}^n) \) and all \( j \in \mathbb{Z} \),
\[ f(x) \in V_j \iff f(2x) \in V_{j+1}; \] (2.15)
for all \( f \in L^2(\mathbb{R}^n) \) and \( k \in \mathbb{Z}^n \),
\[ f(x) \in V_0 \iff f(x-k) \in V_0. \] (2.16)

There exists a function, called the scaling function (SF) \( \phi(x) \in V_0 \), such that the sequence
\[ \{ \phi(x-k), k \in \mathbb{Z}^n \} \] (2.17)
is a Riesz basic of \( V_0 \) (see [5, 9]).

An SF \( \phi \) is called \( \mu \)-regular (\( \mu \in \mathbb{N} \)) if, for each \( m \in \mathbb{N} \), there exists \( c_m \) such that the following condition holds:
\[ |D^\alpha \phi(x)| \leq c_m (1 + |x|)^{-m}, \quad \forall \alpha, |\alpha| \leq \mu. \] (2.18)
Remark 2.2. (1) Denote \( \phi_{jk}(x) = 2^{nj/2}\phi(2^jx - k), \) \( k \in \mathbb{Z}^n \). It follows from (2.14), (2.15), (2.16), and (2.17) that \( V_j = \text{span}\{\phi_{jk}(x), k \in \mathbb{Z}^n\} \), \( j \in \mathbb{Z} \).

(2) For each \( \mu \in \mathbb{N} \), there exists an SF \( \phi(x) \) with compact support, and \( \phi(x) \) is \( \mu \)-regular; so in what follows, we always assume that \( \phi \) has compact support and is \( \mu \)-regular (see [9]).

Using the periodic operator and an MRA of \( L_2(\mathbb{R}^n) \), we can build an MRA of \( L_2(\mathbb{R}^n) \) with the SF \( \phi \) as follows.

Denote
\[
\phi_k^j(x) = 2^{nj/2} \phi(2^jx - k), \quad j \geq 0,
\]
(2.19)
where \( \mathbb{Z}^{nj} = \mathbb{Z}^n/2^j\mathbb{Z}^n \).

Then, the sequence \([V_j]_{j=0}^\infty\) satisfies
\[
[V_0] \subset [V_1] \subset \cdots, \quad \bigcup_{j=0}^\infty [V_j] = L_2(\mathbb{R}^n).
\]
(2.20)

It is clear that \( \dim[V_j] = 2^{nj} \), and if \((\phi_{jk}, \phi_{jl}) = \delta_{kl}, k, l \in \mathbb{Z}^n, \) then \((\phi_k^j, \phi_l^j) = \delta_{kl}, k, l \in \mathbb{Z}^{nj} \) (see [6]).

For each \( j \geq 0 \), let \( P_j : L_2(\mathbb{R}^n) \to [V_j] \) be the orthogonal projection from \( L_2(\mathbb{R}^n) \) on \( [V_j] \), which has the following property.

Theorem 2.3 (see [6, page 600]). Let \(-\mu - 1 \leq s \leq \mu, -\mu \leq q \leq \mu + 1, \) and \( s \leq q, \) then
\[
\|u - P_j u\|_s \leq c2^{j(s-q)}\|u\|_q
\]
(2.22)
for all \( u \in H^q(\mathbb{R}^n) \), where \( c \) is independent of \( j \) and \( u \).

Denoting \( h = 2^{-j} \) and \( V_h = [V_j] \), we can write (2.22) as
\[
\|u - P_j u\|_s \leq c2^{j(s-q)}\|u\|_q.
\]
(2.23)

3. The Galerkin-wavelet solution. Fix a distribution with compact support \( \eta \in H^{-s'}(\Gamma) \), where \( s' \geq 0 \) satisfying \( AV_h \subset H^{s'}(\mathbb{R}^n) \) and where \( \Gamma \subset \mathbb{R}^n \) is some fixed compact domain such as a hypercube. For \( f \in H^{s'}(\mathbb{R}^n) \), define
\[
\eta_k^j(f) = 2^{-nj/2}\eta(f(2^{-j}(\cdot + k))).
\]
(3.1)

The space
\[
X_j := \text{span}\{\eta_k^j, k \in \mathbb{Z}^{nj}\}
\]
(3.2)
is contained in $(AV_h)'$, which is the dual of $AV_h$. The corresponding Galerkin-Petrov-wavelet scheme is then given by

$$
\eta_j^k \left( \frac{\partial u_h}{\partial t} \right) = an_j^k (Au_h), \quad k \in \mathbb{Z}^n,
$$

(3.3)

$$
u_h(x,0) = R_h[u_0](x),
$$

(3.4)

where $R_h \nu$ is a linear approximation of $\nu$ in $V_h$ and $u_h : [0, \infty) \to V_h$ is a differentiable operator.

Set

$$u_h(x,t) = \sum_{k \in \mathbb{Z}^n} c_k(t) \phi_j^k(x),
$$

(3.5)

$$R_h[u_0](x) = [u_0]_h(x) := \sum_{k \in \mathbb{Z}^n} c_k(0) \phi_j^k(x).
$$

(3.6)

Then the scheme (3.3) and (3.4) provides an algebra equation system and the solution can be solved by Fourier series.

**Lemma 3.1.** The following formulas hold true:

$$
\mathcal{F}(\phi_j^k)(\xi) = h^{n/2} \hat{\phi}(h\xi) e^{-2\pi ikh\xi},
$$

(3.7)

\[ \mathcal{F}(A\phi_j^k)(\xi) = h^{n/2} \sigma(\xi) \hat{\phi}(h\xi) e^{-2\pi ikh\xi}. \]

**Proof.** (a) It follows from (2.3) and (2.19) that

$$
\mathcal{F}(\phi_j^k)(\xi) = h^{-n/2} \sum_{l \in \mathbb{Z}^n} \int_{[0,1]^n} e^{-2\pi i x \xi} \phi(2^j(x+l) - k) \, dx
$$

$$
= h^{n/2} \sum_{l \in \mathbb{Z}^n} \int_{2^j(l+[0,1]^n) - k} e^{-2\pi i h x \xi} \phi(x) \, dx e^{-2\pi ikh\xi}
$$

$$
= h^{n/2} \int_{\mathbb{R}^n} e^{-2\pi i h x \xi} \phi(x) \, dx e^{-2\pi ikh\xi}
$$

$$
= h^{n/2} \hat{\phi}(h\xi) e^{-2\pi ikh\xi}.
$$

(3.8)

(b) We have

$$
\mathcal{F}(Au)(\xi) = \sigma(\xi) \hat{u}(\xi);
$$

(3.9)

consequently,

$$
\mathcal{F}(A\phi_j^k)(\xi) = \sigma(\xi) \mathcal{F}(\phi_j^k)(\xi) = h^{n/2} \sigma(\xi) \hat{\phi}(h\xi) e^{-2\pi ikh\xi}.
$$

(3.10)

The proof of the lemma is complete. \qed
**Corollary 3.2.** The following formulas hold true:

\[
\eta_j^k(\phi_l^j) = h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi)\hat{\eta}(h\xi)e^{-2\pi ih(l-k)\xi},
\]

\[
\eta_j^k(A\phi_l^j) = h^n \sum_{\xi \in \mathbb{Z}^n} \sigma(\xi)\hat{\phi}(h\xi)\hat{\eta}(h\xi)e^{-2\pi ih(l-k)\xi}.
\]

**Proof.** (a) Using (2.4), Lemma 3.1, and (3.1), we have

\[
\eta_j^k(\phi_l^j) = \eta_j^k\left(\sum_{\xi \in \mathbb{Z}^n} \mathcal{F}(\phi_l^j)(\xi)e^{2\pi ix\xi}\right)
= \eta_j^k\left(h^{n/2}\hat{\phi}(h\xi)e^{-2\pi ihl\xi}e^{2\pi ix\xi}\right)
= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi)e^{-2\pi hl\xi}\hat{\eta}(h\xi)e^{2\pi ix(k-k)\xi}
= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi)\hat{\eta}(h\xi)e^{-2\pi ih(l-k)\xi}.
\]

(b) Similarly, we can get the second assertion. 

The following lemma is extracted from [6].

**Lemma 3.3.** The following formula holds valid:

\[
\sum_{m \in \mathbb{Z}^{nj}} e^{-2\pi ihm(k-\xi)} = \begin{cases} 2^n \eta_j^k & \text{if } \xi = k + 2^j \theta, \theta \in \mathbb{Z}^n, \\ 0 & \text{otherwise}. \end{cases}
\]

Set

\[
\alpha(k) = \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi)\hat{\eta}(h\xi)e^{2\pi ihk\xi},
\]

\[
\delta(k) = \sum_{\xi \in \mathbb{Z}^n} \sigma(h\xi)\hat{\phi}(h\xi)\hat{\eta}(h\xi)e^{2\pi ihk\xi}, \quad k \in \mathbb{Z}^{nj}.
\]

The series

\[
\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \alpha(k)e^{-2\pi ihk\zeta},
\]

\[
\tilde{\delta}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \delta(k)e^{-2\pi ihk\zeta},
\]

\[
\tilde{c}(\zeta, t) = h^n \sum_{k \in \mathbb{Z}^{nj}} c_k(t)e^{-2\pi ihk\zeta}, \quad \zeta \in \mathbb{Z}^n
\]

are called discrete Fourier series.
It follows from (3.3), (3.5), the positively homogeneous condition, and Corollary 3.2 that

$$
\sum_{k \in \mathbb{Z}^n} c_k^t(t) \alpha(l-k) = ah^{-r} \sum_{k \in \mathbb{Z}^n} c_k(t) \delta(l-k), \quad l \in \mathbb{Z}^n. \quad (3.19)
$$

Thus

$$
\tilde{c}_t^\prime(\zeta, t) \tilde{\alpha}(\zeta) = ah^{-r} \tilde{c}(\zeta, t) \tilde{\delta}(\zeta), \quad (3.20)
$$

$$
\tilde{c}(\zeta, t) = \exp \left( \frac{at \tilde{\delta}(\zeta)}{\hbar r \tilde{\alpha}(\zeta)} \right) \tilde{c}(\zeta, 0). \quad (3.21)
$$

For each \( \tau = 0, 1 \), set

$$
g_{\phi, \tau}(\zeta) = \sum_{k \in \mathbb{Z}^n} \sigma(h\zeta + k) \tau \hat{\phi}(h\zeta + k) \hat{\eta}(h\zeta + k). \quad (3.22)
$$

**Lemma 3.4.** If the series (3.22) converges absolutely, then

$$
\tilde{\alpha}(\zeta) = g_{\phi, 0}(\zeta), \quad \tilde{\delta}(\zeta) = g_{\phi, 1}(\zeta). \quad (3.23)
$$

**Proof.** (a) From (3.14) and (3.16), it follows that

$$
\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^n} \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \hat{\eta}(h\xi) e^{-2\pi ih\xi} \tilde{\alpha}(\zeta). \quad (3.24)
$$

By the hypothesis of the lemma, we can interchange the summation in the above double sum; then by using the variable change and Lemma 3.3, it is easy to see that

$$
\tilde{\alpha}(\zeta) = h^n \sum_{\xi \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi) \hat{\eta}(h\xi) e^{-2\pi ih\xi} \tilde{\alpha}(\zeta). \quad (3.25)
$$

(b) Similarly, the second assertion of the lemma will be checked. From (3.5), (3.6), and (3.21), it follows that

$$
\hat{u}_h(\xi, t) = \exp \left( \frac{at \tilde{\delta}(\xi)}{h r \tilde{\alpha}(\xi)} \right) \hat{\mathcal{F}}([u_0]_h)(\xi). \quad (3.26)
$$

Let \( F_h(t) \) be the operator defined by

$$
\mathcal{F}(F_h(t)v(\cdot))(\xi) = \exp \left( \frac{at \tilde{\delta}(\xi)}{h r \tilde{\alpha}(\xi)} \right) \hat{v}(\xi), \quad (3.27)
$$

then the approximation \( u_h(x) \) can be represented by

$$
u_h(x) = F_h(t)R_h[u_0](x). \quad (3.28)
$$
4. The error estimate of approximation solutions. Now to estimate the error, we need some restrictions on the $\sigma$, $\phi$, and $\eta$ used above. The triplet $(\sigma, \phi, \eta)$ is called admissible if the following properties hold:

(i) there exists $p \in \mathbb{N}$, $p \geq r$, such that the series

$$
\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) \tag{4.1}
$$

converges absolutely and

$$
\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) = \sigma(h\xi) \hat{\phi}(h\xi) \hat{\eta}(h\xi) + o(|h\xi|^p) \tag{4.2}
$$

as $|h\xi| \to 0$,

(ii) $\hat{\phi}(\xi) \hat{\eta}(\xi) \geq 0$, for all $\xi \in \mathbb{R}^n$, $\hat{\phi}(0) \hat{\eta}(0) \neq 0$,

(iii) the series

$$
\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) \tag{4.3}
$$

converges and

$$
\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) = \hat{\phi}(h\xi) \hat{\eta}(h\xi) + o(|h\xi|^p) \tag{4.4}
$$

as $|h\xi| \to 0$.

**Remark 4.1.** (1) If $\eta = \phi$ and $\sigma$ is a pseudodifferential operator with symbol $\sigma(\xi) = |\xi|^r$, $0 < r \leq \mu$, then the triplet $(\sigma, \phi, \phi)$ is automatically admissible at least for $p = \mu$, where $\mu \in \mathbb{N}$ is used in (2.18) (see [7] for detail).

(2) If $\eta = \phi$ and $\sigma$ is a pseudodifferential operator with symbol $\sigma(\xi) = \langle \xi \rangle^2$, then the triplet $((\langle \xi \rangle^2, \phi, \phi)$ is admissible for $p = \mu$ (see [6]).

Write

$$
\begin{aligned}
\epsilon = u - u_h &= \{ u - F_h(t)[u_0] \} + F_h(t) \{ [u_0] - R_h[u_0] \}.
\end{aligned} \tag{4.5}
$$

We have

$$
\begin{aligned}
\mathcal{F}(F_h(t)[u_0](\cdot))(\xi) &= \exp \left( \frac{at}{h^r} \tilde{\delta}(\xi) \right) \mathcal{F}([u_0])(\xi) \\
&= \exp \left( \frac{at}{h^r} \tilde{\alpha}(\xi) \right) \hat{u}_0(\xi), \quad \xi \in \mathbb{Z}^n,
\end{aligned} \tag{4.6}
$$

thus

$$
\begin{aligned}
\mathcal{F}(u - F_h(t)[u_0])(\xi) \\
&= \left\{ \exp(\frac{at}{h^r} \sigma(\xi)) - \exp \left( \frac{at}{h^r} \tilde{\alpha}(\xi) \right) \right\} \hat{u}_0(\xi), \quad \xi \in \mathbb{Z}^n.
\end{aligned} \tag{4.7}
$$
If the triplet \((\sigma, \phi, \eta)\) is admissible, then it follows from (3.22) and Lemma 3.4 that
\[
\frac{\bar{\delta}(\xi)}{\bar{\alpha}(\xi)} = \sigma(h\xi) + O(|h\xi|^p) \quad \text{as} \quad |h\xi| \to 0.
\]

(4.8)

Theorem 4.2. Suppose that \(r + s' \leq s \leq p, 0 \leq m \leq s\), and it is assumed that the triplet \((\sigma, \phi, \eta)\) is admissible. Then, for \(u_0 \in L^2_0 \cap H^m_{2s}(\mathbb{R}^n), 0 \leq t \leq T\), with \(h\) small enough, we get
\[
||u - F_h(t)[u_0]||_m \leq ch^{s-r}||u_0||_{s+m,d},
\]
where \(c\) is independent of \(u, h, \) and \(u_0\).

Proof. It follows from (4.8) that
\[
\left| \frac{at\sigma(\xi) - \frac{at}{h^r\bar{\alpha}(\xi)}}{\bar{\delta}(\xi)} \right| \leq ch^{p-r}|\xi|^p \quad \text{as} \quad |h\xi| \leq 1.
\]

(4.10)
The equality
\[
e^{ta} - e^{tb} = t(a - b) \int_0^1 e^{sta+(1-s)tb} ds,
\]
(4.11), and (1.3) imply that, for \(r \leq s \leq p\) and \(0 \leq t \leq T\),
\[
\left| \exp(at\sigma(\xi)) - \exp \left( \frac{at}{h^r\bar{\alpha}(\xi)} \right) \right| \leq ch^{s-r}|\xi|^s \quad \text{as} \quad |h\xi| \leq 1.
\]

(4.12)
Hence, from (4.7) and (4.12), we obtain
\[
|\mathcal{F}(u(\cdot, t) - F_h(t)[u_0](\cdot))| (\xi) \leq ch^{s-r}|\xi|^s |\hat{u}_0(\xi)| \quad \text{as} \quad |h\xi| \leq 1.
\]

(4.13)
By (1.3) and the admissibility of the triplet \((\sigma, \phi, \eta)\), inequality (4.13) is also valid for all \(\xi \in \mathbb{Z}^n\). Hence, for each \(0 \leq m \leq s, r + s' \leq s \leq p\), and \(0 \leq t \leq T\), we get
\[
||u - F_h(t)[u_0]||_m^2 = \sum_{\xi \in \mathbb{Z}^n} |\langle \xi \rangle|^{2m} |\mathcal{F}[u(\cdot, t) - F_h(t)[u_0](\cdot)](\xi)|^2
\leq ch^{2(s-r)} \sum_{\xi \in \mathbb{Z}^n} |\langle \xi \rangle|^{2(s+m)} |\hat{u}_0(\xi)|^2
\leq ch^{2(s-r)}||u_0||_{m+s,d}^2.
\]

(4.14)
The theorem is thus proved. \(\square\)
From the admissibility of the triplet \((\sigma, \phi, \eta)\) and (1.3), it follows that \(F_h(t) : H^m(\mathbb{R}^n) \rightarrow H^m(\mathbb{R}^n)\), \(0 \leq m \leq s\), is a continuous linear operator. Consequently,

\[ \|F_h(t)([u_0] - R_h[u_0])\|_m \leq c\|[u_0] - R_h[u_0]\|_m, \] (4.15)

Therefore, if we assume that

\[ \|(I - R_h)[u_0]\|_m \leq ch^s\|[u_0]\|_{m+s}, \] (4.16)

then

\[ \|F_h(t)([u_0] - R_h[u_0])\|_m \leq ch^s\|[u_0]\|_{m+s}. \] (4.17)

**Remark 4.3.** It follows from (2.23) that the assumption (4.17) is satisfied, when \(R_h = P_j\) for \(0 \leq m, m + s \leq \mu + 1\).

Thus from (4.5), (4.9), and (4.17), we obtain the following theorem.

**Theorem 4.4.** If all the hypotheses of Theorem 4.2 and assumption (4.17) are satisfied, then

\[ \|u - u_h\|_m \leq ch^{s-r}\|[u_0]\|_{m+s} + ch^s\|[u_0]\|_{m+s}, \] (4.18)

where \(c\) is independent of \(u_0, h\).

**Acknowledgment.** The authors thank the referee and the managing editor for their helpful comments and suggestions.

**References**


Nguyen Minh Chuong: National Centre for Natural Science and Technology, Institute of Mathematics, 18 Hoang Quoc Viet Road, Cau Giay District, Hanoi, Vietnam
E-mail address: nmchuong@thevinh.ncst.ac.vn

Bui Kien Cuong: Department of Mathematics, Hanoi Pedagogical University, Number 2, Xuan Hoa, Me Linh, Vinh Phu, Vietnam
E-mail address: bkhcuong@hn.vnn.vn