KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

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We prove an equivalent relation between Ky Fan-type inequalities and certain bounds for the differences of means. We also generalize a result of Alzer et al. (2001).

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1. Introduction. Let \( P_{n,r}(x) \) be the generalized weighted power means:
\[
P_{n,r}(x) = \left( \sum_{i=1}^{n} \omega_i x_i^r \right)^{1/r},
\]
where \( \omega_i > 0, 1 \leq i \leq n \) with \( \sum_{i=1}^{n} \omega_i = 1 \) and \( x = (x_1, x_2, \ldots, x_n) \). Here, \( P_{n,0}(x) = \prod_{i=1}^{n} x_i^{\omega_i} \) denotes the limit of \( P_{n,r}(x) \) as \( r \to 0^+ \), which can be proved by noting that if \( p(r) = \ln \left( \sum_{i=1}^{n} \omega_i x_i^r \right) \), then \( p'(0) = \ln \left( \prod_{i=1}^{n} x_i^{\omega_i} \right) = \ln(P_{n,0}(x)) \). We write \( P_{n,r} \) for \( P_{n,r}(x) \) when there is no risk of confusion.

In this paper, we assume that \( 0 < x_1 \leq x_2 \leq \cdots \leq x_n \). With any given \( x \), we associate \( x' = (1-x_1, 1-x_2, \ldots, 1-x_n) \) and write \( A_n = P_{n,1}, \ G_n = P_{n,0}, \) and \( H_n = P_{n,-1} \). When \( 1-x_i \geq 0 \) for all \( i \), we define \( A'_n = P_{n,1}(x') \) and similarly for \( G'_n \) and \( H'_n \). We also let \( \sigma_n = \sum_{i=1}^{n} \omega_i [x_i - A_n]^2 \).

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published by Beckenbach and Bellman [7].

**Theorem 1.1.** For \( x_i \in (0, 1/2] \),
\[
\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}
\]
with equality holding if and only if \( x_1 = \cdots = x_n \).

In this paper, we consider the validity of the following additive Ky Fan-type inequalities (with \( x_1 < x_n < 1 \)):
\[
\frac{x_1}{1-x_1} < \frac{P_{n,r} - P_{n,s}}{P_{n,r} - P_{n,s}} < \frac{x_n}{1-x_n}.
\]

Note that by a change of variables \( x_i - 1 - x_i \), the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). We can deduce (see [9]) *Theorem 1.1* from the case \( r = 1, s = 0 \), and \( x_n \leq 1/2 \) in (1.2), which is a result
of Alzer [5]. Gao [9] later proved the validity of (1.2) for \( r = 1, -1 \leq s < 1 \), and \( x_n \leq 1/2 \).

What is worth mentioning is a nice result of Mercer [12] who showed that the validity of \( r = 1 \) and \( s = 0 \) in (1.2) is a consequence of a result of Cartwright and Field [8] who established the validity of \( r = 1 \) and \( s = 0 \) for the following bounds for the differences between power means (\( r > s \)):

\[
\frac{r-s}{2x_1} \sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2x_n} \sigma_n, \tag{1.3}
\]

where the constant \((r-s)/2\) is the best possible (see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all \( r > s \). We refer the reader to the survey article [2] and the references therein for an account of Ky Fan’s inequality, and to [4, 5, 10, 11] for other interesting refinements and extensions of (1.3).

Mercer’s result reveals a close relation between (1.3) and (1.2), and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed \( r \) and \( s \). As a consequence of this result, we give a characterization of the validity of (1.3) for \( r = 1 \) or \( s = 1 \). A solution of an open problem from [11] is also given.

Among the numerous sharpenings of Ky Fan’s inequality in the literature, we have the following inequalities connecting the three classical means (with \( \omega_i = 1/n \) here):

\[
\left( \frac{H_n}{H'_n} \right)^{n-1} \frac{A_n}{A'_n} \leq \left( \frac{G_n}{G'_n} \right)^n \leq \left( \frac{A_n}{A'_n} \right)^{n-1} \frac{H_n}{H'_n}. \tag{1.4}
\]

The right-hand side inequality of (1.4) is due to W. L. Wang and P. F. Wang [14] and the left-hand side inequality was recently proved by Alzer et al. [6].

It is natural to ask whether we can extend the above inequality to the weighted case, and using the same idea as in [6], we show that this is indeed true in Section 5.

2. The main theorem

**Theorem 2.1.** For fixed \( r > s \), the following inequalities are equivalent: (i) inequality (1.2) for \( x_n \leq 1/2 \); (ii) inequality (1.2); (iii) inequality (1.3).

**Proof.** (iii)\( \Rightarrow \) (ii) follows from a similar argument as given in [12], (ii)\( \Rightarrow \) (i) is trivial, so it suffices to show that (i)\( \Rightarrow \) (iii).

Fix \( r > s \) assuming that (1.2) holds for \( x_n \leq 1/2 \). Without loss of generality, we can assume that \( x_1 < x_n \). For a given \( x = (x_1, x_2, \ldots, x_n) \), let \( y = (\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n) \). We can choose \( \varepsilon \) small so that \( \varepsilon x_n \leq 1/2 \). Now, applying the right-hand side inequality (1.2) for \( y \), we get

\[
x_n (P_{n,r}(x) - P_{n,s}(x)) > \frac{1-\varepsilon x_n}{\varepsilon^2} \left( P_{n,r}(y') - P_{n,s}(y') \right). \tag{2.1}
\]
Let \( f(\epsilon) = P_{n,r}(y') - P_{n,s}(y') \), then \( f'(0) = 0 \) and \( f''(0) = (r - s)\sigma_n \). Thus, by letting \( \epsilon \) tend to 0, it is easy to verify that the limit of the expression on the right-hand side of (2.1) is \( (r - s)\sigma_n / 2 \). We can consider the left-hand side of (1.2) by a similar argument and this completes the proof.

3. An application of Theorem 2.1

**Lemma 3.1.** If inequality (1.3) holds for \( r > s \), then \( 0 \leq r + s \leq 3 \).

**Proof.** Let \( n = 2 \), and write \( \omega_1 = 1 - q \), \( \omega_2 = q \), \( x_1 = 1 \), and \( x_2 = 1 + t \) with \( t \geq -1 \). Let

\[
D(t;r,s,q) = \frac{r-s}{2} \sum_{i=1}^{2} w_i [x_i - A_2]^2 - P_{2,r} + P_{2,s}.
\]

For \( t \geq 0 \), \( D(t;r,s,q) \geq 0 \) implies the validity of the left-hand side inequality of (1.3) while for \( -1 \leq t \leq 0 \), \( D(t;r,s,q) \leq 0 \) implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of \( D(t;r,s,q) \) around \( t = 0 \), it is readily seen that \( D(0;r,s,q) = D^{(1)}(0;r,s,q) = D^{(2)}(0;r,s,q) = 0 \). Thus, by the Lagrangian remainder term of the Taylor expansion,

\[
D(t;r,s,q) = \frac{D^{(3)}(\theta t;r,s,q)}{3!} t^3
\]

with \( 0 < \theta < 1 \).

Since

\[
\lim_{t \to 0^+} D^{(3)}(\theta t;r,s,q) = D^{(3)}(0;r,s,q),
\]

a necessary condition for (1.3) to hold is \( D^{(3)}(0;r,s,q) \geq 0 \) for \( 0 \leq q \leq 1 \). The calculation yields

\[
D^{(3)}(0;r,s,q) = (r-s)q(q-1)((3-2r-2s)q-(3-r-s)).
\]

It is easy to check that this is equivalent to \( 0 \leq r + s \leq 3 \).

**Theorem 3.2.** Let \( r > s \). If \( r = 1 \), inequality (1.3) holds if and only if \( -1 \leq s < 1 \). If \( s = 1 \), inequality (1.3) holds if and only if \( 1 < r \leq 2 \).

**Proof.** A result of Gao [9] shows the validity of (1.2) for \( r = 1, -1 \leq s < 1, x_n \leq 1/2 \), and a similar result of his [10] shows the validity of (1.2) for \( s = 1, 1 < r \leq 2, x_n \leq 1/2 \). Thus, it follows from Theorem 2.1 that (1.3) holds for \( r = 1, -1 \leq s < 1, \) and \( s = 1, 1 < r \leq 2 \). This proves the “if” part of the statement, and the “only if” part follows from the previous lemma.
We note here that a special case of Theorem 3.2 answers an open problem of Mercer [11], namely, we have shown that
\[ \frac{1}{x_1} \sigma_n \geq A_n - H_n \geq \frac{1}{x_n} \sigma_n. \] (3.5)

4. Two lemmas

**Lemma 4.1.** Let \( x, b, u, \) and \( v \) be real numbers with \( 0 < x \leq b, u \geq 1, v \geq 0, \) and \( u + v \geq 2, \) then \( f(u, v, x, b) \leq 0, \) where

\[ f(u, v, x, b) = \frac{u + v - 1}{ux + vb} + \frac{1}{x^2(u/x + v/b)} - \frac{1}{x} - \frac{u + v - 2}{b^2(u + v)^2} v(x - b) \] (4.1)

with equality holding if and only if \( x = b \) or \( v = 0 \) or \( u = v = 1. \)

**Proof.** Let \( x < b, u > 1, \) and \( v > 1. \) We have

\[ f(u, v, x, b) = v(b - x) \left( \frac{(u - 1)b + (v - 1)x}{x(bv + ux)(bu + vx)} + \frac{(u - 1) + (v - 1)}{b^2(u + v)^2} \right) \]
\[ < \frac{v(b - x)}{xb^2(u + v)^2} \left( [(u - 1) + (v - 1)x - (u - 1)b - (v - 1)x] \right) \] (4.2)
\[ = - \frac{v(u - 1)(b - x)^2}{xb^2(u + v)^2} < 0 \]

since \( b^2(u + v)^2 > (bv + ux)(bu + vx). \) Thus, we conclude that \( f(u, v, x, b) \leq 0 \) for \( 0 < x \leq b, u \geq 1, v \geq 0, \) and \( u + v \geq 2. \)

**Lemma 4.2.** Let \( x, a, b, u, v, \) and \( s \) be real numbers with \( 0 < x \leq a \leq b, u \geq 1, v \geq 1, u + v \geq 3, \) and \( 0 \leq s \leq v, \) then

\[ \frac{u + v - 1}{ux + sa + (v - s)b} + \frac{1}{x^2(u/x + s/a + (v - s)/b)} - \frac{1}{x} - \frac{u + v - 2}{b^2(u + v)^2} (s(x - a) + (v - s)(x - b)) \leq 0 \] (4.3)

with equality holding if and only if one of the following cases is true: \( 1 \) \( x = a = b; \) \( 2 \) \( s = 0 \) and \( x = b; \) \( 3 \) \( s = v \) and \( x = a. \)

**Proof.** Let \( M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b \}. \) Furthermore, we define \( H(s, a) \) as the expression on the left-hand side of (4.3), where \( (s, a) \in M. \) It suffices to show that \( H(s, a) < 0. \) We denote the absolute minimum of \( H \) by \( m = (s_0, a_0). \) If \( m \) is an interior point of \( M, \) then we obtain

\[ 0 = \frac{1}{s} \frac{\partial H}{\partial a} - \frac{1}{a} \frac{\partial H}{\partial s} \bigg|_{(s, a) = (s_0, a_0)} = \frac{b - a}{x^4 a^2 b (u/x + s/a + (v - s)/b)^2} > 0. \] (4.4)
Hence, \( m \) is a boundary point of \( M \), so we get

\[
m \in \{ (s_0, x), (s_0, b), (0, a_0), (v, a_0) \}.
\] (4.5)

Using Lemma 4.1, we obtain

\[
H(s_0, x) = f(u + s_0, v - s_0, x, b) \leq 0,
\]

\[
H(s_0, b) = H(0, a_0) = f(u, v, x, b) \leq 0,
\]

\[
H(v, a_0) = f(u, v, x, a_0) - \frac{v(u + v - 2)(a_0 - x)(b^2 - a_0^2)}{a_0^2 b^2 (u + v)^2} \leq 0.
\] (4.6)

Thus, we get that if \( (s, a) \in M \), then \( H(s, a) \leq 0 \). The conditions for equality can be easily checked using Lemma 4.1.

5. A sharpening of Ky Fan’s inequality. In this section, we prove the following theorem.

**Theorem 5.1.** For \( 0 < x_1 \leq \cdots \leq x_n, q = \min \{ \omega_i \} \),

\[
\frac{1 - 2q}{2x_1^2} \sigma_n \geq (1 - q) \ln A_n + q \ln H_n - \ln G_n \geq \frac{1 - 2q}{2x_n^2} \sigma_n,
\] (5.1)

\[
\frac{1 - 2q}{2x_1^2} \sigma_n \geq \ln G_n - q \ln A_n - (1 - q) \ln H_n \geq \frac{1 - 2q}{2x_n^2} \sigma_n
\] (5.2)

with equality holding if and only if \( q = 1/2 \) or \( x_1 = \cdots = x_n \).

**Proof.** The proof uses the ideas in [6]. We prove the right-hand side inequality of (5.1); the proofs for other inequalities are similar. Fix \( 0 < x = x_1, x_n = b \) with \( x_1 < x_n, n \geq 2 \); we define

\[
f_n'(x_n, q) = (1 - q) \ln A_n + q \ln H_n - \ln G_n - \frac{1 - 2q}{2x_n^2} \sigma_n,
\] (5.3)

where we regard \( A_n, G_n, \) and \( H_n \) as functions of \( x_n = (x_1, \ldots, x_n) \).

We then have

\[
g_n(x_2, \ldots, x_{n-1}) := \frac{1}{\omega_1} \frac{\partial f_n}{\partial x_1} = \frac{1 - q}{A_n} + \frac{q H_n}{x_1^2} - \frac{1}{x_1} - \frac{1 - 2q}{x_n^2} (x_1 - A_n).
\] (5.4)

We want to show that \( g_n \leq 0 \). Let \( D = \{ (x_2, \ldots, x_{n-1}) \in R^{n-2} \mid 0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b \} \). Let \( a = (a_2, \ldots, a_{n-1}) \in D \) be the point in which the absolute minimum of \( g_n \) is reached. Next, we show that

\[
a = (x, \ldots, x, a, \ldots, a, b, \ldots, b) \quad \text{with} \quad x < a < b,
\] (5.5)

where the numbers \( x, a, \) and \( b \) appear \( r, s, \) and \( t \) times, respectively, with \( r, s, t \geq 0 \) and \( r + s + t = n-2 \).
Suppose not, this implies that two components of $a$ have different values and are interior points of $D$. We denote these values by $a_k$ and $a_l$. Partial differentiation leads to

\[
\frac{B}{a_i^2} + C = 0
\]

for $i = k, l$, where

\[
B = q \frac{H_n^2}{x_1^2}, \quad C = -\frac{1 - q}{A_n^2} + \frac{1 - 2q}{x_n^2}.
\]

Since $z \rightarrow B/z^2 + C$ is strictly monotonic for $z > 0$, then (5.6) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus, (5.5) is valid and it suffices to show that $g_n \leq 0$ for the case $n = 2, 3$.

When $n = 2$, by setting $x_1 = x$, $x_2 = b$, $\omega_1/q = u$, and $\omega_2/q = v$, we can identify $g_2$ as (4.1), and the result follows from Lemma 4.1.

When $n = 3$, by setting $x_1 = x$, $x_2 = a$, $x_3 = b$, $\omega_1/q = u$, $\omega_2/q = s$, and $\omega_3/q = v - s$, we can identify $g_3$ as (4.3), and the result follows from Lemma 4.2.

Thus, we have shown that $g_n = (1/\omega_1) \partial f_n/\partial x_1 \leq 0$ with equality holding if and only if $n = 1$ or $n = 2$, $q = 1/2$. By letting $x_1$ tend to $x_2$, we have

\[
f_n(x_n, q) \geq f_{n-1}(x_{n-1}, q) \geq f_{n-1}(x_{n-1}, q'),
\]

where $x_{n-1} = (x_2, \ldots, x_n)$ with weights $\omega_1 + \omega_2, \ldots, \omega_{n-1}, \omega_n$ and $q' = \min\{\omega_1 + \omega_2, \ldots, \omega_n\}$. Here, we have used the following inequality, which is a consequence of (3.5) (see [9]):

\[
\ln A_n - \ln H_n \geq \frac{1}{x_n^2} \sigma_n.
\]

It then follows by induction that $f_n \geq f_{n-1} \geq \cdots \geq f_2 = 0$ when $q = 1/2$ in $f_2$ or else $f_n \geq f_{n-1} \geq \cdots \geq f_1 = 0$, and this completes the proof.

We note that the above theorem gives a sharpening of Sierpiński’s inequality [13], originally stated for the unweighted case ($\omega_i = 1/n$) as

\[
H_n^{n-1} A_n \leq G_n \leq A_n^{n-1} H_n.
\]

The following corollary gives refinements of (1.4).
Corollary 5.2. For $0 < x_1 \leq \cdots \leq x_n < 1$, $q = \min\{\omega_i\}$,

\[
\left( \frac{A_n (1-q) H_n^{1-q}}{G_n} \right)^{(1-x_1)^2/x_1^2} \geq \frac{A_n^{1-q} H_n^q}{G_n} \geq \left( \frac{A_n^{1} H_n}{G_n} \right)^{(1-x_n)^2/x_n^2},
\]

\[(5.11)\]

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ or $q = 1/2$.

Proof. This is a direct consequence of Theorem 5.1, following from a similar argument as in [12].

6. Concluding remarks. We note that if for $x_n \leq 1/2$, we have

\[
\left( \frac{x_1}{1-x_1} \right)^\beta < \frac{P_{n,r} - P_{n,s}}{P_{n,r} - P_{n,s}} < \left( \frac{x_n}{1-x_n} \right)^\alpha,
\]

\[(6.1)\]

then $\beta \geq 1$ and $\alpha \leq 1$; otherwise, by letting $\epsilon$ tend to 0 in (2.1), we get contradictions.

It was conjectured that an additive companion of (1.4) is true (see [1])

\[
n(G_n - G_n') \leq (n-1)(A_n - A_n') + H_n - H_n'.
\]

\[(6.2)\]

In [3], Alzer asked if the above conjecture is true and whether there exists a weighted version. Based on what we have got in this paper, it is natural to give the following conjecture of the weighed version of (6.2).

Conjecture 6.1. For $0 < x_1 \leq \cdots \leq x_n \leq 1/2$ and $q = \min\{\omega_i\}$,

\[
G_n - G_n' \leq (1-q)(A_n - A_n') + q(H_n - H_n').
\]

\[(6.3)\]

Recently, Alzer et al. [6] asked the following question: what is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

\[
\alpha(A_n - A_n') + (1-\alpha)(H_n - H_n') \leq G_n - G_n' \leq \beta(A_n - A_n') + (1-\beta)(H_n - H_n').
\]

\[(6.4)\]

for all $x_i \in (0,1/2]$ ($i = 1, \ldots, n$)?

We note here that $\alpha \leq 0$ since the left-hand side inequality above can be written as

\[
\alpha A_n + (1-\alpha)H_n - G_n \leq \alpha A_n' + (1-\alpha)H_n' - G_n'.
\]

\[(6.5)\]
By a similar argument as in the proof of Theorem 2.1, replacing \((x_1, \ldots, x_n)\) by \((\epsilon x_1, \ldots, \epsilon x_n)\) and letting \(\epsilon\) tend to 0 in (6.5), we find that (6.5) implies that

\[ \alpha A_n + (1 - \alpha)H_n - G_n \leq 0 \]  
(6.6)

for any \(x\). If we further let \(x_1\) tend to 0 in (6.6), we get

\[ \alpha A_n \leq 0 \]  
(6.7)

which implies that \(\alpha \leq 0\).

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References


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