K-THEORY FOR CUNTZ-KRIEGER ALGEBRAS ARISING FROM REAL QUADRATIC MAPS

NUNO MARTINS, RICARDO SEVERINO, and J. SOUSA RAMOS

Received 20 September 2002

We compute the K -groups for the Cuntz-Krieger algebras $\mathcal{O}_{A_{\mathcal{K}}(f_\mu)}$, where $A_{\mathcal{K}}(f_\mu)$ is the Markov transition matrix arising from the kneading sequence $\mathcal{K}(f_\mu)$ of the one-parameter family of real quadratic maps f_μ .

2000 Mathematics Subject Classification: 37A55, 37B10, 37E05, 46L80.

Consider the one-parameter family of real quadratic maps $f_\mu : [0, 1] \rightarrow [0, 1]$ defined by $f_\mu(x) = \mu x(1 - x)$, with $\mu \in [0, 4]$. Using Milnor-Thurston kneading theory [14], Guckenheimer [5] has classified, up to topological conjugacy, a certain class of maps, which includes the quadratic family. The idea of kneading theory is to encode information about the orbits of a map in terms of infinite sequences of symbols and to exploit the natural order of the interval to establish topological properties of the map. In what follows, I denotes the unit interval $[0, 1]$ and c the unique turning point of f_μ . For $x \in I$, let

$$\varepsilon_n(x) = \begin{cases} -1, & \text{if } f_\mu^n(x) > c, \\ 0, & \text{if } f_\mu^n(x) = c, \\ +1, & \text{if } f_\mu^n(x) < c. \end{cases} \quad (1)$$

The sequence $\varepsilon(x) = (\varepsilon_n(x))_{n=0}^\infty$ is called the itinerary of x . The itinerary of $f_\mu(c)$ is called the *kneading sequence* of f_μ and will be denoted by $\mathcal{K}(f_\mu)$. Observe that $\varepsilon_n(f_\mu(x)) = \varepsilon_{n+1}(x)$, that is, $\varepsilon(f_\mu(x)) = \sigma \varepsilon(x)$, where σ is the shift map. Let $\Sigma = \{-1, 0, +1\}$ be the alphabet set. The sequences on $\Sigma^\mathbb{N}$ are ordered lexicographically. However, this ordering is not reflected by the mapping $x \rightarrow \varepsilon(x)$ because the map f_μ reverses orientation on $[c, 1]$. To take this into account, for a sequence $\varepsilon = (\varepsilon_n)_{n=0}^\infty$ of the symbols -1 , 0 , and $+1$, another sequence $\theta = (\theta_n)_{n=0}^\infty$ is defined by $\theta_n = \prod_{i=0}^n \varepsilon_i$. If $\varepsilon = \varepsilon(x)$ is the itinerary of a point $x \in I$, then $\theta = \theta(x)$ is called the *invariant coordinate* of x . The fundamental observation of Milnor and Thurston [14] is the monotonicity of the invariant coordinates:

$$x < y \implies \theta(x) \leq \theta(y). \quad (2)$$

We now consider only those kneading sequences that are periodic, that is,

$$\begin{aligned}\mathcal{H}(f_\mu) &= \varepsilon_0(f_\mu(c)) \cdots \varepsilon_{n-1}(f_\mu(c)) \varepsilon_0(f_\mu(c)) \cdots \varepsilon_{n-1}(f_\mu(c)) \cdots \\ &= (\varepsilon_0(f_\mu(c)) \cdots \varepsilon_{n-1}(f_\mu(c)))^\infty \equiv (\varepsilon_1(c) \cdots \varepsilon_n(c))^\infty\end{aligned}\quad (3)$$

for some $n \in \mathbb{N}$. The sequences $\sigma^i(\mathcal{H}(f_\mu)) = \varepsilon_{i+1}(c) \varepsilon_{i+2}(c) \cdots$, $i = 0, 1, 2, \dots$, will then determine a Markov partition of I into $n - 1$ line intervals $\{I_1, I_2, \dots, I_{n-1}\}$ [15], whose definitions will be given in the proof of [Theorem 1](#). Thus, we will have a Markov transition matrix $A_{\mathcal{H}(f_\mu)}$ defined by

$$A_{\mathcal{H}(f_\mu)} := (a_{ij}) \quad \text{with } a_{ij} = \begin{cases} 1, & \text{if } f_\mu(\text{int } I_i) \supseteq \text{int } I_j, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

It is easy to see that the matrix $A_{\mathcal{H}(f_\mu)}$ is not a permutation matrix and no row or column of $A_{\mathcal{H}(f_\mu)}$ is zero. Thus, for each one of these matrices and following the work of Cuntz and Krieger [3], one can construct the Cuntz-Krieger algebra $\mathcal{O}_{A_{\mathcal{H}(f_\mu)}}$. In [2], Cuntz proved that

$$K_0(\mathcal{O}_A) \cong \mathbb{Z}^r / (1 - A^T) \mathbb{Z}^r, \quad K_1(\mathcal{O}_A) \cong \ker(I - A^t : \mathbb{Z}^r \longrightarrow \mathbb{Z}^r), \quad (5)$$

for an $r \times r$ matrix A that satisfies a certain condition (I) (see [3]), which is readily verified by the matrices $A_{\mathcal{H}(f_\mu)}$. In [1], Bowen and Franks introduced the group $BF(A) := \mathbb{Z}^r / (1 - A) \mathbb{Z}^r$ as an invariant for flow equivalence of topological Markov subshifts determined by A .

We can now state and prove the following theorem.

THEOREM 1. *Let $\mathcal{H}(f_\mu) = (\varepsilon_1(c) \varepsilon_2(c) \cdots \varepsilon_n(c))^\infty$ for some $n \in \mathbb{N} \setminus \{1\}$. Thus,*

$$\begin{aligned}K_0(\mathcal{O}_{A_{\mathcal{H}(f_\mu)}}) &\cong \mathbb{Z}_a \quad \text{with } a = \left| 1 + \sum_{l=1}^{n-1} \prod_{i=1}^l \varepsilon_i(c) \right|, \\ K_1(\mathcal{O}_{A_{\mathcal{H}(f_\mu)}}) &\cong \begin{cases} \{0\}, & \text{if } a \neq 0, \\ \mathbb{Z}, & \text{if } a = 0. \end{cases}\end{aligned}\quad (6)$$

PROOF. Set $z_i = \varepsilon_i(c) \varepsilon_{i+1}(c) \cdots$ for $i = 1, 2, \dots$. Let $z'_i = f_\mu^i(c)$ be the point on the unit interval $[0, 1]$ represented by the sequence z_i for $i = 1, 2, \dots$. We have $\sigma(z_i) = z_{i+1}$ for $i = 1, \dots, n - 1$ and $\sigma(z_n) = z_1$. Denote by ω the $n \times n$ matrix representing the shift map σ . Let C_0 be the vector space spanned by the formal basis $\{z'_1, \dots, z'_n\}$. Now, let ρ be the permutation of the set $\{1, \dots, n\}$, which allows us to order the points $z'_1, \dots, z'_n$ on the unit interval $[0, 1]$, that is,

$$0 < z'_{\rho(1)} < z'_{\rho(2)} < \cdots < z'_{\rho(n)} < 1. \quad (7)$$

Set $x_i := z'_{\rho(i)}$ with $i = 1, \dots, n$ and let π denote the permutation matrix which takes the formal basis $\{z'_1, \dots, z'_n\}$ to the formal basis $\{x_1, \dots, x_n\}$. We will denote by C_1 the $(n-1)$ -dimensional vector space spanned by the formal basis $\{x_{i+1} - x_i : i = 1, \dots, n-1\}$. Set

$$I_i := [x_i, x_{i+1}] \quad \text{for } i = 1, \dots, n-1. \quad (8)$$

Thus, we can define the Markov transition matrix $A_{\mathcal{K}(f_\mu)}$ as above. Let φ denote the incidence matrix that takes the formal basis $\{x_1, \dots, x_n\}$ of C_0 to the formal basis $\{x_2 - x_1, \dots, x_n - x_{n-1}\}$ of C_1 . Put $\eta := \varphi\pi$. As in [7, 8], we obtain an endomorphism α of C_1 , that makes the following diagram commutative:

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & C_1 \\ \omega \downarrow & & \downarrow \alpha \\ C_0 & \xrightarrow{\eta} & C_1. \end{array} \quad (9)$$

We have $\alpha = \eta\omega\eta^T(\eta\eta^T)^{-1}$. Remark that if we neglect the negative signs on the matrix α , then we will obtain precisely the Markov transition matrix $A_{\mathcal{K}(f_\mu)}$. In fact, consider the $(n-1) \times (n-1)$ matrix

$$\beta := \begin{bmatrix} 1_{n_L} & 0 \\ 0 & -1_{n_R} \end{bmatrix}, \quad (10)$$

where 1_{n_L} and 1_{n_R} are the identity matrices of ranks n_L and n_R , respectively, with n_L (n_R) being the number of intervals I_i of the Markov partition placed on the left- (right-) hand side of the turning point of f_μ . Therefore, we have

$$A_{\mathcal{K}(f_\mu)} = \beta\alpha. \quad (11)$$

Now, consider the following matrix:

$$y_{\mathcal{K}(f_\mu)} := (y_{ij}) \quad \text{with} \quad \begin{cases} y_{ii} = \varepsilon_i(c), & i = 1, \dots, n, \\ y_{in} = -\varepsilon_i(c), & i = 1, \dots, n, \\ y_{ij} = 0, & \text{otherwise.} \end{cases} \quad (12)$$

The matrix $y_{\mathcal{K}(f_\mu)}$ makes the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & C_1 \\ y_{\mathcal{K}(f_\mu)} \downarrow & & \downarrow \beta \\ C_0 & \xrightarrow{\eta} & C_1 \end{array} \quad (13)$$

commutative. Finally, set $\theta_{\mathcal{K}(f_\mu)} := \theta_{\mathcal{K}(f_\mu)} \omega$. Then, the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & C_1 \\ \theta_{\mathcal{K}(f_\mu)} \downarrow & & \downarrow A_{\mathcal{K}(f_\mu)} \\ C_0 & \xrightarrow{\eta} & C_1 \end{array} \tag{14}$$

is also commutative. Now, notice that the transpose of η has the following factorization:

$$\eta^T = Y i X, \tag{15}$$

where Y is an invertible (over $\mathbb{Z}$) $n \times n$ integer matrix given by

$$Y := \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & \vdots & & & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & -1 & \cdots & & -1 & 1 \end{pmatrix}, \tag{16}$$

i is the inclusion $C_1 \hookrightarrow C_0$ given by

$$i := \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ & & & 1 \\ 0 & \cdots & & 0 \end{pmatrix}, \tag{17}$$

and X is an invertible (over $\mathbb{Z}$) $(n-1) \times (n-1)$ integer matrix obtained from the $(n-1) \times n$ matrix η^T by removing the n th row of η^T . Thus, from the commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\eta^T} & C_0 \\ A_{\mathcal{K}(f_\mu)}^T \downarrow & & \downarrow \theta_{\mathcal{K}(f_\mu)}^T \\ C_1 & \xrightarrow{\eta^T} & C_0, \end{array} \tag{18}$$

we will have the following commutative diagram with short exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_1 & \xrightarrow{i} & C_0 & \xrightarrow{p} & C_0/C_1 \longrightarrow 0 \\
 & & \downarrow A' & & \downarrow \theta' & & \downarrow 0 \\
 0 & \longrightarrow & C_1 & \xrightarrow{i} & C_0 & \xrightarrow{p} & C_0/C_1 \longrightarrow 0,
 \end{array} \quad (19)$$

where the map p is represented by the $1 \times n$ matrix $[0 \cdots 0 \ 1]$ and

$$A' = X A_{\mathcal{K}(f_\mu)}^T X^{-1}, \quad \theta' = Y^{-1} \theta_{\mathcal{K}(f_\mu)}^T Y, \quad (20)$$

that is, A' is similar to $A_{\mathcal{K}(f_\mu)}^T$ over $\mathbb{Z}$ and θ' is similar to $\theta_{\mathcal{K}(f_\mu)}^T$ over $\mathbb{Z}$. Hence, for example, by [10] we obtain, respectively,

$$\begin{aligned}
 \mathbb{Z}^{n-1} / (1 - A') \mathbb{Z}^{n-1} &\cong \mathbb{Z}^{n-1} / (1 - A_{\mathcal{K}(f_\mu)}) \mathbb{Z}^{n-1}, \\
 \mathbb{Z}^n / (1 - \theta') \mathbb{Z}^n &\cong \mathbb{Z}^n / (1 - \theta_{\mathcal{K}(f_\mu)}) \mathbb{Z}^n.
 \end{aligned} \quad (21)$$

Now, from the last diagram we have, for example, by [9],

$$\theta' = \begin{bmatrix} A' & * \\ 0 & 0 \end{bmatrix}. \quad (22)$$

Therefore,

$$\begin{aligned}
 \mathbb{Z}^{n-1} / (1 - A') \mathbb{Z}^{n-1} &\cong \mathbb{Z}^n / (1 - \theta') \mathbb{Z}^n, \\
 \mathbb{Z}^{n-1} / (1 - A_{\mathcal{K}(f_\mu)}) \mathbb{Z}^{n-1} &\cong \mathbb{Z}^n / (1 - \theta_{\mathcal{K}(f_\mu)}) \mathbb{Z}^n.
 \end{aligned} \quad (23)$$

Next, we will compute $\mathbb{Z}^n / (1 - \theta_{\mathcal{K}(f_\mu)}) \mathbb{Z}^n$. From the previous discussions and notations, the $n \times n$ matrix $\theta_{\mathcal{K}(f_\mu)}$ is explicitly given by

$$\theta_{\mathcal{K}(f_\mu)} := \begin{pmatrix} -\varepsilon_1(c) & \varepsilon_1(c) & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & \vdots & \ddots & & 0 \\ -\varepsilon_{n-1}(c) & & & & \varepsilon_{n-1}(c) \\ 0 & 0 & \cdots & & 0 \end{pmatrix}. \quad (24)$$

Notice that the matrix $\theta_{\mathcal{K}(f_\mu)}$ completely describes the dynamics of f_μ . Finally, using row and column elementary operations over $\mathbb{Z}$, we can find invertible

(over $\mathbb{Z}$) matrices U_1 and U_2 with integer entries such that

$$1 - \theta_{\mathcal{H}(f_\mu)} = U_1 \begin{pmatrix} 1 + \sum_{l=1}^{n-1} \prod_{i=1}^l \varepsilon_i(c) & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} U_2. \tag{25}$$

Thus, we obtain

$$K_0\left(\mathbb{O}_{A_{\mathcal{H}(f_\mu)}}\right) \cong \mathbb{Z}^{n-1} / \left(1 - A_{\mathcal{H}(f_\mu)}^T\right) \mathbb{Z}^{n-1} \cong \mathbb{Z}_a, \tag{26}$$

where

$$a = \left| 1 + \sum_{l=1}^{n-1} \prod_{i=1}^l \varepsilon_i(c) \right|, \quad n \in \mathbb{N} \setminus \{1\}. \tag{27}$$

□

EXAMPLE 2. Set

$$\mathcal{H}(f_\mu) = (RLLRRC)^\infty, \tag{28}$$

where $R = -1$, $L = +1$, and $C = 0$. Thus, we can construct the 5×5 Markov transition matrix $A_{\mathcal{H}(f_\mu)}$ and the matrices $\theta_{\mathcal{H}(f_\mu)}$, ω , φ , and π :

$$\begin{aligned} A_{\mathcal{H}(f_\mu)} &= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, & \theta_{\mathcal{H}(f_\mu)} &= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \omega &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \varphi &= \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 & 1 \end{pmatrix}, \\ \pi &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{29}$$

We have

$$K_0\left(\mathbb{C}_{A_{\mathcal{K}}(f_\mu)}\right) \cong \mathbb{Z}_2, \quad K_1\left(\mathbb{C}_{A_{\mathcal{K}}(f_\mu)}\right) \cong \{0\}. \quad (30)$$

REMARK 3. In the statement of [Theorem 1](#) the case $a = 0$ may occur. This happens when we have a star product factorizable kneading sequence [\[4\]](#). In this case the correspondent Markov transition matrix is reducible.

REMARK 4. In [\[6\]](#), Katayama et al. have constructed a class of C^* -algebras from the β -expansions of real numbers. In fact, considering a semiconjugacy from the real quadratic map to the tent map [\[14\]](#), we can also obtain [Theorem 1](#) using [\[6\]](#) and the λ -expansions of real numbers introduced in [\[4\]](#).

REMARK 5. In [\[13\]](#) (see also [\[12\]](#)) and [\[11\]](#), the BF-groups are explicitly calculated with respect to another kind of maps on the interval.

REFERENCES

- [1] R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. of Math. (2) **106** (1977), no. 1, 73–92.
- [2] J. Cuntz, *A class of C^* -algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^* -algebras*, Invent. Math. **63** (1981), no. 1, 25–40.
- [3] J. Cuntz and W. Krieger, *A class of C^* -algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268.
- [4] B. Derrida, A. Gervois, and Y. Pomeau, *Iteration of endomorphisms on the real axis and representation of numbers*, Ann. Inst. H. Poincaré Sect. A (N.S.) **29** (1978), no. 3, 305–356.
- [5] J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Phys. **70** (1979), no. 2, 133–160.
- [6] Y. Katayama, K. Matsumoto, and Y. Watatani, *Simple C^* -algebras arising from β -expansion of real numbers*, Ergodic Theory Dynam. Systems **18** (1998), no. 4, 937–962.
- [7] J. P. Lampreia and J. Sousa Ramos, *Trimodal maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **3** (1993), no. 6, 1607–1617.
- [8] ———, *Symbolic dynamics of bimodal maps*, Portugal. Math. **54** (1997), no. 1, 1–18.
- [9] S. Lang, *Algebra*, Addison-Wesley, Massachusetts, 1965.
- [10] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.
- [11] N. Martins, R. Severino, and J. Sousa Ramos, *Bowen-Franks groups for bimodal matrices*, J. Differ. Equations Appl. **9** (2003), no. 3-4, 423–433.
- [12] N. Martins and J. Sousa Ramos, *Cuntz-Krieger algebras arising from linear mod one transformations*, Differential Equations and Dynamical Systems (Lisbon, 2000), Fields Inst. Commun., vol. 31, American Mathematical Society, Rhode Island, 2002, pp. 265–273.
- [13] ———, *Bowen-Franks groups associated with linear mod one transformations*, to appear in Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2003.

- [14] J. Milnor and W. Thurston, *On iterated maps of the interval*, Dynamical Systems (College Park, Md, 1986–87), Lecture Notes in Math., vol. 1342, Springer, Berlin, 1988, pp. 465–563.
- [15] P. Štefan, *A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), no. 3, 237–248.

Nuno Martins: Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

E-mail address: nmartins@math.ist.utl.pt

Ricardo Severino: Departamento de Matemática, Universidade do Minho, 4710-057 Braga, Portugal

E-mail address: ricardo@math.uminho.pt

J. Sousa Ramos: Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

E-mail address: sramos@math.ist.utl.pt