

ON NILPOTENT FILIFORM LIE ALGEBRAS OF DIMENSION EIGHT

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The aim of this paper is to determine both the Zariski constructible set of characteristically nilpotent filiform Lie algebras \mathfrak{g} of dimension 8 and that of the set of nilpotent filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms, in the variety of filiform Lie algebras of dimension 8 over C .

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1. Introduction. Characteristically nilpotent Lie algebras were defined by Dixmier and Lister in [4] and filiform Lie algebras by Vergne in [8]. A complete classification of nilpotent filiform Lie algebras of dimension 8 is available since 1988 in [1] due to Ancochéa-Bermúdez and Goze. Then, Echarte-Reula et al. [6], considering that a filiform Lie algebra \mathfrak{g} is characteristically nilpotent if and only if \mathfrak{g} is not a derived algebra, obtained a list of characteristically nilpotent filiform Lie algebras of dimension 8. Recently, Castro-Jiménez and Núñez-Valdés studied extensively in [2, 3] the cases of dimension 9 and 10 and gave the sets of the corresponding characteristically nilpotent Lie algebras as a finite union of Zariski locally closed subsets. In 1970, Dyer in [5] gave an example of a characteristically nilpotent Lie algebra of dimension 9 and showed that each automorphism of this Lie algebra is unipotent. Some years later, Favre in [7] reached the same result working on an example of a characteristically nilpotent Lie algebra of dimension 7.

In this paper, we study the Lie algebras of dimension 8. We first express the set of characteristically nilpotent filiform Lie algebras \mathfrak{g} as a finite union of locally closed subsets, then we prove that the set of nilpotent filiform Lie algebras \mathfrak{g} , whose group of automorphisms consists of unipotent automorphisms, is a Zariski constructible set in the variety of nilpotent filiform Lie algebras, and we express it as a finite union of locally closed subsets. Furthermore, we prove that the group of automorphisms $\text{Aut}(\mathfrak{g})$ of each one of the above characteristically nilpotent filiform Lie algebras consists of unipotent automorphisms except two, in each of which the set of their unipotent automorphisms forms a proper subgroup of the group $\text{Aut}(\mathfrak{g})$.

2. Preliminaries. Let \mathfrak{g} be a Lie algebra of dimension n over C of characteristic zero. If we consider the descending central sequence $C^1\mathfrak{g} = \mathfrak{g}$,

$C^2g = [g, g], \dots$, and $C^qg = [g, C^{q-1}g], \dots$, of the above Lie algebra, then the Lie algebra g is called filiform if $\dim_C C^qg = n - q$ for $2 \leq q \leq n$ [8].

Let g be a filiform Lie algebra of dimension n . Then, there exists a basis $E = \{e_1, e_2, \dots, e_n\}$ of g such that $e_1 \in g \setminus C^2g$, the matrix of $\text{ad}(e_1)$ with respect to E has a Jordan block of order $n - 1$, and $C^i g$ is the vector space generated by $\{e_2, e_3, \dots, e_{n-i+1}\}$ with $2 \leq i \leq n - 1$. Such a basis is called an adapted basis.

Let W be a vector space over a field C of dimension n . A subset A of W is an algebraic one if there exists a set B of polynomial functions on W such that $A = \{x \in W / p(x) = 0, \text{ for all } p \in B\}$. We consider the set $C[x]$ of all polynomials in n variables $x = \{x_1, \dots, x_n\}$ over C and I an ideal of $C[x]$. We denote by $\mathcal{V}(I)$ the set $\mathcal{V}(I) = \{a \in C^n / p(a) = 0, \text{ for all } p \in I\}$. As a consequence of the above definitions, $\mathcal{V}(I)$ is an algebraic subset of the vector space C^n and so the Zariski topology on the space C^n is the one whose closed sets are $\mathcal{V}(I)$. Finally, we denote by $D(I)$ the complement of $\mathcal{V}(I)$ in C^n .

3. The equations. It has been proved in [1] that there exists a basis $\{e_1, e_2, \dots, e_8\}$ such that every nilpotent filiform Lie algebra g over a field C of characteristic zero of dimension 8 is isomorphic to one of the Lie algebras belonging to the nine-parameter family given in [1].

We now consider a change of the previous base of the nilpotent filiform Lie algebra g such that $Y_i = e_i, i = 1, 2, \dots, 7$, and $Y_8 = e_8 + a_1 e_1$. So, the set of nilpotent filiform Lie algebras over C can be parametrized by the points (a_2, a_3, \dots, a_9) of the algebraic set $V' \in C^8$, and the above-mentioned equations of the nine-parameter family, with respect to the new base $A = \{e_1, e_2, \dots, e_8\}$, takes the form

$$\begin{aligned}
 [e_1, e_i] &= e_{i-1}, \quad i \geq 3, \\
 [e_4, e_7] &= a_2 e_2, \\
 [e_4, e_8] &= a_2 e_3 + a_3 e_2, \\
 [e_5, e_6] &= a_4 e_2, \\
 [e_5, e_7] &= (a_2 + a_4) e_3 + a_5 e_2, \\
 [e_5, e_8] &= (2a_2 + a_4) e_4 + (a_3 + a_5) e_3 + a_6 e_2, \\
 [e_6, e_7] &= (a_2 + a_4) e_4 + a_5 e_3 + a_7 e_2, \\
 [e_6, e_8] &= (3a_2 + 2a_4) e_5 + (a_3 + 2a_5) e_4 + (a_6 + a_7) e_3 + a_8 e_2, \\
 [e_7, e_8] &= (3a_2 + 2a_4) e_6 + (a_3 + 2a_5) e_5 + (a_6 + a_7) e_4 + a_8 e_3 + a_9 e_2,
 \end{aligned}
 \tag{3.1}$$

with $a_j \in C, j = 2, 3, \dots, 9$, verifying the equations

$$\begin{aligned}
 a_2 + a_4 &= 0, \\
 a_2(5a_5 + 2a_3) &= 0.
 \end{aligned}
 \tag{3.2}$$

Those two equations are consequences of the Jacobi's identities.

4. Characteristically nilpotent filiform Lie algebras. Let \mathfrak{g} be a nilpotent Lie algebra of dimension n over C of characteristic zero. A Lie algebra \mathfrak{g} is said to be characteristically nilpotent if the Lie algebra of its derivations D is nilpotent. By $D : \mathfrak{g} \rightarrow \mathfrak{g}$ verifying $D[x, y] = [Dx, y] + [x, Dy]$ for all $(x, y) \in \mathfrak{g}$, we mean a derivation of \mathfrak{g} .

Let $D = (d_{ij}) \in \text{Mat}(8 \times 8, C)$ be the set of matrices representing the derivations D of the filiform Lie algebras over C of dimension 8 with respect to the new base $A = \{e_1, e_2, \dots, e_8\}$.

Suppose that

$$\begin{aligned} De_k &= \sum d_{k\lambda} e_\lambda, \quad 1 \leq k, \lambda \leq 8, \quad d_{k\lambda} \in C, \\ D[e_i, e_j] - De_k &= 0, \quad 1 \leq i < j \leq 8, \quad 1 \leq k \leq 8. \end{aligned} \tag{4.1}$$

From

$$D[e_1, e_2] = 0, \quad D[e_1, e_i] = De_{i-1}, \quad i \geq 3, \tag{4.2}$$

we deduce that

$$d_{ij} = 0, \quad 2 \leq i \leq 7, \quad i < j, \quad 3 \leq j \leq 8, \quad d_{i1} = 0, \quad 2 \leq i \leq 8. \tag{4.3}$$

For each $(i, j, k), 1 \leq i < j \leq 8, 1 \leq k \leq 8$, we denote by $b(i, j, k)$ the coefficient of e_k in the expression $D[e_i, e_j] - [De_i, e_j] - [e_i, De_j]$ with respect to the base A . From above, we obtain a homogeneous linear system defined by

$$S = \{b(i, j, k) = 0, \quad 1 \leq i < j \leq 8, \quad 1 \leq k \leq 8\}. \tag{4.4}$$

The solutions satisfying system (4.4) are elements of the set of matrices $D = (d_{ij}) \in \text{Mat}(8 \times 8, C)$.

In case that D are nilpotent matrices, according to the previous definition, the filiform Lie algebra \mathfrak{g} is characteristically nilpotent.

4.1. The system of equations. Let $t = (a_2, a_3, a_5, a_6, a_7, a_8, a_9)$ be a point of $V \in C^7$, \mathfrak{g}_t the corresponding filiform Lie algebra of dimension 8, and S_t the homogeneous linear system corresponding to (4.4). We consider the linear system S_t as a system with coefficients in the quotient ring R/I where $R = C[a_2, a_3, a_5, a_6, a_7, a_8, a_9]$ and I is the ideal generated by (3.2). In that case,

system S in (4.4) is reduced to the following equivalent system S_I :

$$\begin{aligned}
d_{11} - d_{22} + d_{33} &= 0 \\
d_{11} - d_{77} + d_{88} &= 0 \\
a_2 d_{17} - d_{76} + d_{87} &= 0 \\
a_2 d_{22} - a_2 d_{44} - a_2 d_{77} &= 0 \\
a_2 d_{22} - a_2 d_{55} - a_2 d_{66} &= 0 \\
a_2 d_{33} - a_2 d_{44} - a_2 d_{88} &= 0 \\
a_5 d_{33} - a_5 d_{66} - a_5 d_{77} &= 0 \\
a_2 d_{44} - a_2 d_{55} - a_2 d_{88} &= 0 \\
a_2 d_{55} - a_2 d_{66} - a_2 d_{88} &= 0 \\
a_2 d_{66} - a_2 d_{77} - a_2 d_{88} &= 0 \\
d_{11} - a_2 d_{18} - d_{33} + d_{44} &= 0 \\
d_{11} - a_2 d_{18} - d_{44} + d_{55} &= 0 \\
d_{11} - a_2 d_{18} - d_{55} + d_{66} &= 0 \\
d_{11} - a_2 d_{18} - d_{66} + d_{77} &= 0 \\
(a_3 + a_5) d_{18} + d_{43} - d_{54} &= 0 \\
(a_6 + a_7) d_{18} + d_{64} - d_{75} &= 0 \\
(a_3 + 2a_5) d_{18} + d_{54} - d_{65} &= 0 \\
(a_3 + 2a_5) d_{18} + d_{65} - d_{76} &= 0 \\
a_5 d_{16} - a_8 d_{18} - d_{63} + d_{74} &= 0 \\
a_2 d_{17} + a_3 d_{18} + d_{32} - d_{43} &= 0 \\
a_5 d_{17} + (a_6 + a_7) d_{18} + d_{53} - d_{64} &= 0 \\
a_2 d_{15} + a_7 d_{17} + a_8 d_{18} + d_{52} - d_{63} &= 0 \\
a_2 d_{16} - a_5 d_{17} - a_6 d_{18} - d_{42} + d_{53} &= 0 \\
a_2 d_{16} + (a_3 + 2a_5) d_{17} - d_{75} + d_{86} &= 0 \\
a_3 d_{22} + a_2 d_{32} - a_3 d_{44} - a_2 d_{87} - a_3 d_{88} &= 0 \\
a_5 d_{22} - a_2 d_{54} + a_5 d_{55} + a_2 d_{76} + a_5 d_{77} &= 0 \\
a_2 d_{14} + a_5 d_{15} + a_7 d_{16} - a_9 d_{18} - d_{62} + d_{73} &= 0 \\
a_3 d_{14} + a_6 d_{15} + a_8 d_{16} + a_9 d_{17} - d_{72} + d_{83} &= 0 \\
(a_3 + a_5) d_{15} + (a_6 + a_7) d_{16} + a_8 d_{17} - d_{73} + d_{84} &= 0 \\
a_2 d_{15} + (a_3 + 2a_5) d_{16} + (a_6 + a_7) d_{17} - d_{74} + d_{85} &= 0
\end{aligned}$$

$$\begin{aligned}
& a_7d_{22} + a_5d_{32} - a_2d_{64} - a_5d_{65} - a_7d_{66} - a_2d_{75} - a_7d_{77} = 0 \\
& (a_3 + a_5)d_{33} + a_2d_{43} - a_2d_{54} - (a_3 + a_5)d_{55} - (a_3 + a_5)d_{88} = 0 \\
& (a_3 + 2a_5)d_{44} + a_2d_{54} - a_2d_{65} - (a_3 + 2a_5)d_{66} - (a_3 + 2a_5)d_{88} = 0 \\
& (a_3 + 2a_5)d_{55} + a_2d_{65} - a_2d_{76} - (a_3 + 2a_5)d_{77} - (a_3 + 2a_5)d_{88} = 0 \\
& a_6d_{22} + (a_3 + a_5)d_{32} + a_2d_{42} - a_3d_{54} - a_6d_{55} + a_2d_{86} - a_5d_{87} - a_6d_{88} = 0 \\
& (a_6 + a_7)d_{33} + (a_3 + 2a_5)d_{43} + a_2d_{53} - a_2d_{64} - (a_3 + a_5)d_{65} \\
& \quad - (a_6 + a_7)d_{66} - a_5d_{87} - (a_6 + a_7)d_{88} = 0 \\
& (a_6 + a_7)d_{44} + (a_3 + 2a_5)d_{54} + a_2d_{64} - a_2d_{75} - (a_3 + 2a_5)d_{76} \\
& \quad - (a_6 + a_7)d_{77} - (a_6 + a_7)d_{88} = 0 \\
& a_8d_{22} + (a_6 + a_7)d_{32} + (a_3 + 2a_5)d_{42} + a_2d_{52} - a_3d_{64} - a_6d_{65} \\
& \quad - a_8d_{66} - a_2d_{85} - a_7d_{87} - a_8d_{88} = 0 \\
& a_8d_{33} + (a_6 + a_7)d_{43} + (a_3 + 2a_5)d_{53} + a_2d_{63} - a_2d_{74} - (a_3 + a_5)d_{75} \\
& \quad - (a_6 + a_7)d_{76} - a_8d_{77} + a_5d_{86} - a_8d_{88} = 0 \\
& a_9d_{22} + a_8d_{32} + (a_6 + a_7)d_{42} + (a_3 + 2a_5)d_{52} + a_2d_{62} - a_3d_{74} - a_6d_{75} \\
& \quad - a_8d_{76} - a_9d_{77} + a_2d_{84} + a_5d_{85} + a_7d_{86} - a_9d_{88} = 0.
\end{aligned} \tag{4.5}$$

The solutions satisfying S_t are derivations of the nilpotent filiform Lie algebra \mathfrak{g}_t . If all the derivations of \mathfrak{g}_t are nilpotent, then \mathfrak{g}_t is characteristically nilpotent.

We will prove that the set of points $t \in V \subset C^7$, such that there exists a solution of S_t satisfying the conditions of \mathfrak{g}_t being a characteristically nilpotent filiform Lie algebra, is a Zariski constructible set, and we will express it as a finite union of Zariski locally closed subsets. To realize the above idea, we study S_t in suitable subsets of V .

4.2. Main results. We consider two cases: first, $a_2 \neq 0$ and then, $a_2 = 0$.

4.2.1. $a_2 \neq 0$. Let the open set $V \cap D(a_2)$. Because of the equation $a_2(5a_5 + 2a_3) = 0$, we can distinguish the following two subcases.

(1) ($a_3 \neq 0$). First, we consider the set $T^{(1)} = V \cap D(a_2 \cdot a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = -(2/5)a_3$. By doing the necessary calculations in system S_t , we prove that, in the set of points $T^{(1)} \cap D(Q_1)$ with $Q_1 = 2a_3^2 - 25a_2a_6 - 25a_2a_7$, the corresponding Lie algebra is characteristically nilpotent.

(2) ($a_3 = 0$). Now, we consider the set $T^{(2)} = V \cap D(a_2) \cap \mathcal{V}(a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = 0$. In case that $a_6 + a_7 \neq 0$ and $a_8 \neq 0$, that means in $T^{(2)} \cap D((a_6 + a_7) \cdot a_8)$, only one Lie algebra is characteristically nilpotent.

From the above, we can state the following theorem.

THEOREM 4.1. *Consider the set of complex filiform Lie algebras. Consider C^8 with (a_2, a_3, \dots, a_9) as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). In the Zariski open set $V \cap D(a_2)$, the Zariski constructible subset of characteristically nilpotent Lie algebras is defined as the union of the following subsets:*

$$D(a_3 \cdot (2a_3^2 - 25a_2a_6 - 25a_2a_7)), \tag{4.6}$$

$$\mathcal{V}(a_3) \cap D((a_6 + a_7) \cdot a_8).$$

4.2.2. $a_2 = 0$. We consider the set $T^{(3)} = V \cap \mathcal{V}(a_2)$. Because of the equation $a_2(5a_5 + 2a_3) = 0$, we can distinguish the following subcases.

(1) ($a_5 \neq 0$). So, we obtain the set $T^{(3)} \cap D(a_5)$ and we distinguish the following:

(1A) ($a_3 + 2a_5 \neq 0$). In the subset $T^{(3)} \cap D(a_5 \cdot (a_3 + 2a_5) \cdot Q_2)$ with $Q_2 = 2a_3^2a_7 - 3a_3a_5a_6 + 5a_3a_5a_7 - 3a_5^2a_6 + 5a_5^2a_7$, the corresponding Lie algebra is characteristically nilpotent,

(1B) ($a_3 + 2a_5 = 0$). The corresponding Lie algebra in the set of points $T^{(3)} \cap D(a_5 \cdot (a_6 + a_7)) \cap \mathcal{V}(a_3 + 2a_5)$ is characteristically nilpotent.

(2) ($a_5 = 0$). First, we distinguish two subcases $a_3 \neq 0$ and $a_3 = 0$.

(2A) ($a_3 \neq 0$). Then, we consider the set $T^{(3)} \cap \mathcal{V}(a_5) \cap D(a_3)$. By doing some calculations, we distinguish two more subcases.

(i) ($a_7 \neq 0$). In this case, the Lie algebra corresponding to the set of points $T^{(3)} \cap \mathcal{V}(a_5) \cap D(a_3 \cdot a_7)$ is characteristically nilpotent.

(ii) ($a_7 = 0$). Now, we study S_t in the set of points $Z = T^{(3)} \cap \mathcal{V}(a_5, a_7) \cap D(a_3)$. The Lie algebras corresponding to the set of points $Z \cap (D(Q_{31}) \cup D(Q_{32}))$, with $Q_{31} = 4a_3a_8 - 5a_6^2$ and $Q_{32} = 2a_3^2a_9 - 2a_3a_6a_8 - a_6^3$, are characteristically nilpotent.

(2B) ($a_3 = 0$). We operate in $T^{(3)} \cap \mathcal{V}(a_3, a_5)$ and we distinguish the cases $a_7 \neq 0$ and $a_7 = 0$.

(i) ($a_7 \neq 0$). The Lie algebra corresponding to the set of points $T^{(3)} \cap \mathcal{V}(a_3, a_5) \cap D(a_7 \cdot a_8)$ is characteristically nilpotent.

(ii) ($a_7 = 0$). We now consider the subset $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7)$. We distinguish another two subcases, $a_6 \neq 0$ and $a_6 = 0$.

(iiA) ($a_6 \neq 0$). The Lie algebra corresponding to $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7) \cap D(a_6 \cdot a_8)$ is characteristically nilpotent.

(iiB) ($a_6 = 0$). The Lie algebra in the set $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7) \cap D(a_8 \cdot a_9)$ is characteristically nilpotent.

So, we have proved the following theorem.

THEOREM 4.2. *Consider the set of complex filiform Lie algebras. Consider C^8 with (a_2, a_3, \dots, a_9) as coordinates given by (3.1), and let V be the hypersurface defined in C^7 by (3.2). The Zariski constructible subset of characteristically nilpotent Lie algebras in the Zariski closed set $V \cap \mathcal{V}(a_2)$ is defined as the union*

of the following subsets:

$$\left\{ \begin{array}{l} D(a_5) \cap \left\{ \begin{array}{l} D((a_3 + 2a_5) \cdot (2a_3^2 a_7 - 3a_3 a_5 a_6 + 5a_3 a_5 a_7 - 3a_5^2 a_6 + 5a_5^2 a_7)), \\ \mathcal{V}(a_3 + 2a_5) \cap D(a_6 + a_7), \end{array} \right. \\ \\ \mathcal{V}(a_5) \cap \left\{ \begin{array}{l} D(a_3) \cap \left\{ \begin{array}{l} D(a_7), \\ \mathcal{V}(a_7) \cap (D(4a_3 a_8 - 5a_6^2) \cup D(2a_3^2 a_9 - 2a_3 a_6 a_8 - a_6^3)), \end{array} \right. \\ \\ \mathcal{V}(a_3) \cap \left\{ \begin{array}{l} D(a_7 \cdot a_8), \\ \mathcal{V}(a_7) \cap \left\{ \begin{array}{l} D(a_6 \cdot a_8), \\ \mathcal{V}(a_6) \cap D(a_8 \cdot a_9). \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \quad (4.7)$$

By $\{\}$, we mean the union of the corresponding sets.

5. Unipotent automorphisms of nilpotent filiform Lie algebras. Let g be a nilpotent Lie algebra of dimension n over C of characteristic zero. The automorphism θ of a Lie algebra g over C is defined by the mapping $[x, y] \rightarrow \theta([x, y]) = [\theta(x), \theta(y)]$ for all $(x, y) \in g$. An automorphism θ is called unipotent if its representation, with respect to the base $\{e_1, e_2, \dots, e_n\}$, has the form

$$B = (b_{ij}) \in \text{Mat}(n \times n, C), \quad b_{ij} = 0, \quad j < i, \quad b_{ii} = 1, \quad 1 \leq i, j \leq n. \quad (5.1)$$

Let $B = (b_{ij}) \in \text{Mat}(8 \times 8, C)$ be the set of matrices representing the automorphisms θ of the filiform Lie algebras over C of dimension 8 with respect to the new base $A = \{e_1, e_2, \dots, e_8\}$.

Suppose that

$$\begin{aligned} \theta(e_k) &= \sum b_{k\lambda} e_\lambda, \quad 1 \leq k, \lambda \leq 8, \quad b_{k\lambda} \in C, \\ \theta([e_i, e_j]) - \theta(e_k) &= 0, \quad 1 \leq i < j \leq 8, \quad 1 \leq k \leq 8. \end{aligned} \quad (5.2)$$

From

$$\theta([e_1, e_2]) = 0, \quad \theta([e_1, e_i]) = \theta(e_{i-1}), \quad i \geq 3, \quad \theta([e_3, e_8]) = 0, \quad (5.3)$$

we deduce that

$$b_{ij} = 0, \quad 3 \leq i \leq 8, \quad j < i, \quad 2 \leq j \leq 7, \quad b_{1j} = 0, \quad 2 \leq j \leq 8. \quad (5.4)$$

For each $(i, j, k), 1 \leq i < j \leq 8, 1 \leq k \leq 8$, we denote by $c(i, j, k)$ the coefficient of e_k in the expression $\theta([e_i, e_j]) - [\theta(e_i), \theta(e_j)]$ with respect to the base A .

From the above, we obtain a homogeneous system defined by

$$S' = \{c(i, j, k) = 0, \quad 1 \leq i < j \leq 8, \quad 1 \leq k \leq 8\}. \quad (5.5)$$

The solutions satisfying system (5.5) are elements of the set of matrices $B = (b_{ij}) \in \text{Mat}(8 \times 8, C)$.

In case that $B = (b_{ij}) \in \text{Mat}(8 \times 8, C)$ are matrices of the form (5.1), according to the above definition, the group of automorphisms $\text{Aut}(\mathfrak{g})$ of the corresponding filiform Lie algebra \mathfrak{g} consists of unipotent automorphisms.

5.1. The system of equations. Let $t = (a_2, a_3, a_5, a_6, a_7, a_8, a_9)$ be a point of $V \in C^7$, \mathfrak{g}_t the corresponding filiform Lie algebra of dimension 8, and S'_t the homogeneous system corresponding to (5.5). We will consider the linear system S'_t as a system with coefficients in the quotient ring R/I where $R = C[a_2, a_3, a_5, a_6, a_7, a_8, a_9]$ and I is the ideal generated by (3.2). In that case, system S' in (5.5) is reduced to the following equivalent system S'_t :

$$\begin{aligned}
b_{22} - b_{11}b_{33} &= 0 \\
b_{77} - b_{11}b_{88} &= 0 \\
a_2b_{22} - a_2b_{44}b_{77} &= 0 \\
a_2b_{22} - a_2b_{55}b_{66} &= 0 \\
a_2b_{33} - a_2b_{44}b_{88} &= 0 \\
a_2b_{44} - a_2b_{55}b_{88} &= 0 \\
a_2b_{55} - a_2b_{66}b_{88} &= 0 \\
a_2b_{66} - a_2b_{77}b_{88} &= 0 \\
a_5b_{33} - a_5b_{66}b_{77} &= 0 \\
b_{33} - b_{11}b_{44} + a_2b_{44}b_{81} &= 0 \\
b_{44} - b_{11}b_{55} + a_2b_{55}b_{81} &= 0 \\
b_{55} - b_{11}b_{66} + a_2b_{66}b_{81} &= 0 \\
b_{66} - b_{11}b_{77} + a_2b_{77}b_{81} &= 0 \\
b_{23} - b_{11}b_{34} + a_2b_{44}b_{71} + a_3b_{44}b_{81} &= 0 \\
b_{67} - b_{11}b_{78} - a_2b_{71}b_{88} + a_2b_{78}b_{81} &= 0 \\
a_3b_{22} + a_2b_{23} - a_2b_{44}b_{78} - a_3b_{44}b_{88} &= 0 \\
a_5b_{22} - a_2b_{45}b_{77} + a_2b_{55}b_{67} - a_5b_{55}b_{77} &= 0 \\
b_{34} - b_{11}b_{45} + a_2b_{45}b_{81} + (a_3 + a_5)b_{55}b_{81} &= 0 \\
b_{45} - b_{11}b_{56} + a_2b_{56}b_{81} + (a_3 + 2a_5)b_{66}b_{81} &= 0 \\
b_{56} - b_{11}b_{67} + a_2b_{67}b_{81} + (a_3 + 2a_5)b_{77}b_{81} &= 0 \\
(a_3 + a_5)b_{33} + a_2b_{34} - a_2b_{45}b_{88} - (a_3 + a_5)b_{55}b_{88} &= 0 \\
(a_3 + 2a_5)b_{44} + a_2b_{45} - a_2b_{56}b_{88} - (a_3 + 2a_5)b_{66}b_{88} &= 0 \\
(a_3 + 2a_5)b_{55} + a_2b_{56} - a_2b_{67}b_{88} - (a_3 + 2a_5)b_{77}b_{88} &= 0 \\
b_{46} - b_{11}b_{57} + a_2b_{57}b_{81} + (a_3 + 2a_5)b_{67}b_{81} + (a_6 + a_7)b_{77}b_{81} &= 0 \\
b_{24} - b_{11}b_{35} + a_2b_{45}b_{71} + a_3b_{45}b_{81} - a_2b_{55}b_{61} + a_5b_{55}b_{71} + a_6b_{55}b_{81} &= 0
\end{aligned}$$

$$\begin{aligned}
& a_7b_{22} + a_5b_{23} - a_2b_{46}b_{77} + a_2b_{56}b_{67} - a_5b_{56}b_{77} - a_2b_{57}b_{66} - a_7b_{66}b_{77} = 0 \\
& b_{35} - b_{11}b_{46} + a_2b_{46}b_{81} + (a_3 + a_5)b_{56}b_{81} + a_5b_{66}b_{71} + (a_6 + a_7)b_{66}b_{81} = 0 \\
& b_{57} - b_{11}b_{68} - a_2b_{61}b_{88} + a_2b_{68}b_{81} - (a_3 + 2a_5)b_{71}b_{88} + (a_3 + 2a_5)b_{78}b_{81} = 0 \\
& (a_6 + a_7)b_{44} + (a_3 + 2a_5)b_{45} + a_2b_{46} - a_2b_{57}b_{88} - (a_3 + 2a_5)b_{67}b_{88} \\
& \quad - (a_6 + a_7)b_{77}b_{88} = 0 \\
& b_{36} - b_{11}b_{47} + a_2b_{47}b_{81} + (a_3 + a_5)b_{57}b_{81} - a_5b_{61}b_{77} + a_5b_{67}b_{71} \\
& \quad + (a_6 + a_7)b_{67}b_{81} + a_8b_{77}b_{81} = 0 \\
& a_6b_{22} + (a_3 + 2a_5)b_{23} + a_2b_{24} - a_2b_{45}b_{78} - a_3b_{45}b_{88} + a_2b_{55}b_{68} \\
& \quad - a_5b_{55}b_{78} - a_6b_{55}b_{88} = 0 \\
& (a_6 + a_7)b_{33} + (a_3 + 2a_5)b_{34} + a_2b_{35} - a_2b_{46}b_{88} - (a_3 + a_5)b_{56}b_{88} \\
& \quad - a_5b_{66}b_{78} - (a_6 + a_7)b_{66}b_{88} = 0 \\
& b_{25} - b_{11}b_{36} + a_2b_{46}b_{71} + a_3b_{46}b_{81} + a_2b_{51}b_{66} - a_2b_{56}b_{61} + a_5b_{56}b_{71} \\
& \quad + a_6b_{56}b_{81} + a_7b_{66}b_{71} + a_8b_{66}b_{81} = 0 \\
& b_{47} - b_{11}b_{58} - a_2b_{51}b_{88} + a_2b_{58}b_{81} - (a_3 + 2a_5)b_{61}b_{88} + (a_3 + 2a_5)b_{68}b_{81} \\
& \quad - (a_6 + a_7)b_{71}b_{88} + (a_6 + a_7)b_{78}b_{81} = 0 \\
& a_8b_{33} + (a_6 + a_7)b_{34} + (a_3 + 2a_5)b_{35} + a_2b_{36} - a_2b_{47}b_{88} - (a_3 + a_5)b_{57}b_{88} \\
& \quad - a_5b_{67}b_{78} - (a_6 + a_7)b_{67}b_{88} + a_5b_{68}b_{77} - a_8b_{77}b_{88} = 0 \\
& a_8b_{22} + (a_6 + a_7)b_{23} + (a_3 + 2a_5)b_{24} + a_2b_{25} - a_2b_{46}b_{78} - a_3b_{46}b_{88} \\
& \quad + a_2b_{56}b_{68} - a_5b_{56}b_{78} - a_6b_{56}b_{88} - a_2b_{58}b_{66} \\
& \quad - a_7b_{66}b_{78} - a_8b_{66}b_{88} = 0 \\
& b_{26} - b_{11}b_{37} - a_2b_{41}b_{77} + a_2b_{47}b_{71} + a_3b_{47}b_{81} + a_2b_{51}b_{67} - a_5b_{51}b_{77} \\
& \quad - a_2b_{57}b_{61} + a_5b_{57}b_{71} + a_6b_{57}b_{81} - a_7b_{61}b_{77} + a_7b_{67}b_{71} \\
& \quad + a_8b_{67}b_{81} + a_9b_{77}b_{81} = 0 \\
& b_{37} - b_{11}b_{48} - a_2b_{41}b_{88} + a_2b_{48}b_{81} - (a_3 + a_5)b_{51}b_{88} + (a_3 + a_5)b_{58}b_{81} \\
& \quad - a_5b_{61}b_{78} - (a_6 + a_7)b_{61}b_{88} + a_5b_{68}b_{71} + (a_6 + a_7)b_{68}b_{81} \\
& \quad - a_8b_{71}b_{88} + a_8b_{78}b_{81} = 0 \\
& a_9b_{22} + a_8b_{23} + (a_6 + a_7)b_{24} + (a_3 + 2a_5)b_{25} + a_2b_{26} - a_2b_{47}b_{78} - a_3b_{47}b_{88} \\
& \quad + a_2b_{48}b_{77} + a_2b_{57}b_{68} - a_5b_{57}b_{78} - a_6b_{57}b_{88} - a_2b_{58}b_{67} + a_5b_{58}b_{77} \\
& \quad - a_7b_{67}b_{78} - a_8b_{67}b_{88} + a_7b_{68}b_{77} - a_9b_{77}b_{88} = 0 \\
& b_{27} - b_{11}b_{38} - a_2b_{41}b_{78} - a_3b_{41}b_{88} + a_2b_{48}b_{71} + a_3b_{48}b_{81} + a_2b_{51}b_{68} \\
& \quad - a_5b_{51}b_{78} - a_6b_{51}b_{88} - a_2b_{58}b_{61} + a_5b_{58}b_{71} + a_6b_{58}b_{81} \\
& \quad - a_7b_{61}b_{78} - a_8b_{61}b_{88} + a_7b_{68}b_{71} + a_8b_{68}b_{81} \\
& \quad - a_9b_{71}b_{88} + a_9b_{78}b_{81} = 0.
\end{aligned}$$

(5.6)

The solutions satisfying S'_t (5.6) are elements of the set of matrices $B = (b_{ij}) \in \text{Mat}(8 \times 8, C)$. In case that B are matrices of the form

$$B = (b_{ij}) \in \text{Mat}(8 \times 8, C), \quad b_{ij} = 0, \quad j < i, \quad b_{ii} = 1, \tag{5.7}$$

the group of automorphisms $\text{Aut}(g_t)$ of g_t consists of unipotent automorphisms.

We will prove that the set of points $t \in V \subset C^7$, such that there exists a solution $B = (b_{ij}) \in \text{Mat}(8 \times 8, C)$ of S'_t satisfying the conditions (5.7), is a Zariski constructible set and we will express it as a finite union of Zariski locally closed subsets.

To realize the above idea, we study S'_t in suitable subsets of V .

5.2. Main results. We consider two cases: first, $a_2 \neq 0$ and then, $a_2 = 0$.

5.2.1. $a_2 \neq 0$. Let the open set $V \cap D(a_2)$. Because of the equation $a_2(5a_5 + 2a_3) = 0$, we can distinguish the following two subcases.

(1) ($a_3 \neq 0$). First, we consider the set $T^{(1)} = V \cap D(a_2 \cdot a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = -(2/5)a_3$. By doing the necessary calculations in S'_t , we can deduce

$$b_{11} = b_{88}^2, \quad b_{22} = b_{88}^9, \quad b_{ii} = b_{88}^{10-i}, \quad i = 3, \dots, 7, \quad Q_4(b_{88} - 1) = 0 \tag{5.8}$$

with $Q_4 = 2a_3^2 - 25a_2a_6 - 25a_2a_7$.

So, the set of points in $T^{(1)}$ in which the group $\text{Aut}(g_t)$ of the corresponding Lie algebra consists of unipotent automorphisms is $T^{(1)} \cap D(Q_4)$.

(2) ($a_3 = 0$). Now, we consider the set $T^{(2)} = V \cap D(a_2) \cap \mathcal{V}(a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = 0$. By doing some calculations as above, in case that $a_6 + a_7 \neq 0$, we deduce

$$b_{11} = b_{88}^3, \quad b_{22} = b_{88}^{11}, \quad b_{ii} = b_{88}^{11-i}, \quad i = 3, \dots, 7, \quad a_8(b_{88} - 1) = 0. \tag{5.9}$$

So, in the set $T^{(2)} \cap D((a_6 + a_7) \cdot a_8)$, the group $\text{Aut}(g_t)$ of only one Lie algebra consists of unipotent automorphisms. On the other hand, the group $\text{Aut}(g_t)$ of each of the corresponding Lie algebras in the set $T^{(2)} \cap D(a_6 + a_7) \cap \mathcal{V}(a_8)$ and $T^{(2)} \cap \mathcal{V}(a_6 + a_7)$ do not contain unipotent automorphisms.

From the above, we can state the following theorem.

THEOREM 5.1. *Consider the set of complex filiform Lie algebras. Consider C^8 with (a_2, a_3, \dots, a_9) as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). In the Zariski open set $V \cap D(a_2)$, the Zariski constructible*

subset of filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms is defined as the union of the following subsets:

$$\begin{aligned}
 &D(a_3 \cdot (2a_3^2 - 25a_2a_6 - 25a_2a_7)), \\
 &\mathcal{V}(a_3) \cap D((a_6 + a_7) \cdot a_8).
 \end{aligned}
 \tag{5.10}$$

5.2.2. $a_2 = 0$. We consider the set $T^{(3)} = V \cap \mathcal{V}(a_2)$. Because of the equation $a_2(5a_5 + 2a_3) = 0$, we can distinguish the following subcases.

(1) ($a_5 \neq 0$). So, we obtain the set $T^{(3)} \cap D(a_5)$ and we distinguish the following:

(1A) ($a_3 + 2a_5 \neq 0$). In the subset $T^{(3)} \cap D(a_5 \cdot (a_3 + 2a_5))$, after the necessary calculations in system (5.6), we deduce

$$b_{ii} = b_{11}^{10-i}, \quad i = 2, \dots, 8, \quad Q_5(b_{11} - 1) = 0
 \tag{5.11}$$

with $Q_5 = 2a_3^2a_7 - 3a_3a_5a_6 + 5a_3a_5a_7 - 3a_5^2a_6 + 5a_5^2a_7$.

So, in the set of points $T^{(3)} \cap D(a_5 \cdot (a_3 + 2a_5) \cdot Q_5)$, the group $\text{Aut}(g_t)$ of the corresponding Lie algebra consists of unipotent automorphisms, whereas, in the set $T^{(3)} \cap D(a_5 \cdot (a_3 + 2a_5)) \cap \mathcal{V}(Q_5)$, the group $\text{Aut}(g_t)$ of the corresponding Lie algebra does not contain unipotent automorphisms.

(1B) ($a_3 + 2a_5 = 0$). We study S'_t in $T^{(3)} \cap D(a_5) \cap \mathcal{V}(a_3 + 2a_5)$ and we obtain

$$b_{ii} = b_{11}^{10-i}, \quad i = 2, \dots, 8, \quad (a_6 + a_7)(b_{11} - 1) = 0.
 \tag{5.12}$$

So, the group $\text{Aut}(g_t)$ of the corresponding Lie algebra in the set of points $T^{(3)} \cap D(a_5 \cdot (a_6 + a_7)) \cap \mathcal{V}(a_3 + 2a_5)$ consists of unipotent automorphisms, but the group $\text{Aut}(g_t)$ of the Lie algebra corresponding to the set $T^{(3)} \cap D(a_5) \cap \mathcal{V}(a_3 + 2a_5, a_6 + a_7)$ does not contain unipotent automorphisms.

(2) ($a_5 = 0$). First, we distinguish two subcases $a_3 \neq 0$ and $a_3 = 0$.

(2A) ($a_3 \neq 0$). Then, we consider the set $T^{(3)} \cap \mathcal{V}(a_5) \cap D(a_3)$. By doing some calculations in S'_t , we obtain

$$b_{ii} = b_{11}^{10-i}, \quad i = 2, \dots, 8, \quad a_7(b_{11} - 1) = 0.
 \tag{5.13}$$

In this case, the group $\text{Aut}(g_t)$ of the Lie algebra corresponding to the set of points $T^{(3)} \cap \mathcal{V}(a_5) \cap D(a_3 \cdot a_7)$ consists of unipotent automorphisms.

(2B) ($a_3 = 0$). In $T^{(3)} \cap \mathcal{V}(a_3, a_5)$, we distinguish the cases $a_7 \neq 0$ and $a_7 = 0$.

(i) ($a_7 \neq 0$). In the subset $T^{(3)} \cap \mathcal{V}(a_3, a_5) \cap D(a_7)$, after the necessary calculations in system S'_t , we deduce

$$b_{ii} = b_{11}^{11-i}, \quad i = 2, \dots, 8, \quad a_8(b_{11} - 1) = 0.
 \tag{5.14}$$

So, the group $\text{Aut}(g_t)$ of the Lie algebra corresponding to the set of points $T^{(3)} \cap \mathcal{V}(a_3, a_5) \cap D(a_7 \cdot a_8)$ consists of unipotent automorphisms.

(ii) ($a_7 = 0$). We now consider the subset $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7)$. We distinguish another two subcases $a_6 \neq 0$ and $a_6 = 0$.

(iiA) ($a_6 \neq 0$). So, we obtain the set $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7) \cap D(a_6)$. By doing some calculations in S'_t , we deduce

$$b_{ii} = b_{11}^{11-i}, \quad i = 2, \dots, 8, \quad a_8(b_{11} - 1) = 0. \tag{5.15}$$

From the above, we conclude that the group $\text{Aut}(g_t)$ of the Lie algebra corresponding to $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7) \cap D(a_6 \cdot a_8)$ consists of unipotent automorphisms, but the groups $\text{Aut}(g_t)$ of those that correspond to $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7, a_8) \cap D(a_6)$ do not contain unipotent automorphisms.

(iiB) ($a_6 = 0$). The set of points in which we are acting now is $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7)$. In case that $a_8 \neq 0$ and by using similar techniques as we did previously in system S'_t , we deduce

$$b_{ii} = b_{11}^{12-i}, \quad i = 2, \dots, 8, \quad a_9(b_{11} - 1) = 0. \tag{5.16}$$

Hence, the group $\text{Aut}(g_t)$ of the Lie algebra in the set $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7) \cap D(a_8 \cdot a_9)$ consists of unipotent automorphisms. On the other hand, the groups $\text{Aut}(g_t)$ of each of the algebras in the subsets $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_9) \cap D(a_8)$, $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_8) \cap D(a_9)$, and $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_8, a_9)$ do not contain unipotent automorphisms.

So, we have proved the following theorem.

THEOREM 5.2. *Consider the set of complex filiform Lie algebras. Consider C^8 with (a_2, a_3, \dots, a_9) as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). The Zariski constructible subset of filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms in the Zariski closed set $V \cap \mathcal{V}(a_2)$ is defined as the union of the following subsets:*

$$\left\{ \begin{array}{l} D(a_5) \cap \left\{ \begin{array}{l} D((a_3 + 2a_5) \cdot (2a_3^2a_7 - 3a_3a_5a_6 + 5a_3a_5a_7 - 3a_5^2a_6 + 5a_5^2a_7)), \\ \mathcal{V}(a_3 + 2a_5) \cap D(a_6 + a_7), \end{array} \right. \\ \mathcal{V}(a_5) \cap \left\{ \begin{array}{l} D(a_3 \cdot a_7), \\ \mathcal{V}(a_3) \cap \left\{ \begin{array}{l} D(a_7 \cdot a_8), \\ \mathcal{V}(a_7) \cap \left\{ \begin{array}{l} D(a_6 \cdot a_8), \\ \mathcal{V}(a_6) \cap D(a_8 \cdot a_9). \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \tag{5.17}$$

By (\cup) , we mean the union of the corresponding sets.

Let $g_1 = \mu_8^{10,a}$ and $g_2 = \mu_8^{11}$ be the following Lie algebras belonging to the family (3.1) as they were defined in [1]:

$$\begin{aligned} [e_1, e_i] &= e_{i-1}, \quad i \geq 3, & [e_4, e_8] &= e_2, \\ [e_5, e_8] &= e_3, & [e_6, e_8] &= e_2 + e_4, \\ [e_7, e_8] &= ae_2 + e_3 + e_5, \quad a \in C, \end{aligned} \tag{5.18}$$

$$\begin{aligned} [e_1, e_i] &= e_{i-1}, \quad i \geq 3, \\ [e_i, e_8] &= e_{i-2}, \quad 4 \leq i \leq 6, \quad [e_7, e_8] = e_2 + e_5. \end{aligned} \tag{5.19}$$

Now, we can state the following theorems.

THEOREM 5.3. *Consider the set of complex filiform Lie algebras. If $C^8 = \{(a_2, \dots, a_9) / a_i \in C \text{ and } a_i \text{ satisfy (3.1)}\}$, we define the hypersurface V in C^7 by (3.2). The group $\text{Aut}(g_1)$ of g_1 corresponding to the set $V \cap \mathcal{V}(a_2, a_5, a_7) \cap D(a_3 \cdot (4a_3a_8 - 5a_6^2))$ consists of automorphisms of the type*

$$\begin{aligned} L &= (l_{ij}), \quad l_{ij} = 0, \quad j < i, \quad l_{ii} = 1, \quad 1 \leq i, j \leq 8. \\ L &= (l_{ij}), \quad l_{ij} = 0, \quad j < i, \quad l_{ii} = \begin{cases} 1, & i \text{ is even,} \\ -1, & i \text{ is odd,} \end{cases} \end{aligned} \tag{5.20}$$

where $1 \leq i, j \leq 8$.

So, the set of unipotent automorphisms form a proper subgroup of the group $\text{Aut}(g_1)$.

PROOF. If we consider the set of points $V \cap \mathcal{V}(a_2, a_5, a_7) \cap D(a_3)$ after the necessary calculations in system S'_i , we deduce

$$b_{ii} = b_{11}^{10-i}, \quad i = 2, \dots, 8, \quad (4a_3a_8 - 5a_6^2)(b_{11}^2 - 1) = 0. \tag{5.21}$$

Obviously, the group $\text{Aut}(g_1)$ of the Lie algebras corresponding to the set of points $V \cap \mathcal{V}(a_2, a_5, a_7) \cap D(a_3 \cdot (4a_3a_8 - 5a_6^2))$ does not contain only unipotent automorphisms. So, the group $\text{Aut}(g_1)$, except the unipotent, contains automorphisms of the following type:

$$L = (l_{ij}), \quad l_{ij} = 0, \quad j < i, \quad l_{ii} = \begin{cases} 1, & i \text{ is even,} \\ -1, & i \text{ is odd,} \end{cases} \tag{5.22}$$

□

where $1 \leq i, j \leq 8$.

In case that $4a_3a_8 - 5a_6^2 = 0$, by acting as we previously did in system S'_t , we obtain

$$b_{ii} = b_{11}^{10-i}, \quad i = 2, \dots, 8, \quad (4a_3^2a_9 - 7a_6^3)(b_{11}^3 - 1) = 0. \quad (5.23)$$

Thus, the group $\text{Aut}(g_2)$ of the Lie algebra corresponding to the set of points $V \cap \mathcal{V}(a_2, a_5, a_7, 4a_3a_8 - 5a_6^2) \cap D(a_3 \cdot (4a_3^2a_9 - 7a_6^3))$ does not contain only unipotent automorphisms but also automorphisms of the type

$$K = (k_{ij}), \quad k_{ij} = 0, \quad j < i, \quad k_{ii} = \begin{cases} z, & i = 1, 3, 6, \\ \bar{z}, & i = 2, 5, 8, \\ 1, & i = 4, 7, \end{cases} \quad (5.24)$$

where $j = 1, \dots, 8$ and z is a cubic root of 1.

So, we have proved the following theorem.

THEOREM 5.4. *Consider the set of complex filiform Lie algebras. If $C^8 = \{(a_2, \dots, a_9) / a_i \in C \text{ and } a_i \text{ satisfy (3.1)}\}$, we define the hypersurface V in C^7 by (3.2). The group $\text{Aut}(g_2)$ of the filiform Lie algebra g_2 corresponding to the set $V \cap \mathcal{V}(a_2, a_5, a_7, 4a_3a_8 - 5a_6^2) \cap D(a_3 \cdot (4a_3^2a_9 - 7a_6^3))$ consists of automorphisms of the type*

$$K = (k_{ij}), \quad k_{ij} = 0, \quad j < i, \quad k_{ii} = \begin{cases} z, & i = 1, 3, 6, \\ \bar{z}, & i = 2, 5, 8, \\ 1, & i = 4, 7, \end{cases} \quad (5.25)$$

where $j = 1, \dots, 8$ and z is a cubic root of 1.

REMARK 5.5. The group $\text{Aut}(g)$ of the Lie algebra corresponding to the set of points $V \cap \mathcal{V}(a_2, a_5, a_7, 4a_3a_8 - 5a_6^2, 4a_3^2a_9 - 7a_6^3) \cap D(a_3)$ does not contain unipotent automorphisms.

6. General conclusions. From Theorems 4.1, 4.2, 5.1, 5.2, 5.3, and 5.4, we conclude.

THEOREM 6.1. *The group of automorphisms $\text{Aut}(g)$ of each one of the characteristically nilpotent filiform Lie algebras of dimension 8 over C given by (4.6) and (4.7) consists of unipotent automorphisms, except that of the Lie algebras $g_1 = \mu_8^{10,a}$, $a \in C$, and $g_2 = \mu_8^{11}$ given by (5.18), and (5.19).*

THEOREM 6.2. *The unipotent automorphisms of each one of the characteristically nilpotent filiform Lie algebras of dimension 8 over C , $g_1 = \mu_8^{10,a}$, $a \in C$,*

and $g_2 = \mu_8^{11}$, given by (5.18) and (5.19), form a proper subgroup of the group $\text{Aut}(g_1)$ and $\text{Aut}(g_2)$, respectively.

THEOREM 6.3. *The group of automorphisms $\text{Aut}(g_1)$ of the characteristically nilpotent filiform Lie algebras of dimension 8 over C , $g_1 = \mu_8^{10,a}$, $a \in C$, given by (5.18), consists of automorphisms of the type*

$$L = (l_{ij}), \quad l_{ij} = 0, \quad j < i, \quad l_{ii} = 1, \quad 1 \leq i, j \leq 8,$$

$$L = (l_{ij}), \quad l_{ij} = 0, \quad j < i, \quad l_{ii} = \begin{cases} 1, & i \text{ is even,} \\ -1, & i \text{ is odd,} \end{cases} \quad (6.1)$$

where $1 \leq i, j \leq 8$.

THEOREM 6.4. *The group of automorphisms $\text{Aut}(g_2)$ of the characteristically nilpotent filiform Lie algebra of dimension 8 over C , $g_2 = \mu_8^{11}$, given by (5.19), consist of automorphisms of the type*

$$K = (k_{ij}), \quad k_{ij} = 0, \quad j < i, \quad k_{ii} = \begin{cases} z, & i = 1, 3, 6, \\ \bar{z}, & i = 2, 5, 8, \\ 1, & i = 4, 7, \end{cases} \quad (6.2)$$

where $j = 1, \dots, 8$ and z is a cubic root of 1.

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