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RING HOMOMORPHISMS ON REAL BANACH ALGEBRAS

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Let B be a strictly real commutative real Banach algebra with the carrier space Φ_B. If A is a commutative real Banach algebra, then we give a representation of a ring homomorphism $\rho : A \to B$, which needs not be linear nor continuous. If A is a commutative complex Banach algebra, then $\rho(A)$ is contained in the radical of B.

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1. Introduction and results. Ring homomorphisms are mappings between two rings that preserve both addition and multiplication. In particular, we are concerned with ring homomorphisms between two commutative Banach algebras. If $\mathbb{R}$ is the real number field, then the zero map and the identity are typical examples of ring homomorphisms on $\mathbb{R}$. Furthermore, the converse is true: if ρ is a nonzero ring homomorphism on $\mathbb{R}$, then $\rho(t) = t$ for every $t \in \mathbb{R}$. For if ρ is nonzero, then $\rho(1) = \rho(1)^2$ implies $\rho(1) = 1$, and hence ρ preserves every rational number. Suppose that $a \geq 0$. Then we have $\rho(a) = \rho(\sqrt{a})^2 \geq 0$. It follows that ρ preserves the order. Fix $t \in \mathbb{R}$ and choose rational sequences $\{p_n\}$ and $\{q_n\}$ converging to t such that $p_n \leq t \leq q_n$. Since ρ preserves both rational numbers and the order $p_n \leq \rho(t) \leq q_n$, thus $\rho(t) = t$.

Let $C_{\mathbb{R}}(K)$ be the commutative real Banach algebra of all real-valued continuous functions on a compact Hausdorff space K. In the proof of [11, Theorem 3.1], Šemrl essentially gave a representation of a ring homomorphism on $C_{\mathbb{R}}(X)$ into $C_{\mathbb{R}}(Y)$ which states that ring homomorphisms preserve scalar multiplication automatically.

Theorem 1.1 (Šemrl [11]). *If $\rho : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ is a ring homomorphism, then there exist a closed and open subset $Y_0 \subset Y$ and a continuous map $\varphi : Y \setminus Y_0 \to X$ such that*

$$\rho(f)(y) = \begin{cases} 0, & y \in Y_0, \\ f(\varphi(y)), & y \in Y \setminus Y_0, \end{cases} \tag{1.1}$$

for every $f \in C_{\mathbb{R}}(X)$.

Recall that a commutative real Banach algebra A is said to be strictly real if $\phi(A) \subset \mathbb{R}$ for all $\phi \in \Phi_A$ (cf. [4]), where Φ_A denotes the carrier space of A. We generalize the above result as follows.

Theorem 1.2. *Suppose that A is a commutative real Banach algebra with carrier space Φ_A and that B is a commutative strictly real Banach algebra with carrier space Φ_B. If ρ is a ring homomorphism on A into B, then there exist a closed subset $\Phi_0 \subset \Phi_B$ and a continuous map $\varphi : \Phi_B \setminus \Phi_0 \to \Phi_A$ such that*

$$\rho(a)\hat{}(\psi) = \begin{cases} 0, & \psi \in \Phi_0, \\ \hat{a}(\varphi(\psi)), & \psi \in \Phi_B \setminus \Phi_0, \end{cases} \tag{1.2}$$

for every $a \in A$, where $\hat{\cdot}$ denotes the Gelfand transform.

If, in addition, A is unital, then the above Φ_0 is closed and open.

Let $C(K)$ be the commutative complex Banach algebra of all complex-valued continuous functions on a compact Hausdorff space K. One might expect that a similar result holds for ring homomorphisms on $C(X)$ into $C(Y)$. Unfortunately, this is not the case. Indeed, there exists a nonzero ring homomorphism τ on $\mathbb{C}$ such that τ is not the identity nor complex conjugate (cf. [6]); such a map is called nontrivial. More precisely, there exist $2^{\mathfrak{c}}$ nontrivial ring homomorphisms on $\mathbb{C}$ (cf. [2]), where $\mathfrak{c}$ denotes the cardinal number of the continuum. However, many authors treat ring homomorphisms between two complex Banach algebras (cf. [1, 3, 5, 7, 8, 9, 10, 11, 12]).

On the other hand, it is easy to see that the zero map is the only ring homomorphism on $\mathbb{C}$ into $\mathbb{R}$. This fact can be generalized as follows.

Theorem 1.3. *Suppose that A is a commutative complex Banach algebra and that B is a commutative strictly real Banach algebra with carrier space Φ_B. If $\rho : A \to B$ is a ring homomorphism, then $\rho(a)\hat{} = 0$ for all $a \in A$, or equivalently, A is mapped into the radical of B.*

2. Proof of results. Suppose that $\mathcal{A}$ is a commutative algebra. We define $\mathcal{A}_e = \mathcal{A}$ if $\mathcal{A}$ is unital; otherwise, $\mathcal{A}_e$ denotes the commutative algebra adjoining a unit element e to $\mathcal{A}$.

Lemma 2.1. *If $\mathcal{A}$ is a commutative algebra over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and if ϕ is a nonzero ring homomorphism on $\mathcal{A}$ into $\mathbb{C}$, then ϕ can be extended to a unique ring homomorphism $\tilde{\phi}$ on $\mathcal{A}_e$ into $\mathbb{C}$.*

Proof. Choose $a \in \mathcal{A}$ so that $\phi(a) \neq 0$. If we define $\tilde{\phi} : \mathcal{A}_e \to \mathbb{C}$ by

$$\tilde{\phi}(f, \lambda) \overset{\text{def}}{=} \phi(f) + \frac{\phi(\lambda a)}{\phi(a)}, \quad (f, \lambda) \in \mathcal{A}_e, \tag{2.1}$$

it is trivial to verify that $\tilde{\phi}$ is additive. Identifying f with $(f, 0)$, we obtain $\tilde{\phi}|_{\mathcal{A}} = \phi$. We show that $\tilde{\phi}$ is multiplicative. For every $\nu, \lambda, \mu \in \mathbb{F}$ and $h \in \mathcal{A}$,

we have

$$\phi(\nu h) = \phi(h)\frac{\phi(\nu a)}{\phi(a)}, \qquad \frac{\phi(\lambda\mu a)}{\phi(a)} = \frac{\phi(\lambda a)}{\phi(a)}\frac{\phi(\mu a)}{\phi(a)}. \tag{2.2}$$

Hence

$$\begin{aligned}\tilde{\phi}((f,\lambda)(g,\mu)) &= \tilde{\phi}(fg+\mu f+\lambda g,\lambda\mu)\\ &= \phi(f)\phi(g)+\phi(\mu f)+\phi(\lambda g)+\frac{\phi(\lambda\mu a)}{\phi(a)}\\ &= \left\{\phi(f)+\frac{\phi(\lambda a)}{\phi(a)}\right\}\left\{\phi(g)+\frac{\phi(\mu a)}{\phi(a)}\right\}\\ &= \tilde{\phi}(f,\lambda)\tilde{\phi}(g,\mu)\end{aligned} \tag{2.3}$$

whenever $(f,\lambda),(g,\mu)\in\mathcal{A}_e$, and thus $\tilde{\phi}$ is multiplicative.

We have now proved that there exists an extension $\tilde{\phi}$ of ϕ on $\mathcal{A}_e$.

It remains to prove that $\tilde{\phi}=\tilde{\phi}'$ whenever $\tilde{\phi}'$ is a ring homomorphism which extends ϕ on $\mathcal{A}_e$. So, fix $(f,\lambda)\in\mathcal{A}_e$. Since

$$\phi(\lambda a) = \tilde{\phi}'(\lambda a) = \tilde{\phi}'(\lambda e)\phi(a), \tag{2.4}$$

it follows that

$$\tilde{\phi}'(f,\lambda) = \tilde{\phi}'(f)+\tilde{\phi}'(\lambda e) = \phi(f)+\frac{\phi(\lambda a)}{\phi(a)} = \tilde{\phi}(f,\lambda), \tag{2.5}$$

and the uniqueness is proved. □

Definition 2.2. Let A be a commutative Banach algebra over $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ and let B be a commutative real or complex Banach algebra with carrier space Φ_B. If ρ is a ring homomorphism on A into B, then the formula

$$\rho_\psi(f) \overset{\text{def}}{=} \rho(f)\hat{}(\psi), \quad f\in A, \tag{2.6}$$

assigns to each $\psi\in\Phi_B$ a ring homomorphism $\rho_\psi : A\to\mathbb{C}$.

If ρ_ψ is nonzero, then there is a unique extension $\widetilde{\rho_\psi}$ of ρ_ψ on A_e (Lemma 2.1). We define a ring homomorphism $\sigma_\psi:\mathbb{F}\to\mathbb{C}$ by

$$\sigma_\psi(\lambda) = \widetilde{\rho_\psi}(\lambda e), \quad \lambda\in\mathbb{F}. \tag{2.7}$$

It follows from this definition that

$$\rho_\psi(\lambda f) = \widetilde{\rho_\psi}(\lambda f) = \widetilde{\rho_\psi}(\lambda e)\widetilde{\rho_\psi}(f) = \sigma_\psi(\lambda)\rho_\psi(f) \tag{2.8}$$

whenever ρ_ψ is nonzero, $\lambda\in\mathbb{F}$, and $f\in A$.

Proof of Theorem 1.2. Put $\Phi_0 \overset{\text{def}}{=} \{\psi \in \Phi_B : \ker \rho_\psi = A\}$ (possibly empty), and let $\{\psi_\alpha\} \subset \Phi_0$ be a net converging to $\psi_0 \in \Phi_B$. Fix $a \in A$. Since $\rho(a)\hat{}$ is a continuous function on Φ_B, then $0 = \rho(a)\hat{}(\psi_\alpha) \to \rho(a)\hat{}(\psi_0)$. Since a was arbitrary, it follows that $\psi_0 \in \Phi_0$, and hence, Φ_0 is a closed subset of Φ_B. Moreover, if A has a unit e, then $\rho_\psi(e) = 0$ or 1, and hence,

$$\Phi_0 = \{\psi \in \Phi_B : \rho_\psi(e) = 0\} = \{\psi \in \Phi_B : \rho_\psi(e) < 2^{-1}\}. \tag{2.9}$$

It follows that Φ_0 is closed and open whenever A is unital.

Pick $\psi \in \Phi_B \setminus \Phi_0$. Since B is strictly real, the map σ_ψ as in Definition 2.2 is a nonzero ring homomorphism on $\mathbb{R}$ into $\mathbb{R}$ so that σ_ψ is the identity map on $\mathbb{R}$. It follows from (2.8) that

$$\rho_\psi(ta) = \sigma_\psi(t)\rho_\psi(a) = t\rho_\psi(a), \quad t \in \mathbb{R},\ a \in A, \tag{2.10}$$

proving that $\rho_\psi \in \Phi_A$ for every $\psi \in \Phi_B \setminus \Phi_0$. Let $\varphi : \Phi_B \setminus \Phi_0 \to \Phi_A$ be the map defined by $\varphi(\psi) \overset{\text{def}}{=} \rho_\psi$. Then we have (1.2) for every $a \in A$. Finally, we show the continuity of φ. Suppose that $\{\psi_\beta\} \subset \Phi_B \setminus \Phi_0$ is a net converging to $\psi_1 \in \Phi_B \setminus \Phi_0$. Then (1.2) gives

$$\hat{a}(\varphi(\psi_\beta)) = \rho(a)\hat{}(\psi_\beta) \longrightarrow \rho(a)\hat{}(\psi_1) = \hat{a}(\varphi(\psi_1)) \tag{2.11}$$

for every $a \in A$. Hence $\varphi(\psi_\beta)$ converges to $\varphi(\psi_1)$. This implies that φ is continuous on $\Phi_B \setminus \Phi_0$. □

Proof of Theorem 1.3. Pick $a \in A$ and $\psi \in \Phi_B$. If $\rho(a)\hat{}(\psi) \neq 0$, then $\sigma_\psi(i) = \pm i$, and hence, $\rho(ia)\hat{}(\psi)$ would be a nonzero pure imaginary number by (2.8), in contradiction to B being strictly real. □

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