A REGRESSION CHARACTERIZATION OF INVERSE GAUSSIAN DISTRIBUTIONS AND APPLICATION TO EDF GOODNESS-OF-FIT TESTS

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We give a new characterization of inverse Gaussian distributions using the regression of a suitable statistic based on a given random sample. A corollary of this result is a characterization of inverse Gaussian distribution based on a conditional joint density function of the sample. Application of this corollary as a transformation in the procedure to construct EDF (empirical distribution function) goodness-of-fit tests for inverse Gaussian distributions is also studied.

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1. Introduction. A distribution is an inverse Gaussian distribution with parameters $m > 0$ and $\lambda > 0$, denoted $IG(m, \lambda)$, if it has a density function given by

$$f(x; m, \lambda) = \begin{cases} \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp \left\{ -\frac{\lambda(x-m)^2}{2m^2 x} \right\} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

(See Tweedie [11].)

The characteristic function of an $IG(m, \lambda)$ distribution is

$$\varphi(t) = \exp \left\{ \frac{\lambda \left[1 - (1 - 2im^2 t \lambda^{-1})^{1/2} \right]}{m} \right\}. \quad (1.2)$$

Let X_j , $j = 1, \dots, n$, $n \geq 2$, be a random sample from an $IG(m, \lambda)$ distribution. Then, the statistics $Y = \sum_{j=1}^n X_j$ and $Z = \sum_{j=1}^n X_j^{-1} - n^2 Y^{-1}$ are jointly complete sufficient for m and λ . Y and Z are independently distributed, Y has an $IG(nm, n^2\lambda)$ distribution, and λZ has a chi-square distribution with $(n-1)$ degrees of freedom. Khatri [4] gave a characterization of the inverse Gaussian distributions based on the independence between Y and Z , then Seshadri [9]

gave several characterizations of the inverse Gaussian distributions based on the constant regression of several different statistics given Y . In this note, we give a characterization of the inverse Gaussian distributions based on the regression of a statistic given Y and Z . The corollary of this result is a characterization of the inverse Gaussian distributions based on the conditional joint density function of $X_1, \dots, X_{n-2}$, given Y and Z . The result of this corollary can be used as a transformation in the procedure to construct EDF (empirical distribution function) goodness-of-fit tests for inverse Gaussian distributions.

2. Characterization results. The conditional joint density function of $X_1, \dots, X_{n-2}$, given $Y = y > 0$, $Z = z > 0$, is

$$f_{X_1, \dots, X_{n-2} | Y, Z}(x_1, \dots, x_{n-2} | y, z) = \begin{cases} \frac{2\Gamma((n-1)/2)y^{3/2}}{n\pi^{(n-1)/2} \prod_{j=1}^{n-2} x_j^{3/2} (y - \sum_{j=1}^{n-2} x_j)^{1/2} z^{(n-1)/2}} \\ \times \left[\left(y - \sum_{j=1}^{n-2} x_j \right)^2 \left(z + n^2 y^{-1} - \sum_{j=1}^{n-2} x_j^{-1} \right) - 4 \left(y - \sum_{j=1}^{n-2} x_j \right) \right]^{-1/2} \\ \text{for } \sum_{j=1}^{n-2} x_j < y, \left(y - \sum_{j=1}^{n-2} x_j \right) \left(z + n^2 y^{-1} - \sum_{j=1}^{n-2} x_j^{-1} \right) < 4, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

From (2.1), the UMVUE of the density function at a point $x_1 > 0$ is given by

$$f_{X_1 | Y, Z}(x_1 | y, z) = \begin{cases} \frac{(n-1)\Gamma((n-1)/2)}{n\sqrt{\pi}\Gamma((n-2)/2)} \\ \times \frac{y^{3/2} [z + n^2 y^{-1} - x_1^{-1} - (n-1)^2 (y - x_1)^{-1}]^{(n-2)/2-1}}{x_1^{3/2} (y - x_1)^{3/2} z^{(n-1)/2}} \\ \text{for } x_1 < y, z + n^2 y^{-1} - x_1^{-1} - (n-1)^2 (y - x_1)^{-1} > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.2)$$

where $y, z > 0$. (See Chhikara and Folks [1].)

Let $T = X_1[Z + n^2Y^{-1} - X_1^{-1} - (n-1)^2(Y - X_1)^{-1}]$. $E[T|Y, Z]$ can be computed in two different ways. On the one hand,

$$\begin{aligned} E[T|Y, Z] &= E\left[X_1\left[Z + n^2Y^{-1} - X_1^{-1} - (n-1)^2(Y - X_1)^{-1}|Y, Z\right]\right] \\ &= \frac{YZ}{n} + (n-1) - (n-1)^2E\left[X_1(Y - X_1)^{-1}|Y, Z\right] \\ &= \frac{YZ}{n} + n(n-1) - (n-1)^2YE\left[(Y - X_1)^{-1}|Y, Z\right]. \end{aligned} \quad (2.3)$$

On the other hand, this expectation can be computed using the conditional density function of X_1 given by (2.2), and the following integral is taken on the support of this conditional density function:

$$E[T|Y, Z] = \int t(x)f_{X_1|Y, Z(x)} dx = \int u dv, \quad (2.4)$$

where

$$\begin{aligned} u(x) &= \frac{n-1}{n} \times \frac{\Gamma((n-1)/2)y^{3/2}[z + n^2y^{-1} - x^{-1} - (n-1)^2(y-x)^{-1}]^{(n-2)/2}}{\sqrt{\pi}\Gamma((n-2)/2)z^{(n-1)/2-1}}, \\ dv &= \frac{dx}{x^{1/2}(y-x)^{3/2}}. \end{aligned} \quad (2.5)$$

Hence,

$$v = \frac{2x^{1/2}}{y(y-x)^{1/2}}. \quad (2.6)$$

Using integration by parts,

$$\begin{aligned} E[T|Y, Z] &= -\left[\frac{n-2}{Y}\right]E[(Y - X_1) - (n-1)^2X_1^2(Y - X_1)^{-1}|Y, Z] \\ &= -n(n-1)(n-2) + (n-1)^2(n-2)YE[(Y - X_1)^{-1}|Y, Z]. \end{aligned} \quad (2.7)$$

Comparing (2.3) and (2.7),

$$E[(Y - X_1)^{-1}|Y, Z] = \frac{nY^{-1}}{n-1} + \frac{Z}{n(n-1)^3}. \quad (2.8)$$

In the following part, we construct a characterization of inverse Gaussian distributions based on regression (2.8).

If X has an inverse Gaussian distribution with the characteristic function $\varphi(t)$ given by (1.2), then take logarithm of this characteristic function following three successive differentiations and several simplifications, then $\varphi(t)$ satisfies the differential equation

$$\varphi'^4(t) - 3\varphi(t)\varphi'^2(t)\varphi''(t) - \varphi^2(t)\varphi'(t)\varphi'''(t) + 3\varphi^2(t)\varphi''^2(t) = 0. \quad (2.9)$$

Conversely, if $\varphi(t)$ is the characteristic function of a random variable X with finite $E[X^{-1}]$ and $E[X^3]$, that is, a solution of the differential equation (2.9), then, by the continuity of a characteristic function using the reverse procedure for getting (2.9), this characteristic function is (1.2). Hence, the following result is obtained.

LEMMA 2.1. *Let X be a nonnegative random variable with a nondegenerate distribution F and with finite $E[X^{-1}]$ and $E[X^3]$. Assume that $E[X] = m$ and $\text{Var}(X) = m^3/\lambda$ for some positive numbers m and λ , then F is an $\text{IG}(m, \lambda)$ if and only if its characteristic function is a solution of the differential equation (2.9).*

The following theorem is the main result of this note.

THEOREM 2.2. *Let $X_j, j = 1, \dots, n, n \geq 2$, be a random sample of n nonnegative random variables from a nondegenerate distribution F with finite $E[X]$ and $\text{Var}(X)$. Then, F is an inverse Gaussian distribution if and only if regression (2.8) holds.*

PROOF. We only need to show that if (2.8) holds, then F is an inverse Gaussian distribution.

From (2.8),

$$E\left[e^{itY}\left\{n(n-1)^3(Y-X_1)^{-1} - n^3(n-2)Y^{-1} - \sum_{j=1}^n X_j^{-1}\right\}\right] = 0. \quad (2.10)$$

From the fact that X is a random variable with finite $E[X^{-1}]$,

$$E[X^{-1}e^{itX}] = i \int_{-T}^t \varphi(u) du + \int_R x^{-1} e^{-iT x} dF(x), \quad (2.11)$$

for any constant T such that $-T < t$, where φ is the characteristic function of X (Khatri [4]), then

$$\begin{aligned} I_1(t) &= E\left[e^{it(Y-X_1)}(Y-X_1)^{-1}\right] \\ &= i \int_{-T}^t \varphi^{n-1}(u) du + \int_R x^{-1} e^{-iT x} dF^{*(n-1)}(x), \\ I_2(t) &= E[e^{itY}Y^{-1}] = i \int_{-T}^t \varphi^n(u) du + \int_R x^{-1} e^{-iT x} dF^{*(n)}(x), \\ I_3(t) &= E[e^{itX}jX_j^{-1}] = i \int_{-T}^t \varphi(u) du + \int_R x^{-1} e^{-iT x} dF(x), \end{aligned} \quad (2.12)$$

where $F^{*(k)}$ denotes the k times convolution of F . Substitute (2.12) into (2.10), simplify, and differentiate three times, the differential equation (2.9) is obtained. Then by Lemma 2.1, F is an inverse Gaussian distribution. $\square$

The following characterization of inverse Gaussian distributions based on (2.1) or (2.2) can be obtained directly from Theorem 2.2. This result will be used as a transformation in the procedure to construct EDF goodness-of-fit tests for inverse Gaussian distributions. The application of this result is discussed in Section 3.

COROLLARY 2.3. *Let X_j , $j = 1, \dots, n$, $n \geq 2$, be a random sample of nonnegative random variables from a nondegenerate distribution F with finite $E[X^{-1}]$ and $E[X^3]$. F is an inverse Gaussian distribution if and only if the conditional joint density function of $X_1, \dots, X_{n-2}$, given $Y = y > 0$ and $Z = z > 0$, is (2.1), or the conditional density function of X_1 , given $Y = y > 0$ and $Z = z > 0$, is (2.2).*

3. Application to goodness-of-fit test. Let X_j , $j = 1, \dots, n$, $n \geq 2$, be a sample of nonnegative random variables from a nondegenerate distribution F with finite $E[X^{-1}]$ and $E[X^3]$. To test whether F is an inverse Gaussian distribution, by Corollary 2.3, it is to test the equivalent simple hypothesis that whether the conditional joint density of $X_1, \dots, X_{n-2}$, given $Y = y > 0$ and $Z = z > 0$, is (1.1). The results of Rosenblatt [8] and then of Chhikara and Folks [1] are used to change the X 's sample to the U 's random sample from a distribution over the interval $(0,1)$, and the equivalent hypothesis now is whether the U 's sample is from the uniform distribution over the interval $(0,1)$. Then, any EDF test statistics can be used (D'Agostino and Stephens [2]). Nguyen and Dinh [5] used this transformation and studied the first exact EDF goodness-of-fit tests for inverse Gaussian distributions. In their study, at some alternative distributions, and with reasonable, not large, sample sizes, the exact EDF goodness-of-fit tests based on this transformation behave pretty well comparing with the other approximate EDF goodness-of-fit tests. The other goodness-of-fit tests for inverse Gaussian distributions using EDF statistics were given by Edgeman et al. [3], O'Reilly and Rueda [6], and Pavur et al. [7]. For detailed references, see Seshadri [10].

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