KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

PENG GAO

Received 23 July 2002

We prove an equivalent relation between Ky Fan-type inequalities and certain bounds for the differences of means. We also generalize a result of Alzer et al. (2001).

2000 Mathematics Subject Classification: 26D15, 26D20.

1. Introduction. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^n \omega_i x_i^r)^{1/r}$, where $\omega_i > 0$, $1 \leq i \leq n$ with $\sum_{i=1}^n \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Here, $P_{n,0}(\mathbf{x}) = \prod_{i=1}^n x_i^{\omega_i}$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \rightarrow 0^+$, which can be proved by noting that if $p(r) = \ln(\sum_{i=1}^n \omega_i x_i^r)$, then $p'(0) = \ln(\prod_{i=1}^n x_i^{\omega_i}) = \ln(P_{n,0}(\mathbf{x}))$. We write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$ when there is no risk of confusion.

In this paper, we assume that $0 < x_1 \leq x_2 \leq \dots \leq x_n$. With any given $\mathbf{x}$, we associate $\mathbf{x}' = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$ and write $A_n = P_{n,1}$, $G_n = P_{n,0}$, and $H_n = P_{n,-1}$. When $1 - x_i \geq 0$ for all i , we define $A'_n = P_{n,1}(\mathbf{x}')$ and similarly for G'_n and H'_n . We also let $\sigma_n = \sum_{i=1}^n \omega_i [x_i - A_n]^2$.

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published by Beckenbach and Bellman [7].

THEOREM 1.1. For $x_i \in (0, 1/2]$,

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n} \quad (1.1)$$

with equality holding if and only if $x_1 = \dots = x_n$.

In this paper, we consider the validity of the following additive Ky Fan-type inequalities (with $x_1 < x_n < 1$):

$$\frac{x_1}{1 - x_1} < \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} < \frac{x_n}{1 - x_n}. \quad (1.2)$$

Note that by a change of variables $x_i \rightarrow 1 - x_i$, the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). We can deduce (see [9]) Theorem 1.1 from the case $r = 1$, $s = 0$, and $x_n \leq 1/2$ in (1.2), which is a result

of Alzer [5]. Gao [9] later proved the validity of (1.2) for $r = 1$, $-1 \leq s < 1$, and $x_n \leq 1/2$.

What is worth mentioning is a nice result of Mercer [12] who showed that the validity of $r = 1$ and $s = 0$ in (1.2) is a consequence of a result of Cartwright and Field [8] who established the validity of $r = 1$ and $s = 0$ for the following bounds for the differences between power means ($r > s$):

$$\frac{r-s}{2x_1} \sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2x_n} \sigma_n, \quad (1.3)$$

where the constant $(r-s)/2$ is the best possible (see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all $r > s$. We refer the reader to the survey article [2] and the references therein for an account of Ky Fan's inequality, and to [4, 5, 10, 11] for other interesting refinements and extensions of (1.3).

Mercer's result reveals a close relation between (1.3) and (1.2), and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed r and s . As a consequence of this result, we give a characterization of the validity of (1.3) for $r = 1$ or $s = 1$. A solution of an open problem from [11] is also given.

Among the numerous sharpenings of Ky Fan's inequality in the literature, we have the following inequalities connecting the three classical means (with $\omega_i = 1/n$ here):

$$\left(\frac{H_n}{H'_n}\right)^{n-1} \frac{A_n}{A'_n} \leq \left(\frac{G_n}{G'_n}\right)^n \leq \left(\frac{A_n}{A'_n}\right)^{n-1} \frac{H_n}{H'_n}. \quad (1.4)$$

The right-hand side inequality of (1.4) is due to W. L. Wang and P. F. Wang [14] and the left-hand side inequality was recently proved by Alzer et al. [6].

It is natural to ask whether we can extend the above inequality to the weighted case, and using the same idea as in [6], we show that this is indeed true in Section 5.

2. The main theorem

THEOREM 2.1. *For fixed $r > s$, the following inequalities are equivalent: (i) inequality (1.2) for $x_n \leq 1/2$; (ii) inequality (1.2); (iii) inequality (1.3).*

PROOF. (iii) $\Rightarrow$ (ii) follows from a similar argument as given in [12], (ii) $\Rightarrow$ (i) is trivial, so it suffices to show that (i) $\Rightarrow$ (iii).

Fix $r > s$ assuming that (1.2) holds for $x_n \leq 1/2$. Without loss of generality, we can assume that $x_1 < x_n$. For a given $\mathbf{x} = (x_1, x_2, \dots, x_n)$, let $\mathbf{y} = (\epsilon x_1, \epsilon x_2, \dots, \epsilon x_n)$. We can choose ϵ small so that $\epsilon x_n \leq 1/2$. Now, applying the right-hand side inequality (1.2) for $\mathbf{y}$, we get

$$x_n(P_{n,r}(\mathbf{x}) - P_{n,s}(\mathbf{x})) > \frac{1 - \epsilon x_n}{\epsilon^2} (P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}')). \quad (2.1)$$

Let $f(\epsilon) = P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}')$, then $f'(0) = 0$ and $f''(0) = (r-s)\sigma_n$. Thus, by letting ϵ tend to 0, it is easy to verify that the limit of the expression on the right-hand side of (2.1) is $(r-s)\sigma_n/2$. We can consider the left-hand side of (1.2) by a similar argument and this completes the proof. $\square$

3. An application of Theorem 2.1

LEMMA 3.1. *If inequality (1.3) holds for $r > s$, then $0 \leq r + s \leq 3$.*

PROOF. Let $n = 2$, and write $\omega_1 = 1 - q$, $\omega_2 = q$, $x_1 = 1$, and $x_2 = 1 + t$ with $t \geq -1$. Let

$$D(t; r, s, q) = \frac{r-s}{2} \sum_{i=1}^2 w_i [x_i - A_2]^2 - P_{2,r} + P_{2,s}. \quad (3.1)$$

For $t \geq 0$, $D(t; r, s, q) \geq 0$ implies the validity of the left-hand side inequality of (1.3) while for $-1 \leq t \leq 0$, $D(t; r, s, q) \leq 0$ implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of $D(t; r, s, q)$ around $t = 0$, it is readily seen that $D(0; r, s, q) = D^{(1)}(0; r, s, q) = D^{(2)}(0; r, s, q) = 0$. Thus, by the Lagrangian remainder term of the Taylor expansion,

$$D(t; r, s, q) = \frac{D^{(3)}(\theta t; r, s, q)}{3!} t^3 \quad (3.2)$$

with $0 < \theta < 1$.

Since

$$\lim_{t \rightarrow 0^+} D^{(3)}(\theta t; r, s, q) = D^{(3)}(0; r, s, q), \quad (3.3)$$

a necessary condition for (1.3) to hold is $D^{(3)}(0; r, s, q) \geq 0$ for $0 \leq q \leq 1$. The calculation yields

$$D^{(3)}(0; r, s, q) = (r-s)q(q-1)((3-2r-2s)q - (3-r-s)). \quad (3.4)$$

It is easy to check that this is equivalent to $0 \leq r + s \leq 3$. $\square$

THEOREM 3.2. *Let $r > s$. If $r = 1$, inequality (1.3) holds if and only if $-1 \leq s < 1$. If $s = 1$, inequality (1.3) holds if and only if $1 < r \leq 2$.*

PROOF. A result of Gao [9] shows the validity of (1.2) for $r = 1$, $-1 \leq s < 1$, $x_n \leq 1/2$, and a similar result of his [10] shows the validity of (1.2) for $s = 1$, $1 < r \leq 2$, $x_n \leq 1/2$. Thus, it follows from Theorem 2.1 that (1.3) holds for $r = 1$, $-1 \leq s < 1$, and $s = 1$, $1 < r \leq 2$. This proves the “if” part of the statement, and the “only if” part follows from the previous lemma. $\square$

We note here that a special case of [Theorem 3.2](#) answers an open problem of Mercer [\[11\]](#), namely, we have shown that

$$\frac{1}{x_1}\sigma_n \geq A_n - H_n \geq \frac{1}{x_n}\sigma_n. \quad (3.5)$$

4. Two lemmas

LEMMA 4.1. *Let x, b, u , and v be real numbers with $0 < x \leq b$, $u \geq 1$, $v \geq 0$, and $u + v \geq 2$, then $f(u, v, x, b) \leq 0$, where*

$$f(u, v, x, b) = \frac{u+v-1}{ux+vb} + \frac{1}{x^2(u/x+v/b)} - \frac{1}{x} - \frac{u+v-2}{b^2(u+v)^2}v(x-b) \quad (4.1)$$

with equality holding if and only if $x = b$ or $v = 0$ or $u = v = 1$.

PROOF. Let $x < b$, $u > 1$, and $v > 1$. We have

$$\begin{aligned} f(u, v, x, b) &= v(b-x) \left(-\frac{(u-1)b+(v-1)x}{x(bv+ux)(bu+vx)} + \frac{(u-1)+(v-1)}{b^2(u+v)^2} \right) \\ &< \frac{v(b-x)}{xb^2(u+v)^2} [((u-1)+(v-1))x - (u-1)b - (v-1)x] \quad (4.2) \\ &= -\frac{v(u-1)(b-x)^2}{xb^2(u+v)^2} < 0 \end{aligned}$$

since $b^2(u+v)^2 > (bv+ux)(bu+vx)$. Thus, we conclude that $f(u, v, x, b) \leq 0$ for $0 < x \leq b$, $u \geq 1$, $v \geq 0$, and $u + v \geq 2$. $\square$

LEMMA 4.2. *Let x, a, b, u, v , and s be real numbers with $0 < x \leq a \leq b$, $u \geq 1$, $v \geq 1$, $u + v \geq 3$, and $0 \leq s \leq v$, then*

$$\begin{aligned} &\frac{u+v-1}{ux+sa+(v-s)b} + \frac{1}{x^2(u/x+s/a+(v-s)/b)} - \frac{1}{x} \\ &- \frac{u+v-2}{b^2(u+v)^2}(s(x-a)+(v-s)(x-b)) \leq 0 \end{aligned} \quad (4.3)$$

with equality holding if and only if one of the following cases is true: (1) $x = a = b$; (2) $s = 0$ and $x = b$; (3) $s = v$ and $x = a$.

PROOF. Let $M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b\}$. Furthermore, we define $H(s, a)$ as the expression on the left-hand side of [\(4.3\)](#), where $(s, a) \in M$. It suffices to show that $H(s, a) < 0$. We denote the absolute minimum of H by $m = (s_0, a_0)$. If m is an interior point of M , then we obtain

$$0 = \frac{1}{s} \frac{\partial H}{\partial a} - \frac{1}{a-b} \frac{\partial H}{\partial s} \Big|_{(s,a)=(s_0,a_0)} = \frac{b-a}{x^4 a^2 b (u/x + s/a + (v-s)/b)^2} > 0. \quad (4.4)$$

Hence, m is a boundary point of M , so we get

$$m \in \{(s_0, x), (s_0, b), (0, a_0), (v, a_0)\}. \quad (4.5)$$

Using Lemma 4.1, we obtain

$$\begin{aligned} H(s_0, x) &= f(u + s_0, v - s_0, x, b) \leq 0, \\ H(s_0, b) &= H(0, a_0) = f(u, v, x, b) \leq 0, \\ H(v, a_0) &= f(u, v, x, a_0) - \frac{v(u + v - 2)(a_0 - x)(b^2 - a_0^2)}{a_0^2 b^2 (u + v)^2} \leq 0. \end{aligned} \quad (4.6)$$

Thus, we get that if $(s, a) \in M$, then $H(s, a) \leq 0$. The conditions for equality can be easily checked using Lemma 4.1. $\square$

5. A sharpening of Ky Fan's inequality. In this section, we prove the following theorem.

THEOREM 5.1. For $0 < x_1 \leq \dots \leq x_n$, $q = \min\{\omega_i\}$,

$$\frac{1-2q}{2x_1^2} \sigma_n \geq (1-q) \ln A_n + q \ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2} \sigma_n, \quad (5.1)$$

$$\frac{1-2q}{2x_1^2} \sigma_n \geq \ln G_n - q \ln A_n - (1-q) \ln H_n \geq \frac{1-2q}{2x_n^2} \sigma_n \quad (5.2)$$

with equality holding if and only if $q = 1/2$ or $x_1 = \dots = x_n$.

PROOF. The proof uses the ideas in [6]. We prove the right-hand side inequality of (5.1); the proofs for other inequalities are similar. Fix $0 < x = x_1$, $x_n = b$ with $x_1 < x_n$, $n \geq 2$; we define

$$f_n(x_n, q) = (1-q) \ln A_n + q \ln H_n - \ln G_n - \frac{1-2q}{2x_n^2} \sigma_n, \quad (5.3)$$

where we regard A_n , G_n , and H_n as functions of $\mathbf{x}_n = (x_1, \dots, x_n)$.

We then have

$$g_n(x_2, \dots, x_{n-1}) := \frac{1}{\omega_1} \frac{\partial f_n}{\partial x_1} = \frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1} - \frac{1-2q}{x_n^2} (x_1 - A_n). \quad (5.4)$$

We want to show that $g_n \leq 0$. Let $D = \{(x_2, \dots, x_{n-1}) \in R^{n-2} \mid 0 < x \leq x_2 \leq \dots \leq x_{n-1} \leq b\}$. Let $\mathbf{a} = (a_2, \dots, a_{n-1}) \in D$ be the point in which the absolute minimum of g_n is reached. Next, we show that

$$\mathbf{a} = (x, \dots, x, a, \dots, a, b, \dots, b) \quad \text{with } x < a < b, \quad (5.5)$$

where the numbers x , a , and b appear r , s , and t times, respectively, with $r, s, t \geq 0$ and $r + s + t = n - 2$.

Suppose not, this implies that two components of $\mathbf{a}$ have different values and are interior points of D . We denote these values by a_k and a_l . Partial differentiation leads to

$$\frac{B}{a_i^2} + C = 0 \quad (5.6)$$

for $i = k, l$, where

$$B = q \frac{H_n^2}{x_1^2}, \quad C = -\frac{1-q}{A_n^2} + \frac{1-2q}{x_n^2}. \quad (5.7)$$

Since $z \mapsto B/z^2 + C$ is strictly monotonic for $z > 0$, then (5.6) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus, (5.5) is valid and it suffices to show that $g_n \leq 0$ for the case $n = 2, 3$.

When $n = 2$, by setting $x_1 = x$, $x_2 = b$, $\omega_1/q = u$, and $\omega_2/q = v$, we can identify g_2 as (4.1), and the result follows from Lemma 4.1.

When $n = 3$, by setting $x_1 = x$, $x_2 = a$, $x_3 = b$, $\omega_1/q = u$, $\omega_2/q = s$, and $\omega_3/q = v - s$, we can identify g_3 as (4.3), and the result follows from Lemma 4.2.

Thus, we have shown that $g_n = (1/\omega_1)\partial f_n/\partial x_1 \leq 0$ with equality holding if and only if $n = 1$ or $n = 2$, $q = 1/2$. By letting x_1 tend to x_2 , we have

$$f_n(\mathbf{x}_n, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q'), \quad (5.8)$$

where $\mathbf{x}_{n-1} = (x_2, \dots, x_n)$ with weights $\omega_1 + \omega_2, \dots, \omega_{n-1}, \omega_n$ and $q' = \min\{\omega_1 + \omega_2, \dots, \omega_n\}$. Here, we have used the following inequality, which is a consequence of (3.5) (see [9]):

$$\ln A_n - \ln H_n \geq \frac{1}{x_n^2} \sigma_n. \quad (5.9)$$

It then follows by induction that $f_n \geq f_{n-1} \geq \dots \geq f_2 = 0$ when $q = 1/2$ in f_2 or else $f_n \geq f_{n-1} \geq \dots \geq f_1 = 0$, and this completes the proof. $\square$

We note that the above theorem gives a sharpening of Sierpiński's inequality [13], originally stated for the unweighted case ($\omega_i = 1/n$) as

$$H_n^{n-1} A_n \leq G_n \leq A_n^{n-1} H_n. \quad (5.10)$$

The following corollary gives refinements of (1.4).

COROLLARY 5.2. For $0 < x_1 \leq \cdots \leq x_n < 1$, $q = \min\{\omega_i\}$,

$$\begin{aligned} \left(\frac{A_n^{(1-q)} H_n^{q'}}{G_n} \right)^{(1-x_1)^2/x_1^2} &\geq \frac{A_n^{1-q} H_n^q}{G_n} \geq \left(\frac{A_n^{(1-q)} H_n^{q'}}{G_n} \right)^{(1-x_n)^2/x_n^2}, \\ \left(\frac{G_n}{A_n^{q'} H_n^{(1-q)'}} \right)^{(1-x_1)^2/x_1^2} &\geq \frac{G_n}{A_n^q H_n^{1-q}} \geq \left(\frac{G_n}{A_n^{q'} H_n^{(1-q)'}} \right)^{(1-x_n)^2/x_n^2}, \end{aligned} \quad (5.11)$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ or $q = 1/2$.

PROOF. This is a direct consequence of [Theorem 5.1](#), following from a similar argument as in [\[12\]](#). $\square$

6. Concluding remarks. We note that if for $x_n \leq 1/2$, we have

$$\left(\frac{x_1}{1-x_1} \right)^\beta < \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} < \left(\frac{x_n}{1-x_n} \right)^\alpha, \quad (6.1)$$

then $\beta \geq 1$ and $\alpha \leq 1$; otherwise, by letting ϵ tend to 0 in [\(2.1\)](#), we get contradictions.

It was conjectured that an additive companion of [\(1.4\)](#) is true (see [\[1\]](#))

$$n(G_n - G'_n) \leq (n-1)(A_n - A'_n) + H_n - H'_n. \quad (6.2)$$

In [\[3\]](#), Alzer asked if the above conjecture is true and whether there exists a weighted version. Based on what we have got in this paper, it is natural to give the following conjecture of the weighed version of [\(6.2\)](#).

CONJECTURE 6.1. For $0 < x_1 \leq \cdots \leq x_n \leq 1/2$ and $q = \min\{\omega_i\}$,

$$G_n - G'_n \leq (1-q)(A_n - A'_n) + q(H_n - H'_n). \quad (6.3)$$

Recently, Alzer et al. [\[6\]](#) asked the following question: what is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

$$\alpha(A_n - A'_n) + (1-\alpha)(H_n - H'_n) \leq G_n - G'_n \leq \beta(A_n - A'_n) + (1-\beta)(H_n - H'_n) \quad (6.4)$$

for all $x_i \in (0, 1/2]$ ($i = 1, \dots, n$)?

We note here that $\alpha \leq 0$ since the left-hand side inequality above can be written as

$$\alpha A_n + (1-\alpha)H_n - G_n \leq \alpha A'_n + (1-\alpha)H'_n - G'_n. \quad (6.5)$$

By a similar argument as in the proof of [Theorem 2.1](#), replacing $(x_1, \dots, x_n)$ by $(\epsilon x_1, \dots, \epsilon x_n)$ and letting ϵ tend to 0 in (6.5), we find that (6.5) implies that

$$\alpha A_n + (1 - \alpha)H_n - G_n \leq 0 \quad (6.6)$$

for any $\mathbf{x}$. If we further let x_1 tend to 0 in (6.6), we get

$$\alpha A_n \leq 0 \quad (6.7)$$

which implies that $\alpha \leq 0$.

ACKNOWLEDGMENT. The author is grateful to the referees for their helpful comments and suggestions.

REFERENCES

- [1] H. Alzer, *An inequality for arithmetic and harmonic means*, Aequationes Math. **46** (1993), no. 3, 257–263.
- [2] ———, *The inequality of Ky Fan and related results*, Acta Appl. Math. **38** (1995), no. 3, 305–354.
- [3] ———, *On Ky Fan's inequality and its additive analogue*, J. Math. Anal. Appl. **204** (1996), no. 1, 291–297.
- [4] ———, *A new refinement of the arithmetic mean–geometric mean inequality*, Rocky Mountain J. Math. **27** (1997), no. 3, 663–667.
- [5] ———, *On an additive analogue of Ky Fan's inequality*, Indag. Math. (N.S.) **8** (1997), no. 1, 1–6.
- [6] H. Alzer, S. Ruscheweyh, and L. Salinas, *On Ky Fan-type inequalities*, Aequationes Math. **62** (2001), no. 3, 310–320.
- [7] E. F. Beckenbach and R. Bellman, *Inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Bd. 30, Springer-Verlag, Berlin, 1961.
- [8] D. I. Cartwright and M. J. Field, *A refinement of the arithmetic mean–geometric mean inequality*, Proc. Amer. Math. Soc. **71** (1978), no. 1, 36–38.
- [9] P. Gao, *A generalization of Ky Fan's inequality*, Int. J. Math. Math. Sci. **28** (2001), no. 7, 419–425.
- [10] ———, *Certain bounds for the differences of means*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Article 7.
- [11] A. McD. Mercer, *Bounds for A–G, A–H, G–H, and a family of inequalities of Ky Fan's type, using a general method*, J. Math. Anal. Appl. **243** (2000), no. 1, 163–173.
- [12] P. R. Mercer, *A note on Alzer's refinement of an additive Ky Fan inequality*, Math. Inequal. Appl. **3** (2000), no. 1, 147–148.
- [13] W. Sierpiński, *On an inequality for arithmetic, geometric and harmonic means*, Warsch. Sitzungsber. **2** (1909), 354–358 (Polish).
- [14] W. L. Wang and P. F. Wang, *A class of inequalities for the symmetric functions*, Acta Math. Sinica **27** (1984), no. 4, 485–497 (Chinese).

Peng Gao: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: penggao@umich.edu