WAVELET ANALYSIS ON A BOEHMIAN SPACE

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We extend the wavelet transform to the space of periodic Boehmians and discuss some of its properties.

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1. Introduction. The concept of Boehmians was introduced by J. Mikusiński and P. Mikusiński [7], and the space of Boehmians with two notions of convergences was well established in [8]. Many integral transforms have been extended to the context of Boehmian spaces, for example, Fourier transform [9, 10, 11], Laplace transform [13, 17], Radon transform [14], and Hilbert transform [3, 5].

On the other hand, the theory of wavelet transform is recently developed, and it has various applications in signal processing, especially to analyze non-stationary signals by providing the time-frequency representation of the signal. For a fixed \( g \in \mathcal{L}^2(\mathbb{R}) \), called a mother wavelet, the wavelet transform \( \Phi_g : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R} \times \mathbb{R}^+) \) is defined by

\[
\Phi_g(f)(a,b) = \int_{-\infty}^{\infty} f(x) \overline{g_{a,b}(x)} \, dx \quad \text{for } a > 0, \ b \in \mathbb{R},
\]

where \( g_{a,b}(x) = (1/\sqrt{a})g((x-b)/a), \ x \in \mathbb{R}, \) are called wavelets. For more details, we refer the reader to [6]. In [4], we extended the wavelet transform to a Boehmian space which properly contains \( \mathcal{L}^2(\mathbb{R}) \) and studied its properties.

Holschneider [2] introduced the wavelet transform on the space \( C^\infty(\mathbb{T}) \) of smooth functions on the unit circle \( \mathbb{T} \) of the complex plane and gave an extension to the space of periodic distributions. In Section 2, we fix some notations and discuss the theory of wavelet transform on \( C^\infty(\mathbb{T}) \). In Section 3, we briefly recall the periodic Boehmians, construct a new Boehmian space \( \mathcal{B}(\mathcal{Y}), (C^\infty(\mathbb{T}), \star, \diamond, \Delta) \), and verify some auxiliary results. In Section 4, we define wavelet transform on the space of periodic Boehmians and prove that it is consistent with the wavelet transform on \( C^\infty(\mathbb{T}) \). Further, we establish that the extended wavelet transform is linear and continuous with respect to \( \delta \)-convergence as well as \( \Delta \)-convergence.
2. Preliminaries. The space $C^\infty(\mathbb{T})$ consists of infinitely differentiable, periodic functions on $\mathbb{R}$ of period $2\pi$, with the Fréchet space topology induced by the increasing sequence of seminorms

$$
\|\phi\|_{C^\infty(\mathbb{T});n} = \sum_{p=0}^{n} \sup_{t \in [0,2\pi]} |\partial^p \phi(t)|. \tag{2.1}
$$

We know that

$$
C^\infty(\mathbb{T}) = C^\infty_+(\mathbb{T}) \oplus C^\infty_-(\mathbb{T}) \oplus K(\mathbb{T}), \tag{2.2}
$$

where $C^\infty_+(\mathbb{T})$ and $C^\infty_-(\mathbb{T})$ are the subspaces consisting of functions with positive and negative Fourier coefficients, respectively, and $K(\mathbb{T})$ is the space of constant functions.

Let $\mathcal{S}(\mathbb{R})$ denote the space of rapidly decreasing functions on $\mathbb{R}$. (See [1].) Given $f \in \mathcal{S}(\mathbb{R})$, $b \in [0,2\pi]$, and $a > 0$, define $f_a, f_b,a \in C^\infty(\mathbb{T})$ by

$$
\begin{align*}
    f_a(x) &= \sum_{n \in \mathbb{Z}} \frac{1}{a} f\left(\frac{x+2n\pi}{a}\right), & x \in [0,2\pi], \\
    f_{b,a}(x) &= f_a(x-b), & x \in [0,2\pi].
\end{align*} \tag{2.3}
$$

Let $\mathcal{S}(\mathbb{R} \times \mathbb{R}^+)$ denote the Fréchet space of all smooth functions $\eta(b,a)$ of rapid descent on $\mathbb{R} \times \mathbb{R}^+$ which are periodic functions in the variable $b$ of period $2\pi$, with the following directed family of seminorms:

$$
\|\eta\|_{\mathcal{S}(\mathbb{R} \times \mathbb{R}^+);n,\alpha,\beta} = \sum_{0 \leq p, l \leq n} \sup_{0 \leq k \leq \beta} \sup_{a > 0} \sup_{b \in [0,2\pi]} |a^p l^k \partial^l a^k b \eta(b,a)|. \tag{2.4}
$$

We choose a mother wavelet $g \in \mathcal{S}(\mathbb{R})$ with all moments $\int_{-\infty}^{\infty} x^n g(x)dx$ are equal to zero.

**Definition 2.1.** The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{R})$ is defined by

$$
T_g(\phi) = \int_0^{2\pi} \phi(x) \overline{g_{b,a}}(x) dx, \quad b \in \mathbb{R}, \ a > 0. \tag{2.5}
$$

**Theorem 2.2.** The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{R})$ is continuous and linear.

**Definition 2.3.** The map $R_g : \mathcal{S}(\mathbb{R}) \to C^\infty(\mathbb{T})$ is defined by

$$
(R_g \eta)(x) = \int_0^{2\pi} \int_0^{\infty} g_{b,a}(x) \eta(b,a) \frac{dadb}{a}. \tag{2.6}
$$
Theorem 2.4. The map \( R_\hat{g} : \mathcal{S}(\mathbb{R}) \to C^\infty(\mathbb{T}) \) is continuous and linear.

A partial inversion formula is given by the following theorem.

Theorem 2.5. If \( \hat{\varphi} \) is the Fourier transform of \( \varphi \) and \( C_+^\infty g = \int_0^\infty |\hat{\varphi}(a)|^2 (da/a) \), \( C_-^\infty g = \int_0^\infty |\hat{\varphi}(-a)|^2 (da/a) \), then

\[
R_\hat{g} \circ T g \varphi = C_+^\infty g \varphi, \quad \forall \varphi \in C_+^\infty(\mathbb{T}),
\]

\[
R_\hat{g} \circ T g \varphi = C_-^\infty g \varphi, \quad \forall \varphi \in C_-^\infty(\mathbb{T}).
\]

3. Boehmian spaces. The triplet \((C_\infty(\mathbb{T}), \ast, \Delta)\), where \( \ast : C_\infty(\mathbb{T}) \times C_\infty(\mathbb{T}) \to C_\infty(\mathbb{T}) \) is defined by

\[
(\varphi \ast \psi)(x) = \int_0^{2\pi} \varphi(x-t)\psi(t)dt, \quad x \in [0, 2\pi]
\]

and \( \Delta \) is the collection of all sequences \((\delta_k)\) from \( C_\infty(\mathbb{T}) \) satisfying

1. \( \int_0^{2\pi} \delta_k(t)dt = 1 \) for all \( k \in \mathbb{N} \),
2. \( \int_0^{2\pi} |\delta_k(t)|dt \leq M \) for all \( k \in \mathbb{N} \), for some \( M > 0 \),
3. \( s(\delta_k) \to 0 \) as \( n \to \infty \) where \( s(\delta_k) = \sup\{t \in [0, 2\pi] : \delta_k(t) \neq 0\} \),

is the collection of all equivalence classes \([\varphi_k/\delta_k]\) given by the equivalence relation \( \sim \) defined by

\[
((\varphi_k), (\delta_k)) \sim ((\psi_k), (\epsilon_k)) \text{ if } \varphi_k \ast \epsilon_j = \psi_j \ast \delta_k \quad \forall k, j \in \mathbb{N}
\]

on the collection \( \mathcal{A} \) of pair of sequences \((\varphi_k), (\delta_k)\), \( \varphi_n \in C_\infty(\mathbb{T}) \), \( (\delta_k) \in \Delta \)

satisfying

\[
\varphi_k \ast \delta_j = \varphi_j \ast \delta_k, \quad \forall k, j \in \mathbb{N}.
\]

This triplet with addition and scalar multiplication, defined by

\[
\begin{bmatrix} \varphi_k \\ \delta_k \end{bmatrix} + \begin{bmatrix} \psi_k \\ \epsilon_k \end{bmatrix} = \begin{bmatrix} \varphi_k \ast \epsilon_k + \psi_k \ast \delta_k \\ \delta_k \ast \epsilon_k \end{bmatrix},
\]

\[
\alpha \begin{bmatrix} \varphi_k \\ \delta_k \end{bmatrix} = \begin{bmatrix} \alpha \varphi_k \\ \alpha \delta_k \end{bmatrix},
\]

is called the periodic Boehmian space [15, 16], and we denote it by \( \mathcal{B}_\mathbb{T} \).

Definition 3.1 (\( \delta \)-convergence). A sequence \((x_n)\) \( \delta \)-converges to \( x \) in \( \mathcal{B}_\mathbb{T} \), denoted by \( x_n \overset{\delta}{\to} x \) as \( n \to \infty \) in \( \mathcal{B}_\mathbb{T} \) if there exists \( (\delta_k) \in \Delta \) such that
\( x_n \ast \delta_k, x \ast \delta_k \in C^\infty(\mathbb{T}) \), and for each \( k \in \mathbb{N} \),

\[
x_n \ast \delta_k \rightharpoonup x \ast \delta_k \text{ as } n \to \infty \text{ in } C^\infty(\mathbb{T}).
\] (3.5)

The following theorem is proved in [8].

**Theorem 3.2.** Let \( x_n, x \in \mathcal{B}_\mathbb{T} \), \( n \in \mathbb{N} \). \( x_n \overset{\delta}{\rightharpoonup} x \) as \( n \to \infty \) in \( \mathcal{B}_\mathbb{T} \) if and only if there exist \( \phi_{n,k}, \phi_k \in C^\infty(\mathbb{T}) \) such that \( x_n = [\phi_{n,k}/\delta_k], [\phi_k/\delta_k] \) and, for each \( k \in \mathbb{N} \),

\[
\phi_{n,k} \rightharpoonup \phi_k \text{ as } n \to \infty \text{ in } C^\infty(\mathbb{T}).
\] (3.6)

**Definition 3.3 (\( \Delta \)-convergence).** A sequence \( (x_n) \) \( \Delta \)-converges to \( x \) in \( \mathcal{B}_\mathbb{T} \), denoted by \( x_n \overset{\Delta}{\rightharpoonup} x \) as \( n \to \infty \) in \( \mathcal{B}_\mathbb{T} \) if there exists a delta-sequence \( (\delta_n) \) such that \( (x_n - x) \ast \delta_n \in C^\infty(\mathbb{T}) \) for each \( n \in \mathbb{N} \) and

\[
(x_n - x) \ast \delta_n \to 0 \text{ as } n \to \infty \text{ in } C^\infty(\mathbb{T}).
\] (3.7)

Now, we construct a new Boehmian space as follows.

As in the context of Boehmian space defined in [12], we take the vector space \( \Gamma \) and the commutative semi-group as \( \mathcal{F}(\mathbb{Y}) \) and \( (C^\infty(\mathbb{T}), *) \), respectively.

**Definition 3.4.** Given \( \eta \in \mathcal{F}(\mathbb{Y}) \) and \( \phi \in C^\infty(\mathbb{T}) \), define

\[
(\eta \circ \phi)(b,a) = \int_0^{2\pi} \eta(b - t,a)\phi(t)dt.
\] (3.8)

**Lemma 3.5.** If \( \eta \in \mathcal{F}(\mathbb{Y}) \) and \( \phi \in C^\infty(\mathbb{T}) \), then \( \eta \circ \phi \in \mathcal{F}(\mathbb{Y}) \).

**Proof.** To prove that \( (\eta \circ \phi)(b,a) \) is infinitely differentiable, we show that

\[
\partial_a(\eta \circ \phi)(b,a) = (\partial_a \eta \circ \phi)(b,a),
\]

\[
\partial_b(\eta \circ \phi)(b,a) = (\partial_b \eta \circ \phi)(b,a).
\] (3.9)

Fix \( a_0 > 0, b_0 \in \mathbb{R} \) arbitrarily.

Consider \( ((\eta \circ \phi)(b_0,a) - (\eta \circ \phi)(b_0,a_0))/(a - a_0) = \int_0^{2\pi} (\eta(b_0 - t,a) - \eta(b_0 - t,a_0))/(a - a_0)\phi(t)dt \). Using the mean-value theorem (in the variable \( a \)), we get that the integrand is dominated by \( \|\eta\|_{\mathcal{F}(\mathbb{Y}),0,1,0}\|\phi\|_{C^\infty(\mathbb{T}),0} \). Therefore, we can apply Lebesgue dominated convergence theorem [18], and we get

\[
\partial_a(\eta \circ \phi)(b_0,a_0) = \lim_{a \to a_0} \int_0^{2\pi} \frac{\eta(b_0 - t,a) - \eta(b_0 - t,a_0)}{a - a_0}\phi(t)dt
\]

\[
= \int_0^{2\pi} \lim_{a \to a_0} \frac{\eta(b_0 - t,a) - \eta(b_0 - t,a_0)}{a - a_0}\phi(t)dt
\]

\[
= \int_0^{2\pi} \partial_a \eta(b_0 - t,a_0)\phi(t)dt
\]

\[
= (\partial_a \eta \circ \phi)(b_0,a_0).
\] (3.10)
By a similar argument, we can prove that
\[ \partial_b (\eta \odot \phi)(b,a) = (\partial_b \eta \odot \phi)(b,a). \]
Finally by a routine manipulation, we get
\[ \| \eta \odot \phi \|_{\mathcal{F}(\mathbb{Y});\alpha,\beta} \leq \| \phi \|_{\mathcal{F}(\mathbb{Y});\alpha,\beta}, \tag{3.11} \]
where \( \| \phi \|_{\mathcal{F}(\mathbb{Y});\alpha,\beta} = \int_{0}^{2\pi} |\phi(t)| \, dt \). Hence, \( \eta \odot \phi \in \mathcal{F}(\mathbb{Y}) \).

**Lemma 3.6.** If \( \eta \in \mathcal{F}(\mathbb{Y}) \) and \( (\delta_n) \in \Delta \), then \( \eta \odot \delta_n \to \phi \) as \( n \to \infty \) in \( \mathcal{F}(\mathbb{Y}) \).

**Proof.** Let \( p,k,l \in \mathbb{N}_0 \) be arbitrary. Using the mean-value theorem and a property of \( \delta \)-sequence, we get
\[ |a^p \partial_a^l \partial_b^k (\eta \odot \delta_n - \eta)(b,a)| = |a^p ((\partial_a^l \partial_b^k \eta) \odot \delta_n)(b,a) - a^p \partial_a^l \partial_b^k \eta(b,a)| \]
\[ \leq \int_{0}^{2\pi} |a^p \partial_a^l \partial_b^k \eta(b - t,a) - \partial_a^l \partial_b^k \eta(b,a))| \delta_n(t) \, dt \]
\[ \leq \| \eta \|_{\mathcal{F}(\mathbb{Y});p,l,k+1} \int_{0}^{2\pi} |t| |\delta_n(t)| \, dt \]
\[ \leq MS(\delta_n) \| \eta \|_{\mathcal{F}(\mathbb{Y});p,l,k+1}, \tag{3.12} \]
which tends to 0 as \( n \to \infty \). This completes the proof of the lemma.

**Lemma 3.7.** If \( \eta_n \to \eta \) as \( n \to \infty \) in \( \mathcal{F}(\mathbb{Y}) \) and \( \psi \in C^\infty(\mathbb{T}) \), then \( \eta_n \odot \psi \to \eta \odot \psi \) as \( n \to \infty \).

**Proof.** Let \( p,k,l \in \mathbb{N}_0 \) be arbitrary. Now,
\[ |a^p \partial_a^l \partial_b^k (\eta_n \odot \psi - \eta \odot \psi)(b,a)| \]
\[ \leq \int_{0}^{2\pi} |a^p \partial_a^l \partial_b^k (\eta_n - \eta)(b,a)| |\psi(t)| \, dt \]
\[ \leq \| \psi \|_{\mathcal{F}(\mathbb{Y});p,l,k} \| \eta_n - \eta \|_{\mathcal{F}(\mathbb{Y});p,l,k} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13} \]
Hence, the lemma follows.

**Lemma 3.8.** If \( \eta_n \to \eta \) as \( n \to \infty \) in \( \mathcal{F}(\mathbb{Y}) \) and \( \delta_n \in \Delta \), then \( \eta_n \odot \delta_n \to \eta \) as \( n \to \infty \).

**Proof.** Since we have \( \eta_n \odot \delta_n - \eta = \eta_n \odot \delta_n - \eta \odot \delta_n + \eta \odot \delta_n - \eta \) and **Lemma 3.6**, we merely prove that \( \eta_n \odot \delta_n - \eta \odot \delta_n \to 0 \) as \( n \to \infty \).

If \( p,k,l \in \mathbb{N}_0 \), then, using a property of \( \delta \)-sequence, we get
\[ |a^p \partial_a^l \partial_b^k (\eta_n - \eta) \odot \delta_n(b,a)| \]
\[ \leq \| \eta_n - \eta \|_{\mathcal{F}(\mathbb{Y});p,l,k} \int_{0}^{2\pi} |\delta_n(t)| \, dt \leq M \| \eta_n - \eta \|_{\mathcal{F}(\mathbb{Y});p,l,k}. \tag{3.14} \]
The above inequalities prove the lemma.
Now using the above lemmas we can construct the Boehmian space $\mathcal{B}_Y = (\mathcal{F}_Y, (C^\infty, \ast), \odot, \Delta)$ in a canonical way.

4. Generalized wavelet transform

**Definition 4.1.** Define $\mathcal{T}_g : \mathcal{B}_T \to \mathcal{B}_Y$ by

$$\mathcal{T}_g \left( \left[ \frac{\phi_n}{\delta_n} \right] \right) = \left[ \frac{T_g\phi_n}{\delta_n} \right].$$

(4.1)

**Theorem 4.2.** The generalized wavelet transform $\mathcal{T}_g : \mathcal{B}_T \to \mathcal{B}_Y$ is well defined.

First, we state and prove a lemma that will be useful.

**Lemma 4.3.** If $\phi, \psi \in C^\infty(\mathbb{T})$, then $T_g(\phi \ast \psi) = T_g\phi \odot \psi$.

**Proof.** Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ be arbitrary. Now

$$T_g(\phi \ast \psi)(b,a) = \int_0^{2\pi} (\phi \ast \psi)(x) g_a(x-b) dx$$

$$= \int_0^{2\pi} g_a(x-b) dx \int_0^{2\pi} \phi(x-t) \psi(t) dt. \quad (4.2)$$

By an easy verification, we can apply Fubini’s theorem and the last integral equals

$$\int_0^{2\pi} \psi(t) dt \int_0^{2\pi} \phi(x-t) g_a(x-b) dx$$

$$= \int_0^{2\pi} \psi(t) dt \int_0^{2\pi} \phi(x) g_a(x-(b-t)) dx$$

$$= (T_g\phi \odot \psi)(b,a). \quad (4.3)$$

**Proof of Theorem 4.2.** First, we show that $((T_g\phi_n), (\delta_n))$ is a quotient. Since $[\phi_n/\delta_n] \in \mathcal{B}_T$, we have

$$\phi_k \ast \delta_j = \phi_j \ast \delta_k, \quad \forall j, k \in \mathbb{N}. \quad (4.4)$$

Applying the classical wavelet transform $T_g$ on both sides, we get

$$T_g\phi_k \odot \delta_j = T_g\phi_j \odot \phi_k, \quad \forall j, k \in \mathbb{N} \text{ (by Lemma 4.3).} \quad (4.5)$$

Next, we show that the definition of $\mathcal{T}_g$ is independent of the choice of the representative.
Let \( [\phi_k/\epsilon_k] = [\psi_k/\delta_k] \) in \( \mathcal{B}_T \). Then, we have
\[
\phi_k \ast \epsilon_j = \psi_j \ast \delta_k, \quad \forall j, k \in \mathbb{N}.
\] (4.6)

Again, applying the wavelet transform and using Lemma 4.3, we get
\[
\mathcal{I}_g \phi_k \circ \epsilon_j = \mathcal{I}_g \psi_j \circ \delta_k, \quad \forall j, k \in \mathbb{N}.
\] (4.7)

Hence, the theorem follows. \( \Box \)

**Theorem 4.4** (consistency). Let \( \mathcal{I}_T : C^\infty(\mathbb{T}) \to \mathcal{B}_T \) and \( \mathcal{I}_Y : \mathcal{I}(\mathbb{Y}) \to \mathcal{B}_Y \) be the canonical identification defined, respectively, by
\[
\phi \mapsto \left[ \frac{\phi \ast \delta_n}{\delta_n} \right], \quad \eta \mapsto \left[ \frac{\eta \circ \delta_n}{\delta_n} \right],
\] (4.8)
where \( (\delta_n) \in \Delta \), then \( \mathcal{I}_g \circ \mathcal{I}_T = \mathcal{I}_Y \circ T_g \).

**Proof.** Let \( \phi \in C^\infty(\mathbb{T}) \), then
\[
\mathcal{I}_g (\mathcal{I}_T (\phi)) = \mathcal{I}_g \left( \left[ \frac{\phi \ast \delta_n}{\delta_n} \right] \right) = \left[ \frac{T_g (\phi \ast \delta_n)}{\delta_n} \right] = \left[ \frac{T_g \phi \circ \delta_n}{\delta_n} \right] \quad \text{(by Lemma 4.3)}
\] (4.9)
\[
= \mathcal{I}_Y (T_g (\phi)). \quad \Box
\]

**Theorem 4.5.** The wavelet transform \( \mathcal{I}_g : \mathcal{B}_T \to \mathcal{B}_Y \) is a linear map.

**Proof.** If \( [\phi_n/\delta_n], [\psi_n/\epsilon_n] \in \mathcal{B}_T \), then
\[
\mathcal{I}_g \left( \left[ \frac{\phi_n}{\delta_n} \right] + \left[ \frac{\psi_n}{\epsilon_n} \right] \right) = \mathcal{I}_g \left( \left[ \frac{\phi_n \ast \epsilon_n + \psi_n \ast \delta_n}{\delta_n \ast \epsilon_n} \right] \right) = \left[ \frac{T_g (\phi_n \ast \epsilon_n + \psi_n \ast \delta_n)}{\delta_n \ast \epsilon_n} \right]
\]
\[
= \left[ \frac{T_g \phi_n \circ \epsilon_n + T_g \psi_n \circ \delta_n}{\delta_n \ast \epsilon_n} \right] = \left[ \frac{T_g \phi_n}{\delta_n} \right] + \left[ \frac{T_g \psi_n}{\epsilon_n} \right]
\]
\[
= \mathcal{I}_g \left( \left[ \frac{\phi_n}{\delta_n} \right] \right) + \mathcal{I}_g \left( \left[ \frac{\psi_n}{\epsilon_n} \right] \right).
\] (4.10)

If \( \alpha \in \mathbb{C} \) and \( [\phi_n/\delta_n] \in \mathcal{B}_T \), then
\[
\mathcal{I}_g \left( \alpha \left[ \frac{\phi_n}{\delta_n} \right] \right) = \mathcal{I}_g \left( \left[ \frac{\alpha \phi_n}{\delta_n} \right] \right) = \left[ \frac{T_g (\alpha \phi_n)}{\delta_n} \right] = \left[ \frac{\alpha T_g \phi_n}{\delta_n} \right]
\]
\[
= \alpha \left[ \frac{T_g \phi_n}{\delta_n} \right] = \alpha \mathcal{I}_g \left( \left[ \frac{\phi_n}{\delta_n} \right] \right). \quad \Box
\] (4.11)

In the above proof, we have used the fact that \( T_g \) is linear wherever it is required.
From the following two theorems, we say that the generalized wavelet transform is continuous with respect to $\delta$-convergence as well as $\Delta$-convergence.

**Theorem 4.6.** If $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$, then $\mathcal{T}_g x_n \xrightarrow{\delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$.

**Proof.** If $x_n \xrightarrow{\delta} x$ as $n \to \infty$, then, by Theorem 3.2, there exist $\phi_{n,k}, \phi_k \in C^\infty(\mathbb{T})$ and $(\delta_k) \in \Delta$ such that $x_n = [\phi_{n,k}/\delta_k]$ and $x = [\phi_k/\delta_k]$ and, for each $k \in \mathbb{N}$, $\phi_{n,k} \to \phi_k$ as $n \to \infty$ in $C^\infty(\mathbb{T})$.

By the continuity of the classical wavelet transform, we have, for each $k \in \mathbb{N}$,

$$T_g \phi_{n,k} \to T_g \phi_k \text{ as } n \to \infty \text{ in } \mathcal{F}_Y.$$  \hfill (4.12)

Since $\mathcal{T}_g (x_n) = [T_g \phi_{n,k}/\delta_k]$ and $\mathcal{T}_g (x) = [T_g \phi_k/\delta_k]$, we get $\mathcal{T}_g (x_n) \xrightarrow{\delta} \mathcal{T}_g (x)$ as $n \to \infty$. Hence, the theorem follows. \hfill \Box

**Theorem 4.7.** If $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$, then $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$.

**Proof.** Let $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_Y$. Then, by definition, we can find $\phi_n \in C^\infty(\mathbb{T})$ and $(\delta_n) \in \Delta$ such that $(x_n - x) \ast \delta_n = [\phi_n \ast \delta_k/\delta_k]$ and

$$\phi_n \to 0 \text{ as } n \to 0 \text{ in } C^\infty(\mathbb{T}).$$ \hfill (4.13)

Applying the classical wavelet transform and using Lemma 4.3, we get

$$T_g \phi_n \to 0 \text{ as } n \to 0 \text{ in } \mathcal{F}(\mathcal{Y}).$$ \hfill (4.14)

Hence, we get $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_Y$. \hfill \Box

**Lemma 4.8.** If $\eta \in \mathcal{F}(\mathcal{Y})$ and $\phi \in C^\infty(\mathbb{T})$, then $R_g (\eta \ast \phi) = R_g \eta \ast \phi$.

**Proof.** Using Fubini’s theorem, we get

$$R_g (\eta \ast \phi)(x) = \int_0^{2\pi} \int_0^\infty g_a(x-b)(\eta \ast \phi)(b,a) \frac{dadb}{a}$$

$$= \int_0^{2\pi} \phi(t) dt \int_0^\infty g_a(x-b) \eta(b-t,a) \frac{dadb}{a}$$

$$= \int_0^{2\pi} \phi(t) dt \int_0^\infty g_a((x-t) - c) \eta(c,a) \frac{dadc}{a} \quad (b-t = c)$$

$$= \int_0^{2\pi} R_g \eta(x-t) \phi(t) dt$$

$$= (R_g \eta \ast \phi)(x).$$ \hfill (4.15)
Therefore, we can give the following definition.

**Definition 4.9.** Define $R_g : B_Y → B_T$ by

$$R_g \left( \begin{bmatrix} \eta_n \\ \delta_n \end{bmatrix} \right) = \begin{bmatrix} R_g \eta_n \\ \delta_n \end{bmatrix}.$$  \hfill (4.16)

**Theorem 4.10.** The map $R_g : B_Y → B_T$ is linear.

**Theorem 4.11.** The map $R_g : B_Y → B_T$ is continuous with respect to $\delta$-convergence as well as $\Delta$-convergence.

Using Lemma 4.8 and Theorem 2.4, we get a proof similar to that of Theorems 4.6 and 4.7.

**Theorem 4.12** (an inversion formula). If $x = [\phi_n / \delta_n] ∈ B_T$ such that $\phi_n ∈ C^∞_{+,(-)}(T)$ for all $n ∈ N$, then

$$R_g \circ T_g(x) = C_g^{+,(-)} x.$$ \hfill (4.17)

**Proof.** Now,

$$R_g \circ T_g(x) = R_g \left( \begin{bmatrix} T_g \phi_n \\ \delta_n \end{bmatrix} \right) = \begin{bmatrix} (R_g \circ T_g) \phi_n \\ \delta_n \end{bmatrix}$$

$$= \begin{bmatrix} C_g^{+,(-)} \phi_n \\ \delta_n \end{bmatrix} = C_g^{+,(-)} \begin{bmatrix} \phi_n \\ \delta_n \end{bmatrix} = C_g^{+,(-)} x.$$ \hfill (4.18)

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**References**


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