RING HOMOMORPHISMS ON REAL BANACH ALGEBRAS

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Let $B$ be a strictly real commutative real Banach algebra with the carrier space $\Phi_B$. If $A$ is a commutative real Banach algebra, then we give a representation of a ring homomorphism $\rho : A \rightarrow B$, which needs not be linear nor continuous. If $A$ is a commutative complex Banach algebra, then $\rho(A)$ is contained in the radical of $B$.

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1. Introduction and results. Ring homomorphisms are mappings between two rings that preserve both addition and multiplication. In particular, we are concerned with ring homomorphisms between two commutative Banach algebras. If $\mathbb{R}$ is the real number field, then the zero map and the identity are typical examples of ring homomorphisms on $\mathbb{R}$. Furthermore, the converse is true: if $\rho$ is a nonzero ring homomorphism on $\mathbb{R}$, then $\rho(t) = t$ for every $t \in \mathbb{R}$. For if $\rho$ is nonzero, then $\rho(1) = \rho(1)^2$ implies $\rho(1) = 1$, and hence $\rho$ preserves every rational number. Suppose that $a \geq 0$. Then we have $\rho(a) = \rho(\sqrt{a})^2 \geq 0$. It follows that $\rho$ preserves the order. Fix $t \in \mathbb{R}$ and choose rational sequences $\{p_n\}$ and $\{q_n\}$ converging to $t$ such that $p_n \leq t \leq q_n$. Since $\rho$ preserves both rational numbers and the order $p_n \leq \rho(t) \leq q_n$, thus $\rho(t) = t$.

Let $C_\mathbb{R}(K)$ be the commutative real Banach algebra of all real-valued continuous functions on a compact Hausdorff space $K$. In the proof of [11, Theorem 3.1], Šemrl essentially gave a representation of a ring homomorphism on $C_\mathbb{R}(X)$ into $C_\mathbb{R}(Y)$ which states that ring homomorphisms preserve scalar multiplication automatically.

**Theorem 1.1** (Šemrl [11]). If $\rho : C_\mathbb{R}(X) \rightarrow C_\mathbb{R}(Y)$ is a ring homomorphism, then there exist a closed and open subset $Y_0 \subset Y$ and a continuous map $\varphi : Y \setminus Y_0 \rightarrow X$ such that

$$
\rho(f)(y) = \begin{cases} 
0, & y \in Y_0, \\
\varphi(y), & y \in Y \setminus Y_0,
\end{cases}
$$

(1.1)

for every $f \in C_\mathbb{R}(X)$. 

Recall that a commutative real Banach algebra $A$ is said to be strictly real if $\phi(A) \subset \mathbb{R}$ for all $\phi \in \Phi_A$ (cf. [4]), where $\Phi_A$ denotes the carrier space of $A$. We generalize the above result as follows.

**Theorem 1.2.** Suppose that $A$ is a commutative real Banach algebra with carrier space $\Phi_A$ and that $B$ is a commutative strictly real Banach algebra with carrier space $\Phi_B$. If $\rho$ is a ring homomorphism on $A$ into $B$, then there exist a closed subset $\Phi_0 \subset \Phi_B$ and a continuous map $\phi : \Phi_B \setminus \Phi_0 \to \Phi_A$ such that

$$\rho(a)\hat{\psi} = \begin{cases} 0, & \psi \in \Phi_0, \\ \hat{a}(\Phi(\psi)), & \psi \in \Phi_B \setminus \Phi_0, \end{cases}$$

(1.2)

for every $a \in A$, where $\hat{\cdot}$ denotes the Gelfand transform.

If, in addition, $A$ is unital, then the above $\Phi_0$ is closed and open.

Let $C(K)$ be the commutative complex Banach algebra of all complex-valued continuous functions on a compact Hausdorff space $K$. One might expect that a similar result holds for ring homomorphisms on $C(X)$ into $C(Y)$. Unfortunately, this is not the case. Indeed, there exists a nonzero ring homomorphism $\tau$ on $C(K)$ such that $\tau$ is not the identity nor complex conjugate (cf. [6]); such a map is called nontrivial. More precisely, there exist $2^c$ nontrivial ring homomorphisms on $C$ (cf. [2]), where $c$ denotes the cardinal number of the continuum. However, many authors treat ring homomorphisms between two complex Banach algebras (cf. [1, 3, 5, 7, 8, 9, 10, 11, 12]).

On the other hand, it is easy to see that the zero map is the only ring homomorphism on $C$ into $\mathbb{R}$. This fact can be generalized as follows.

**Theorem 1.3.** Suppose that $A$ is a commutative complex Banach algebra and that $B$ is a commutative strictly real Banach algebra with carrier space $\Phi_B$. If $\rho : A \to B$ is a ring homomorphism, then $\rho(a)\hat{\psi} = 0$ for all $a \in A$, or equivalently, $A$ is mapped into the radical of $B$.
we have
\[ \phi(\nu h) = \phi(h) \frac{\phi(\nu a)}{\phi(a)}, \quad \phi(\lambda \mu a) = \phi(\lambda a) \frac{\phi(\mu a)}{\phi(a)}. \] (2.2)

Hence
\[ \tilde{\phi}((f,\lambda)(g,\mu)) = \tilde{\phi}(fg + \mu f + \lambda g,\lambda \mu) \]
\[ = \phi(f)\phi(g) + \phi(\mu f) + \phi(\lambda g) + \frac{\phi(\lambda \mu a)}{\phi(a)} \]
\[ = \left\{ \phi(f) + \frac{\phi(\lambda a)}{\phi(a)} \right\} \left\{ \phi(g) + \frac{\phi(\mu a)}{\phi(a)} \right\} \] (2.3)
whenever \((f,\lambda), (g,\mu) \in \mathcal{A}_e\), and thus \(\tilde{\phi}\) is multiplicative.

We have now proved that there exists an extension \(\tilde{\phi}\) of \(\phi\) on \(\mathcal{A}_e\).

It remains to prove that \(\tilde{\phi} = \tilde{\phi}'\) whenever \(\tilde{\phi}'\) is a ring homomorphism which extends \(\phi\) on \(\mathcal{A}_e\). So, fix \((f,\lambda) \in \mathcal{A}_e\). Since
\[ \phi(\lambda a) = \tilde{\phi}'(\lambda a) = \tilde{\phi}'(\lambda e) \phi(a), \] (2.4)

it follows that
\[ \tilde{\phi}'(f,\lambda) = \tilde{\phi}'(f) + \tilde{\phi}'(\lambda e) = \phi(f) + \frac{\phi(\lambda a)}{\phi(a)} = \tilde{\phi}(f,\lambda), \] (2.5)
and the uniqueness is proved.

**Definition 2.2.** Let \(A\) be a commutative Banach algebra over \(F \in \{\mathbb{R}, \mathbb{C}\}\) and let \(B\) be a commutative real or complex Banach algebra with carrier space \(\Phi_B\). If \(\rho\) is a ring homomorphism on \(A\) into \(B\), then the formula
\[ \rho_\psi(f) \overset{\text{def}}{=} \rho(f\psi), \quad f \in A, \] (2.6)
assigns to each \(\psi \in \Phi_B\) a ring homomorphism \(\rho_\psi : A \to \mathbb{C}\).

If \(\rho_\psi\) is nonzero, then there is a unique extension \(\tilde{\rho}_\psi\) of \(\rho_\psi\) on \(A_e\) (Lemma 2.1). We define a ring homomorphism \(\sigma_\psi : F \to \mathbb{C}\) by
\[ \sigma_\psi(\lambda) = \tilde{\rho}_\psi(\lambda e), \quad \lambda \in F. \] (2.7)

It follows from this definition that
\[ \rho_\psi(\lambda f) = \tilde{\rho}_\psi(\lambda f) = \tilde{\rho}_\psi(\lambda e) \tilde{\rho}_\psi(f) = \sigma_\psi(\lambda) \rho_\psi(f) \] (2.8)
whenever \(\rho_\psi\) is nonzero, \(\lambda \in F\), and \(f \in A\).
Proof of Theorem 1.2. Put $\Phi_0 \overset{\text{def}}{=} \{ \psi \in \Phi_B : \ker \rho_{\psi} = A \}$ (possibly empty), and let $\{ \psi_\alpha \} \subset \Phi_0$ be a net converging to $\psi_0 \in \Phi_B$. Fix $a \in A$. Since $\rho(a)$ is a continuous function on $\Phi_B$, then $0 = \rho(a)(\psi_\alpha) \to \rho(a)(\psi_0)$. Since $a$ was arbitrary, it follows that $\psi_0 \in \Phi_0$, and hence, $\Phi_0$ is a closed subset of $\Phi_B$. Moreover, if $A$ has a unit $e$, then $\rho_{\psi}(e) = 0$ or 1, and hence,

\[ \Phi_0 = \{ \psi \in \Phi_B : \rho_{\psi}(e) = 0 \} = \{ \psi \in \Phi_B : \rho_{\psi}(e) < 2^{-1} \}. \]  

(2.9)

It follows that $\Phi_0$ is closed and open whenever $A$ is unital.

Pick $\psi \in \Phi_B \setminus \Phi_0$. Since $B$ is strictly real, the map $\sigma_{\psi}$ as in Definition 2.2 is a nonzero ring homomorphism on $\mathbb{R}$ into $\mathbb{R}$ so that $\sigma_{\psi}$ is the identity map on $\mathbb{R}$. It follows from (2.8) that

\[ \rho_{\psi}(ta) = \sigma_{\psi}(t)\rho_{\psi}(a) = t\rho_{\psi}(a), \quad t \in \mathbb{R}, \ a \in A, \]  

(2.10)

proving that $\rho_{\psi} \in \Phi_A$ for every $\psi \in \Phi_B \setminus \Phi_0$. Let $\varphi : \Phi_B \setminus \Phi_0 \to \Phi_A$ be the map defined by $\varphi(\psi) \overset{\text{def}}{=} \rho_{\psi}$. Then we have (1.2) for every $a \in A$. Finally, we show the continuity of $\varphi$. Suppose that $\{ \psi_\beta \} \subset \Phi_B \setminus \Phi_0$ is a net converging to $\psi_1 \in \Phi_B \setminus \Phi_0$. Then (1.2) gives

\[ \hat{a}(\varphi(\psi_\beta)) = \rho(a)(\psi_\beta) \to \rho(a)(\psi_1) = \hat{a}(\varphi(\psi_1)) \]  

(2.11)

for every $a \in A$. Hence $\varphi(\psi_\beta)$ converges to $\varphi(\psi_1)$. This implies that $\varphi$ is continuous on $\Phi_B \setminus \Phi_0$. \hfill \Box

Proof of Theorem 1.3. Pick $a \in A$ and $\psi \in \Phi_B$. If $\rho(a)(\psi) \neq 0$, then $\sigma_{\psi}(i) = \pm i$, and hence, $\rho(ia)(\psi)$ would be a nonzero pure imaginary number by (2.8), in contradiction to $B$ being strictly real. \hfill \Box

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References


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