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STOCHASTIC PROCESSES ON NON-ARCHIMEDEAN BANACH SPACES

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Non-Archimedean analogs of Markov quasimeasures and stochastic processes are investigated. They are used for the development of stochastic antiderivations. The non-Archimedean analog of the Itô formula is proved.

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1. Introduction. Stochastic differential equations on real Banach spaces and manifolds are widely used for the solutions of mathematical and physical problems and also for the construction and investigation of measures on them [4, 17]. In particular, stochastic equations can be used for the construction of quasi-invariant measures on topological groups. On the other hand, non-Archimedean functional analysis has developed rapidly in recent years, as well as its applications in mathematical physics [1, 19, 21, 22]. Wide classes of quasi-invariant measures including analogs of Gaussian type on non-Archimedean Banach spaces, loops, and diffeomorphisms groups were investigated in [7, 8, 9, 11, 12, 15]. Quasi-invariant measures on topological groups and their configuration spaces can be used for the investigation of their unitary representations (see [10, 12, 13, 15] and the references therein).

In view of this developments, non-Archimedean analogs of stochastic equations and diffusion processes need to be investigated. Some steps in this direction were made in [6]. There are different variants for such activity, for example, p-adic parameters analogous to time, but with stochastic processes on spaces of complex-valued functions.

At the same time measures may be real, complex, or with values in a non-Archimedean field.

This work treats the case that was not considered by other authors and that is suitable and helpful for the investigation of stochastic processes and quasi-invariant measures on non-Archimedean topological groups. Here are considered spaces of functions with values in Banach spaces over non-Archimedean local fields, in particular, with values in the field $\mathbf{Q}_p$ of p-adic numbers. For this, non-Archimedean analogs of stochastic processes are considered on spaces of functions with values in the non-Archimedean infinite field with a non-trivial valuation such that a parameter analogous to the time is p-adic (see

Section 4.1 and Definition 4.1). Their existence is proved in Theorem 4.2. Specific antiderivation operators generalizing Schikhof antiderivation operators on spaces of functions C^n are investigated (see Section 2). Their continuity and differentiability properties are given in Lemmas 2.1, 2.2, and Theorem 2.12. Also operators analogous to nuclear operators are studied (see Propositions 2.7, 2.8, and 2.9). In Section 3, non-Archimedean analogs of Markov quasimeasures are defined and Propositions 3.2 and 3.3 about their boundedness and unboundedness are proved. The non-Archimedean stochastic integral is defined in Section 4.2. Its continuity as the operator on the corresponding spaces of functions is proved in Proposition 4.4. In Theorems 4.5, 4.7, and Corollary 4.6, analogs of the Itô formula are proved. Spaces of analytic functions lead to simpler expressions of the Itô formula analog, but the space of analytic functions is very narrow, and though it is helpful in non-Archimedean mathematical physics, it is insufficient for the solutions of all mathematical and physical problems. For example, in many cases of topological groups for non-Archimedean manifolds, spaces of analytic functions are insufficient. On the other hand, for spaces C^n rather simple formulas are found. All results of this paper are obtained for the first time.

2. Specific antiderivation of operators

2.1. Let $X := c_0(\alpha,\mathbf{K}_p)$ be a Banach space over a local field (see [23]) $\mathbf{K}_p$ such that $\mathbf{K}_p \supset \mathbf{Q}_p$, $\{e_j : j \in \alpha\}$ denotes the standard orthonormal base in $c_0(\alpha,\mathbf{K}_p)$, where α is an ordinal [5] and $e_j = (0,\dots,0,1,0,\dots)$ with the unit on the jth place, $j \in \alpha$ [21]. The space $c_0(\alpha,\mathbf{K}_p)$ consists of vectors $x = (x_j : x_j \in \mathbf{K}_p, j \in \alpha)$ such that for each $\epsilon > 0$, a set $\{j : j \in \alpha; |x_j| > \epsilon\}$ is finite, $\|x\| := \sup_j |x_j|$. It is convenient to supply the set α with the ordinal structure due to the Kuratwoski-Zorn lemma. Let F be a continuous function on $B_r \times C^0(B_r,X)^{\otimes k}$ with values in $C^0(B_r,X)$:

$$F \in C^0\left(B_r \times C^0(B_r,X)^{\otimes k}, C^0(B_r,X)\right), \tag{2.1}$$

where $Z^{\otimes k} = Z \otimes \cdots \otimes Z$ is the product of k copies of a normed space Z and $Z^{\otimes k}$ is supplied with the box (maximum norm) topology [5], $B_r := B(\mathbf{K}_p, t_0, r)$ is a ball in $\mathbf{K}_p$ containing t_0 and of radius r, Banach spaces $C^t(M,X)$ of mappings $f : M \to X$ from a C^∞-manifold M with clopen charts modelled on a Banach space Y over $\mathbf{K}_p$ into X of class of smoothness C^t with $0 \le t < \infty$ are the same as in [10, 12, 15] with the supremum-norm of $\bar{\Phi}^v f$, $0 \le v \le t$, when M is closed and bounded in the corresponding Banach space. Such mappings can be written in the form

$$F(v,\xi_1,\dots,\xi_l) = \sum_{j\in\alpha} F^j(v,\xi_1,\dots,\xi_k)e_j, \tag{2.2}$$

where $F^j \in C^0(B_r \times C^0(B_r,X)^{\otimes k},\mathbf{K}_p)$ for each $j \in \alpha$. In particular, let

$$F(v;\xi_1,\dots,\xi_k) = G(v;\xi_1,\dots,\xi_l)\cdot(A_{l+1}(v)\xi_{l+1},\dots,A_k(v)\xi_k), \tag{2.3}$$

where $L(X,Y)$ denotes a Banach space of continuous linear operators $A : X \to Y$ supplied with the operator norm $\|A\| := \sup_{0\neq x\in X}\|Ax\|_Y/\|x\|_X$, and $L(X) := L(X,X)$, $A_i(v)$ are continuous linear operators for each $v \in B_r$ such that $A_i \in C^0(B_r,L(X))$, $G(v,\xi_1,\dots,\xi_l) \in L_{k-l}(X^{\otimes(k-l)};X)$ for each fixed $v \in B_r$ and $\xi_1,\dots,\xi_l \in C^0(B_r,X)$, that is, F is a $(k-l)$-linear operator by $\xi_{l+1},\dots,\xi_k$, where $G = G(v,\xi_1,\dots,\xi_l)$ is the short notation of $G(v,\xi_1(v),\dots,\xi_l(v))$, $L_k(X_1,\dots,X_k;Y)$ denotes the Banach (normed) space of k-linear continuous operators from $X_1\otimes\cdots\otimes X_k$ into Y for Banach (normed) spaces $X_1,\dots,X_k,Y$ over $\mathbf{K}$ and $L_k(X^{\otimes k};Y) := L_k(X_1,\dots,X_k;Y)$ for the particular case $X_1 = \cdots = X_k = X$. When $l = 0$ put $G = G(v)$. There exists the following antiderivation of operators given by (2.3):

$$\begin{aligned}\hat{P}_{(\xi_{l+1},\dots,\xi_k)}&[G(s;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](v)\\ &:= \sum_{n=0}^{\infty} G(v_n;\xi_1,\dots,\xi_l)\cdot(A_{l+1}(v_n)[\xi_{l+1}(v_{n+1})-\xi_{l+1}(v_n)],\dots,\\ &\qquad\qquad A_k(v_n)[\xi_k(v_{n+1})-\xi_k(v_n)]),\end{aligned} \tag{2.4}$$

where $v_n = \sigma_n(t)$, $\{\sigma_n : n = 0,1,2,\dots\}$ is an approximation of the identity in B_r, satisfying the following conditions:

(i) $\sigma_0(t) = t_0$,

(ii) $\sigma_m\circ\sigma_n = \sigma_n\circ\sigma_m$ for each $m \geq n$ and there exists $0 < \rho < 1$ such that from

(iii) $|x-y| < \rho^n$, it follows that $\sigma_n(x) = \sigma_n(y)$ and

(iv) $|\sigma_n(x)-x| < \rho^n$ (see [19, Sections 62 and 79]).

Lemma 2.1. (1) *If $G \in C^0(B_r\times X^{\otimes l},L_{k-l}(X^{\otimes(k-l)};X))$, $\xi_i \in C^0(B_r,X)$ for each $i = 1,\dots,k$, and $A_{l+i} \in C^0(B_r,L(X))$ for each $i = 1,\dots,k-l$, then $\hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(s;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](v) \in C^0(B_r\times C^0(B_r,X)^{\otimes l},C^0(B_r,X))$ as the function by $v,\xi_1,\dots,\xi_l$ for each fixed $\xi_{l+1},\dots,\xi_k$ and $\hat{P}$ is of class C^∞ by $\xi_{l+1},\dots,\xi_k$.*

(2) *Moreover, if G is of class of smoothness C^m by arguments $\xi_1,\dots,\xi_l$, then $\hat{P}_{(\xi_{l+1},\dots,\xi_k)}G$ is also in class of smoothness C^m by $\xi_1,\dots,\xi_l$.*

Proof. Since B_r is compact, then ξ_i are uniformly continuous, together with $A_{l+i}(v)[\xi_{l+i}(v)]$. In addition, $\hat{P}$ is the linear operator by $\xi_{l+1},\dots,\xi_k$. From this and conditions (i), (ii), (iii), and (iv), the first statement follows. The last statement follows from the linearity of $\hat{P}$ by G and applying the operator of difference quotients $\bar{\Phi}^m$ by $\xi_1,\dots,\xi_l$ (see [10, 15]). □

Lemma 2.2. *If $\xi_i \in C^1(B_r,X)$ for each $i = 1,\dots,k$ and condition (1) of Lemma 2.1 are satisfied, then*

$$\hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(s;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](x) \in C^1(B_r,X) \tag{2.5}$$

as a function by the argument $x \in B_r$ and

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(s;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](x)\right)\\
&\quad=\sum_{q=l+1}^{k}\hat{P}_{(\xi_{l+1},\dots,\xi_{q-1},\xi_{q+1},\dots,\xi_k)}G(x;\xi_1,\dots,\xi_l)\\
&\qquad\cdot(A_{l+1}(x)\xi_{l+1}(x),\dots,A_{q-1}(x)\xi_{q-1}(x),\\
&\qquad\quad A_q(x)\xi'_q(x),A_{q+1}(x)\xi_{q+1}(x),\dots,A_k(x)\xi_k(x))
\end{aligned}\tag{2.6}$$

such that

$$\begin{aligned}
&\|\hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(s;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](x)\|_{C^1(B_r,X)}\\
&\quad\le\|G\|_{C^0(B_r\times X^{\otimes l},L_{k-l}(X^{\otimes(k-l)};X))}\prod_{i=l+1}^{k}[\|A_i\|_{C^0(B_r,L(X))}\|\xi_i\|_{C^1(B_r,X)}].
\end{aligned}\tag{2.7}$$

Proof. Let

$$\begin{aligned}
y:=&\ \hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(z;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](x)\\
&-\hat{P}_{(\xi_{l+1},\dots,\xi_k)}[G(z;\xi_1,\dots,\xi_l)\circ(A_{l+1}\otimes\cdots\otimes A_k)](y)\\
&-(x-y)\sum_{q=l+1}^{k}\hat{P}_{(\xi_{l+1},\dots,\xi_{q-1},\xi_{q+1},\dots,\xi_k)}\\
&\qquad\times[G(y;\xi_1,\dots,\xi_l)\cdot(A_{l+1}(y)\xi_{l+1}(y),\dots,A_{q-1}(y)\xi_{q-1}(y),\\
&\qquad\qquad A_q(y)\xi'_q(y),A_{q+1}(y)\xi_{q+1}(y),\dots,\\
&\qquad\qquad A_k(y)\xi_k(y))]
\end{aligned}\tag{2.8}$$

and $\rho^{s+1}\le|x-y|<\rho^s$, where $s\in\mathbb{N}$. Therefore, $x_0=y_0,\dots,x_s=y_s$, $x_{s+1}\ne y_{s+1}$ and

$$\begin{aligned}
y=&\left[\sum_{q=l+1}^{k}E(x_s)(v_{l+1},\dots,v_{q-1},h_q,z_{q+1},\dots,z_k)\right]\\
&+E(x_s)(h_{l+1},h_{l+2},z_{l+3},\dots,z_k)+E(x_s)(h_{l+1},v_{l+2},h_{l+3},z_{l+4},\dots,z_k)\\
&+\cdots+E(x_s)(v_{l+1},\dots,v_{k-2},h_{k-1},h_k)+\cdots+E(x_s)(h_{l+1},\dots,h_k)\\
&+\sum_{j=s+1}^{\infty}\{E(x_j)((\xi_{l+1}(x_{j+1})-\xi_{l+1}(x_j)),\dots,(\xi_k(x_{j+1})-\xi_k(x_j)))\\
&\qquad-E(y_j)((\xi_{l+1}(y_{j+1})-\xi_{l+1}(y_j)),\dots,(\xi_k(y_{j+1})-\xi_k(y_j)))\}\\
&-(x-y)\sum_{q=l+1}^{k}\hat{P}_{(\xi_{l+1},\dots,\xi_{q-1},\xi_{q+1},\dots,\xi_k)}E(y)\\
&\qquad\times(\xi_{l+1}(y),\dots,\xi_{q-1}(y),\xi'_q(y),\xi_{q+1}(y),\dots,\xi_k(y)),
\end{aligned}\tag{2.9}$$

where $v_j = \xi_j(x_{s+1}) - \xi_j(x_s)$, $h_j = \xi_j(x_{s+1}) - \xi_j(y_{s+1})$, and $z_j = \xi_j(y_{s+1}) - \xi_j(y_s)$ for $j = l+1,\dots,k$ and

$$\begin{aligned} E := E(x) &:= G(x;\xi_1,\dots,\xi_l)\cdot(A_{l+1}(x)\otimes\cdots\otimes A_k(x)),\\ E(x)(\xi_{l+1},\dots,\xi_k) &:= G(x;\xi_1,\dots,\xi_l)\cdot(A_{l+1}(x)\xi_{l+1},\dots,A_k(x)\xi_k) \end{aligned} \tag{2.10}$$

in accordance with Formula (2.3). On the other hand,

$$\begin{aligned} &\|\xi_i(y_{j+1}) - \xi_i(y_j) - (y_{j+1}-y_j)\xi_i(y)\|\\ &\quad = \|(y_{j+1}-y_j)[(\bar{\Phi}^1\xi_i)(y_j;1;y_{j+1}-y_j) - \xi_i(y)]\|\\ &\quad \le |y_{j+1}-y_j|\,\|\xi_i\|_{C^1(B_r,X)} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &E(x)\cdot(a_{l+1}+b_{l+1},\dots,a_k+b_k) - E(y)\cdot(a_{l+1},\dots,a_k)\\ &\quad = E(x)\cdot(a_{l+1}+b_{l+1},\dots,a_k+b_k) - E(x)(a_{l+1},\dots,a_k)\\ &\qquad + [E(x)-E(y)]\cdot(a_{l+1},\dots,a_k)\\ &\quad = E(x)\cdot(b_{l+1},a_{l+2},\dots,a_k) + \cdots + E(x)\cdot(a_{l+1},\dots,a_{k-1},b_k)\\ &\qquad + E(x)\cdot(b_{l+1},b_{l+2},a_{l+3},\dots,a_k) + \cdots + E(x)\cdot(a_{l+1},\dots,a_{k-2},b_{k-1},b_k)\\ &\qquad + \cdots + E(x)\cdot(b_{l+1},\dots,b_k) + [E(x)-E(y)]\cdot(a_{l+1},\dots,a_k) \end{aligned} \tag{2.12}$$

for each $a_{l+1},\dots,a_k,b_{l+1},\dots,b_k \in C^0(B_r,X)$, hence

$$\begin{aligned} &\Bigg\|\Bigg[\sum_{q=l+1}^{k} E(x_s)(v_{l+1},\dots,v_{q-1},h_q,z_{q+1},\dots,z_k)\Bigg]\\ &\qquad -(x-y)\sum_{q=l+1}^{k}\hat{P}_{(\xi_{l+1},\dots,\xi_{q-1},\xi_{q+1},\dots,\xi_k)}E(y)\\ &\qquad\qquad \times(\xi_{l+1}(y),\dots,\xi_{q-1}(y),\xi'_q(y),\\ &\qquad\qquad\qquad \xi_{q+1}(y),\dots,\xi_k(y))\Bigg\|\\ &\qquad \le \|E\|_{C^0}\rho^s\prod_{q=l+1}^{k}\|\xi_q\|_{C^1}\alpha(s),\\ &\|E(x_j)((\xi_{l+1}(x_{j+1})-\xi_{l+1}(x_j)),\dots,(\xi_k(x_{j+1})-\xi_k(x_j)))\\ &\quad -E(y_j)((\xi_{l+1}(y_{j+1})-\xi_{l+1}(y_j)),\dots,(\xi_k(y_{j+1})-\xi_k(y_j)))\|\\ &\qquad \le \|E\|_{C^0}\rho^s\prod_{q=l+1}^{k}\|\xi_q\|_{C^1}\alpha(s) \end{aligned} \tag{2.13}$$

for each $j \ge s+1$, where $\lim_{s\to\infty} \alpha(s) = 0$. Consequently, $\lim_{|x-y|\to 0} \gamma = 0$ and $\bar{\Phi}^1(\hat{P}_{(\xi_{l+1},\dots,\xi_k)}E)(x) \in C^0(B_r, X)$, where $\Phi^1\eta(x;h;\zeta) = \{\eta(x+\zeta h) - \eta(x)\}/\zeta$ for $0 \ne \zeta \in \mathbf{K}$, $h \in H$, $\eta \in C^1(U,Y)$, U is open in X, X and Y are Banach spaces over $\mathbf{K}$, and $\bar{\Phi}^1\eta$ is a continuous extension of $\Phi^1\eta$ on $U \times V \times B(\mathbf{K},0,1)$ for a neighborhood V of 0 in X (see [10, Section 2.3] or [15, Section I.2.3]). Then,

$$\begin{aligned}(\hat{P}_{(\xi_{l+1},\dots,\xi_k)}E)(x) = \sum_{n=0}^{\infty} &(x_{n+1}-x_n)^{k-l} G(x_n;\xi_1,\dots,\xi_l) \\ &\cdot (A_{l+1}(x_n)(\bar{\Phi}^1\xi_{l+1})(x_n;1;x_{n+1}-x_n),\dots, \\ &A_k(x_n)(\bar{\Phi}^1\xi_k)(x_n;1;x_{n+1}-x_n)).\end{aligned} \tag{2.14}$$

Let $\eta := (\hat{P}_w E)(x) - (\hat{P}_w E)(y)$, then $\eta = E(x_s)(w(x_{s+1}) - w(y_{s+1})) + \sum_{n=s+1}^{\infty}\{E(x_n)(w(x_{n+1}) - w(x_n)) - E(y_n)(w(y_{n+1}) - w(y_n))\}$. Consequently, $\|\eta\| \le \|E\|_{C^0(B_r\times X^{\otimes l}, L_{k-l}(X^{\otimes(k-l)};X))}(\prod_{i=l+1}^{k}\|\xi_i\|_{C^1}|x-y|)$ since E are polylinear mappings by $\xi_{l+1}(z),\dots,\xi_k(z) \in X$, $|x_{s+1}-y_{s+1}| \le |x-y|$ and $|x_{n+1}-x_n| \le |x-y|$ and $|y_{n+1}-y_n| \le |x-y|$ for each $n > s$, where $\rho^{s+1} \le |x-y| < \rho^s$, $w = (\xi_{l+1},\dots,\xi_k)$. □

Note 2.3. In particular, when $X = \mathbf{K}$, $l = 0$, $k = 1$, $A_1 = 1$, and $\xi(x) = x$, this gives the usual formula, $d[\hat{P}_s G(s)](x)/dx = G(x)$.

2.2. Suppose that X and Y are Banach spaces over a (complete relative-to-its-uniformity) local field $\mathbf{K}$. Let X and Y be isomorphic with the Banach spaces $c_0(\alpha,\mathbf{K})$ and $c_0(\beta,\mathbf{K})$ and there are given the standard orthonormal bases $\{e_j : j \in \alpha\}$ in X and $\{q_j : j \in \beta\}$ in Y, respectively, then each $E \in L(X,Y)$ has its matrix realisation $E_{j,k} := q_k^* E e_j$, where α and β are ordinals, $q_k^* \in Y^*$ is a continuous $\mathbf{K}$-linear functional $q_k^* : Y \to \mathbf{K}$ corresponding to q_k under the natural embedding $Y \hookrightarrow Y^*$ associated with the chosen basis, and Y^* is a topologically conjugated or dual space of $\mathbf{K}$-linear functionals on Y.

2.3. Let A be a commutative Banach algebra and A^+ denote the Gelfand space of A, that is, $A^+ = \mathrm{Sp}(A)$, where $\mathrm{Sp}(A)$ in another words spectrum of A was defined in [21, Chapter 6]. Let $C_\infty(A^+,\mathbf{K})$ be the same space as in [16, 21].

Definition 2.4. A commutative Banach algebra A is called a C-algebra if it is isomorphic with $C_\infty(X,\mathbf{K})$ for a locally compact Hausdorff totally disconnected topological space X, where $f+g$ and fg are defined pointwise for each $f,g \in C_\infty(X,\mathbf{K})$.

Remark 2.5. Fix a Banach space H over a non-Archimedean complete field $\mathbf{F}$ as the above $L(H)$ denotes the Banach algebra of all bounded $\mathbf{F}$-linear operators on H. If $b \in L(H)$, we write shortly $\mathrm{Sp}(b)$ instead of $\mathrm{Sp}_{L(H)}(b) := \mathrm{cl}(\mathrm{Sp}(\mathrm{span}_{\mathbf{F}}\{b^n : n = 1,2,3,\dots\}))$ (see also [21]).

It was proved in [20], in the case of $\mathbf{F}$ with the discrete valuation group, that each continuous $\mathbf{F}$-linear operator $A : E \to H$ with $\|A\| \le 1$ from one Banach

space E into another H has the form

$$A = U \sum_{n=0}^{\infty} \pi^n P_{n,A}, \tag{2.15}$$

where $P_n := P_{n,A}$, $\{P_n : n \geq 0\}$ is a family of projections and $P_n P_m = 0$ for each $n \neq m$, $\|P_n\| \leq 1$ and $P_n^2 = P_n$ for each n, U is a partially isometric operator, that is, $U|_{\mathrm{cl}(\sum_n P_n(E))}$ is isometric, $U|_{E \ominus \mathrm{cl}(\sum_n P_n(E))} = 0$, $\ker(U) \supset \ker(A)$, $\mathrm{Im}(U) = \mathrm{cl}(\mathrm{Im}(A))$, $\pi \in \mathbf{F}$, $|\pi| < 1$ and π is the generator of the valuation group of $\mathbf{F}$.

We restrict our attention to the case of the local field $\mathbf{F}$; consequently, $\mathbf{F}$ has the discrete valuation group. If $\|A\| > 1$, we get

$$A = \lambda_A U \sum_{n=0}^{\infty} \pi^n P_{n,A}, \tag{2.16}$$

where $\lambda_A \in \mathbf{F}$ and $|\lambda_A| = \|A\|$. In view of [16], this is the particular case of the spectral integration on the discrete topological space X. Evidently, for each $1 \leq r < \infty$ there exists $J \in L(H)$ for which

$$\left\{ \sum_{n \geq 0} s_n^r \dim_{\mathbf{F}} P_{n,J}(H) \right\}^{1/r} < \infty \tag{2.17}$$

for $1 \leq r < \infty$, where J has the spectral decomposition given by formula (2.16), $s_n := |\lambda_J||\pi|^n \|P_n\|$. Using this result, it is possible to give the following definition.

Definition 2.6. Let E and H be two normed $\mathbf{F}$-linear spaces, where $\mathbf{F}$ is an infinite spherically complete field with a nontrivial non-Archimedean valuation. The $\mathbf{F}$-linear operator $A \in L(E,H)$ is called of class $L_q(E,H)$ if there exist $a_n \in E^*$ and $y_n \in H$ for each $n \in \mathbb{N}$ such that

$$\left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q \right) < \infty, \tag{2.18}$$

and A has the form

$$Ax = \sum_{n=1}^{\infty} a_n(x) y_n \tag{2.19}$$

for each $x \in E$, where $0 < q < \infty$. For each such A, we put

$$\nu_q(A) = \inf \left\{ \sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q \right\}^{1/q}, \tag{2.20}$$

where the infimum is taken by all such representations (2.19) of A,

$$\nu_\infty(A) := \|A\| \tag{2.21}$$

and $L_\infty(E,H) := L(E,H)$.

Proposition 2.7. *The* **F***-linear space $L_q(E,H)$ is the normed* **F***-linear space with the norm ν_q, when $1 \le q$; it is the metric space when $0 < q < 1$.*

Proof. Let $A \in L_q(E,H)$ and $1 \le q < \infty$ since the case $q = \infty$ follows from its definition. Then, A has the representation (2.16). Then due to the ultrametric inequality,

$$\|Ax\|_H \le \|x\|_E \sup_{n\in\mathbb{N}} (\|a_n\|_{E^*}\|y_n\|_H) \le \|x\|_E \left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q\right)^{1/q}, \tag{2.22}$$

$\sup_{x\neq 0} \|Ax\|_H/\|x\|_E =: \|A\| \le \nu_q(A)$.

Let now $A,S \in L_q(E,H)$, then there exist $0 < \delta < \infty$ and two representations $Ax = \sum_{n=1}^{\infty} a_n(x)y_n$ and $Sx = \sum_{m=1}^{\infty} b_m(x)z_m$, for which

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q\right)^{1/q} &\le \nu_q(A) + \delta, \\ \left(\sum_{n=1}^{\infty} \|b_n\|_{E^*}^q \|z_n\|_H^q\right)^{1/q} &\le \nu_q(S) + \delta, \end{aligned} \tag{2.23}$$

hence

$$\begin{aligned} (A+S)x &= \sum_{n=1}^{\infty} (a_n(x)y_n + b_n(x)z_n), \\ \nu_q(A+S) &\le \left(\sum_{n=1}^{\infty} \|a_n\|^q \|y_n\|^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} \|b_n\|^q \|z_n\|^q\right)^{1/q} \\ &\le \nu_q(A) + \nu_q(S) + 2\delta \end{aligned} \tag{2.24}$$

due to the Hölder inequality. The case $0 < q < 1$ is analogous to the classical one given in [18]. □

Proposition 2.8. *If $J \in L_q(H)$ and $S \in L_r(H)$ are commuting operators, the field* **F** *is with the discrete valuation group, and $1/q + 1/r = 1/v$, then $JS \in L_v(H)$, where $1 \le q, r, v \le \infty$.*

Proof. Since **F** is with the discrete valuation, then J and S have the decompositions (2.16). Certainly, each projector $P_{n,J}$ and $P_{m,S}$ belongs to $L_1(H)$ and have the decomposition (2.19). The **F**-linear span of $\bigcup_{n,m} \operatorname{range}(P_{n,J}P_{m,S})$ is dense in H. In particular, for each $x \in \operatorname{range}(P_{n,J}P_{m,S})$, we have $J^k S^l x = \lambda_J^k \lambda_S^l \pi^{nk+ml} P_{n,J}P_{m,S}x$. Applying Remark 2.5 to the commuting operators

J^k and S^l for each $k,l \in \mathbb{N}$ and using the base of H, we get projectors $P_{n,J}$ and $P_{m,S}$ which commute for each n and m; consequently,

$$JS = U_J U_S \lambda_J \lambda_S \sum_{n\geq 0, m\geq 0} \pi^{n+m} P_{n,J} P_{m,S}, \tag{2.25}$$

hence $U_{JS} = U_J U_S$, $\lambda_{JS} = \lambda_J \lambda_S$, $P_{l,JS} = \sum_{n+m=l} P_{n,J} P_{m,S}$. In view of the Hölder inequality, $\nu_v(JS) = \inf(\sum_{n=0}^{\infty} s_{n,JS}^{v} \dim_{\mathbf{F}} P_{n,JS}(H))^{1/v} \leq \nu_q(J)\nu_r(S)$. □

Proposition 2.9. *If E is the normed space and H is the Banach space over the field* $\mathbf{F}$ *(complete relative to its uniformity), then $L_r(E,H)$ is the Banach space such that if $J,S \in L_r(E,H)$, then*

$$\|J+S\|_r \leq \|J\|_r + \|S\|_r; \qquad \|bJ\|_r = |b|\|J\|_r \quad \text{for each } b \in \mathbf{K}; \tag{2.26}$$

$\|J\|_r = 0$ *if and only if $J = 0$, where $1 \leq r \leq \infty$, $\|*\|_q := \nu_q(*)$.*

Proof. In view of Proposition 2.7, it remains to prove that $L_r(E,H)$ is complete when H is complete. Let $\{T_\alpha\}$ be a Cauchy net in $L_r(E,H)$, then there exists $T \in L(E,H)$ such that $\lim_\alpha T_\alpha x = Tx$ for each $x \in E$ since $L_r(E,H) \subset L(E,H)$ and $L(E,H)$ is complete. We demonstrate that $T \in L_r(E,H)$ and T_α converges to T relative to ν_r for $1 \leq r < \infty$. Let α_k be a monotone subsequence in $\{\alpha\}$ such that $\nu_r^r(T_\alpha - T_\beta) < 2^{-k-2}$ for each $\alpha,\beta \geq \alpha_k$, where $k \in \mathbb{N}$. Since $T_{\alpha_{k+1}} - T_{\alpha_k} \in L_r(E,H)$, then $(T_{\alpha_{k+1}} - T_{\alpha_k})x = \sum_{n=1}^{\infty} a_{n,k}(x) y_{n,k}$ with $\sum_{n=1}^{\infty} \|a_{n,k}\|^r \|y_{n,k}\|^r < 2^{-k-2}$. Therefore,

$$(T_{\alpha_{k+p}} - T_{\alpha_k})x = \sum_{h=k}^{k+p-1} \sum_{n=1}^{\infty} a_{n,h}(x) y_{n,h} \tag{2.27}$$

for each $p \in \mathbb{N}$, consequently, using convergence while p tends to ∞, we get $(T - T_{\alpha_k})x = \sum_{h=k}^{\infty} \sum_{n=1}^{\infty} a_{n,h}(x) y_{n,h}$. Then

$$\nu_r^r(T - T_{\alpha_k}) \leq \sum_{h=k}^{\infty} \sum_{n=1}^{\infty} \|a_{n,h}\|^r \|y_{n,h}\|^r \leq 2^{-k-1}, \tag{2.28}$$

hence $T - T_{\alpha_k} \in L_r(E,H)$ and inevitably $T \in L_r(E,H)$. Moreover, $\nu_r(T - T_\alpha) \leq \nu_r(T - T_{\alpha_k}) + \nu_r(T_{\alpha_k} - T_\alpha) \leq 2^{-(k-1)/r} 2$ for each $\alpha \geq \alpha_k$. □

Proposition 2.10. *Let E, H, and G be normed spaces over spherically complete* $\mathbf{F}$. *If $T \in L(E,H)$ and $S \in L_r(H,G)$, then $ST \in L_r(E,G)$ and $\nu_r(ST) \leq \nu_r(S)\|T\|$. If $T \in L_r(E,H)$ and $S \in L(H,G)$, then $ST \in L_r(E,G)$ and $\nu_r(ST) \leq \|S\|\nu_r(T)$.*

Proof. For each $\delta > 0$, there are $b_n \in H^*$ and $z_n \in G$ such that $Sy = \sum_{n=1}^{\infty} b_n(y) z_n$ for each $y \in H$ and $\sum_{n=1}^{\infty} \|b_n\|^r \|z_n\|^r \leq \nu_r^r(S) + \delta$. Therefore, $STx = \sum_{n=1}^{\infty} T^* b_n(x) z_n$ for each $x \in E$, hence $\nu_r(ST) \leq \sum_{n=1}^{\infty} \|T^* b_n\|^r \|z_n\|^r \leq \|T\|[\nu_r^r(S) + \delta]$ since $\|T^* b_n(x)\| = |b_n(Tx)| \leq \|b_n\|\|Tx\| \leq \|b_n\|\|T\|\|x\|$,

where $T^* \in L(H^*,E^*)$ is the adjoint operator such that $b(Tx) =: (T^*b)(x)$ for each $b \in H^*$ and $x \in E$. The operator T^* exists due to the Hahn-Banach theorem for normed spaces over the spherically complete field $\mathbf{F}$ [21]. □

Proposition 2.11. *If $T \in L_r(E,H)$, then $T^* \in L_r(H^*,E^*)$ and $\nu_r(T^*) \le \nu_r(T)$, where E and H are over the spherically complete field* $\mathbf{F}$.

Proof. For each $\delta > 0$, there are $a_n \in E^*$ and $y_n \in H$ such that $Tx = \sum_{n=1}^{\infty} a_n(x)y_n$ for each $x \in E$ and $\sum_{n=1}^{\infty} \|a_n\|^r \|y_n\|^r \le \nu_r^r(T) + \delta$. Since $(T^*b)(x) = b(Tx) = \sum_{n=1}^{\infty} a_n(x)b(y_n)$ for each $b \in H^*$ and $x \in E$, then $T^*b = \sum_{n=1}^{\infty} y_n^*(b)a_n$, where $y_n^*(b) := b(y_n)$, that is correct due to the Hahn-Banach theorem for E and H over the spherically complete field $\mathbf{F}$ [21]. Therefore, $\nu_r^r(T^*) \le \sum_{n=1}^{\infty} \|y_n\|^r \|a_n\|^r \le \nu_r^r(T) + \delta$ since $\|y^*\|_{H^*} = \|y\|_H$ for each $y \in H$. □

2.4. For a space $L_k(H_1,\dots,H_k;H)$ of k-linear mappings of $H_1 \otimes \cdots \otimes H_k$ into H, we have its embedding into $L(E,H)$, where E is a normed space $H_1 \otimes \cdots \otimes H_k$ in its maximum norm topology for normed spaces $H_1,\dots,H_k,H$ over $\mathbf{F}$ (see Section 2.1, Definition 2.6, and Proposition 2.7). Therefore, we can define the following normed space $L_{k,r}(H_1,\dots,H_k;H) := L_k(H_1,\dots,H_k;H) \cap L_r(E;H)$, in particular, $L_{k,\infty}(H_1,\dots,H_k;H) := L_k(H_1,\dots,H_k;H)$ with the norm $\nu_r(J) =: \|J\|_r$, where $1 \le r \le \infty$. Certainly, $L_{k,r} \subset L_{k,q}$ for each $1 \le r < q \le \infty$.

Suppose that $(\Omega,\mathcal{B},\lambda)$ is a probability space (with a nonnegative measure λ), where $\mathcal{B}$ is a σ-algebra of subsets of Ω. We define a $\mathbf{K}$-linear Banach space $L^q(\Omega,\mathcal{B},\lambda;L_{k,r}(H_1,\dots,H_k;H))$ and $L^q(\Omega,\mathcal{B},\lambda;L_k(H_1,\dots,H_k;H))$ as a completion of a family of mappings $\sum_{j=1}^{n} A_j Ch_{W_j}$ with $A_j \in L_{k,r}(H_1,\dots,H_k;H)$ or $A_j \in L_k(H_1,\dots,H_k;H)$, respectively, and $W_j \in \mathcal{B}$ and $n \in \mathbb{N}$, where Ch_W is the characteristic function of a subset W. That is, as consisting of those mappings $\Omega \ni \nu \mapsto A(\nu) \in L_{k,r}(H_1,\dots,H_k;H)$ for which $\|A(\nu)\|_r$ is λ-measurable and

$$\begin{gathered}\|A\|_{L^q} := \left\{\int_\Omega \|A(\nu)\|_r^q \lambda(d\nu)\right\}^{1/q} < \infty, \quad 1 \le q < \infty; \\ \|A\|_{L^\infty} := \operatorname{ess-sup}_\lambda \|A(\nu)\|_r.\end{gathered} \tag{2.29}$$

2.5. We consider a C^∞-manifold X with an atlas $\mathrm{At}(X) = \{(U_j,\phi_j) : j \in \Lambda_X\}$, where $\bigcup_j U_j = X$, $\phi_j(U_j)$ are open in $c_0(\alpha,\mathbf{K})$ and U_j are open in X, $\phi_j : U_j \to \phi_j(U_j)$ are homeomorphisms, $\phi_i \circ \phi_j^{-1} \in C^\infty$ for each $U_i \cap U_j \ne \varnothing$ and $\|\phi_i \circ \phi_j^{-1}\|_{C^m} < \infty$ for each $m \in \mathbb{N}$, $\phi_j(U_j)$ are bounded in $c_0(\alpha,\mathbf{K})$ for each $j \in \Lambda_X$, Λ_X is a set, $C_b^n(X,H)$ is a completion of a set of all functions $f : X \to H$ such that $f \circ \phi_j^{-1} \in C^n(\phi_j(U_j),H)$ for each $j \in \Lambda_X$, and $\sup_j \|f \circ \phi_j^{-1}\|_{C^n} =: \|f\|_{C^n(X,H)} < \infty$, where H is a Banach space over $\mathbf{K}$. Then, $C^n(X,H)$ is the set of all functions $f : X \to H$ such that for each $x \in X$ there exists a neighborhood $x \in U \subset X$ for which $f|_U \in C_b^n(U,H)$.

By $L^s(\Omega,\mathcal{B},\lambda;C^n(X,H))$, we denote a completion of a space of simple functions $\sum_{j=1}^{n} \xi_j(x) Ch_{W_j}(\nu)$ with $\xi_j(x) \in C^n(X,H)$, $W_j \in \mathcal{B}$ and $n \in \mathbb{N}$, relative

to the norm

$$\|\xi\|_{L^s} := \left\{ \int_\Omega \|\xi(x,\nu)\|^s_{C^n(X,H)} \lambda(d\nu) \right\}^{1/s} < \infty, \tag{2.30}$$

for each $1 \le s < \infty$, or

$$\|\xi\|_{L^\infty} := \operatorname{ess-sup}_\lambda \|\xi(x,\nu)\|_{C^n(X,H)} < \infty, \tag{2.31}$$

where X is the C^∞ Banach manifold on $c_0(\alpha,\mathbf{K})$ and $\|\xi(x,\nu)\|_{C^n(X,H)}$ is attached to ξ as a function by $x \in X$ with parameter $\nu \in \Omega$ such that $\|\xi(x,\nu)\|_{C^n(X,H)}$ is a measurable function by ν.

Theorem 2.12. *Let*

$$\begin{gathered} G \in L^r(\Omega,\mathcal{B},\lambda;C^0(B_R \times L^q(\Omega,\mathcal{B},\lambda;C^0(B_R,H))^{\otimes l}, L_{k-l}(H^{\otimes(k-l)};H))), \\ \xi_1,\dots,\xi_k \in L^q(\Omega,\mathcal{B},\lambda;C^0(B_R,H)), A_{l+i} \in C^0(B_R,L(H)), \end{gathered} \tag{2.32}$$

for each $i = 1,\dots,k-l$ (see Section 2.1), where $B_R = B(\mathbf{K},0,R)$, $G = G(x;\xi_1,\dots,\xi_l;\nu)$, $\xi_i = \xi_i(x,\nu)$ with $x \in B_R$, $\nu \in \Omega$, $1/r + (k-l)/q = 1/s$ with $1 \le r,q,s \le \infty$. Then, $(\hat{P}_{(\xi_{l+1},\dots,\xi_k)} G \circ (A_{l+1} \otimes \cdots \otimes A_k)) \in L^s(\Omega,\mathcal{B},\lambda;C^0(B_R,H))$.

Proof. In $L^q(\Omega,\mathcal{F},\lambda;C^0(B_R \times V,W))$, the family of step functions $f(t,x,\omega) = \sum_{j=1}^n Ch_{U_j}(\omega) f_j(t,x)$ is dense, where $f_j \in C^0(B_R \times V,W)$, Ch_U is the characteristic function of $U \in \mathcal{F}$, $n \in \mathbb{N}$, V and W are Banach spaces over $\mathbf{K}$, $t \in B_R$, $x \in V$, $\omega \in \Omega$, since $\lambda(\Omega) = 1$ and λ is nonnegative [2, 3]. Each matrix element $F_{h,b}(x,\nu)$ is in $L^r(\Omega,\mathcal{B},\lambda;C^0(B_R,\mathbf{K}))$ and $\xi_j \in L^q(\Omega,\mathcal{B},\lambda;C^0(B_R,\mathbf{K}))$, where $F(x,\nu) := G(x;a_1,\dots,a_l;\nu) \cdot (A_{l+1}a_{l+1}(x),\dots,A_k a_k(x))$, $h \in H^*$, $b \in H$, $F_{h,b} := h(Fb)$, $a_i \in C^0(B_R,H)$ for each $i = 1,\dots,k$. Since $\|\xi_j(x,\nu)\|_{C^0(X,H)} \in L^q(\lambda)$, $\|F_{a,b}(x,\nu)\|_{C^n(X,H)} \in L^r(\lambda)$, then $F(x,\nu) \cdot w(x,\nu) \in L^s(\Omega,\mathcal{B},\lambda;C^0(B_R,H))$, where $w = (\xi_1,\dots,\xi_k)$. The operator $\hat{P}_w F$ is linear by w and F, hence it is defined on simple functions. In view of Lemma 2.1,

$$\begin{aligned} \|\hat{P}_w F(x,\nu)\|_H \le{}& \|F(x,\nu)\|_{C^0(B_R \times H^{\otimes l}, L_{k-l}(H^{\otimes(k-l)};H))} \\ &\times \prod_{i=l+1}^k [\|A_i\|_{C^0(B_R,L(H))} \|\xi_i(x,\nu)\|_{C^0(B_R,H)}] \end{aligned} \tag{2.33}$$

for λ-a.e. $\nu \in \Omega$, hence $\|(\hat{P}_w F)(x,\nu)\|_{L^s} \le \|G\|_{L^r} \prod_{i=l+1}^k [\|A_i\|_{C^0} \|\xi_i\|_{L^q}]$. □

Corollary 2.13. *If in Theorem 2.12 $\xi_i \in L^q(\Omega,\mathcal{B},\lambda;C^1(B_R,H))$ for each $i = 1,\dots,k$, then $(\hat{P}_w F) \in L^s(\Omega,\mathcal{B},\lambda;C^1(B_R,H))$ and*

$$\begin{aligned} &\|(\hat{P}_w G \cdot (A_{l+1} \otimes \cdots \otimes A_k))\|_{L^s(\lambda;C^1(B_R,H))} \\ &\quad \le \|G\|_{L^r(\lambda;C^0(B_R \times H^{\otimes l}, L_{k-l}(H^{\otimes(k-l)};H)))} \\ &\qquad \times \prod_{i=l+1}^k [\|A_i\|_{C^0(B_R,L(H))} \|\xi_i\|_{L^q(\lambda;C^1(B_R,H))}]. \end{aligned} \tag{2.34}$$

Proof. In view of Lemma 2.2 and Theorem 2.12,

$$\begin{aligned}\|(\hat{P}_w F)(x,\nu)\|_{C^1(B_R,H)} &\le \|G(x;\xi_1,\dots,\xi_l;\nu)\|_{C^0(B_R\times H^{\otimes l},L_{k-l}(H^{\otimes(k-l)},H))}\\ &\quad\times\prod_{i=l+1}^{k}\left[\|A_i\|_{C^0(B_R,L(H))}\|\xi_i(x,\nu)\|_{C^1(B_R,H)}\right]\end{aligned}\tag{2.35}$$

for λ-almost each $\nu\in\Omega$. From this formula (2.34) follows. □

3. Markov quasimeasures for a non-Archimedean Banach space

Remark 3.1. Let $H=c_0(\alpha,\mathbf{K})$ be a Banach space over a local field $\mathbf{K}$. Let $\mathcal{U}^{\mathcal{P}}$ be a cylindrical algebra generated by projections on finite-dimensional over $\mathbf{K}$ subspaces F in H and Borel σ-algebras $Bf(F)$. Denote by $\mathcal{U}$ the minimal σ-algebra $\sigma(\mathcal{U}^{\mathcal{P}})$ generated by $\mathcal{U}^{\mathcal{P}}$. Each vector $x\in H$ is considered as continuous linear functional on H by the formula $x(y)=\sum_j x^j y^j$ for each $y\in H$, so there is the natural embedding $H\hookrightarrow H^*=l^\infty(\alpha,\mathbf{K})$, where $x=\sum_j x^j e_j$, $x^j\in\mathbf{K}$.

3.1. Notes and definitions. Let $T=B(\mathbf{K},t_0,r)$ and $X_t=X$ be a locally $\mathbf{K}$-convex space for each $t\in T$. Let $(\tilde{X}_T,\tilde{\mathcal{U}}):=\prod_{t\in T}(X_t,\mathcal{U}_t)$ be a product of measurable spaces, where $\mathcal{U}_t$ is a σ-algebra of subsets of X_t, $\tilde{\mathcal{U}}$ is the σ-algebra of cylindrical subsets of $\tilde{X}_T$ generated by projections $\tilde{\pi}_q:\tilde{X}_t\to X^q$, $X^q:=\prod_{t\in q}X_t$, and $q\subset T$ is a finite subset of T (see [4, Section I.1.3]). A function $P(t_1,x_1,t_2,A)$ with values in $\mathbf{C}$ for each $t_1\ne t_2\in T$, $x_1\in X_{t_1}$, $A\in\mathcal{U}_{t_2}$ is called a transition measure if it satisfies the following conditions:

(i) the set function

$$\nu_{x_1,t_1,t_2,}(A):=P(t_1,x_1,t_2,A)\tag{3.1}$$

is a σ-additive measure on $(X_{t_2},\mathcal{U}_{t_2})$;

(ii) the function

$$\alpha_{t_1,t_2,A}(x_1):=P(t_1,x_1,t_2,A)\tag{3.2}$$

of the variable x_1 is $\mathcal{U}_{t_1}$-measurable;

(iii)

$$P(t_1,x_1,t_2,A)=\int_{X_s}P(t_1,x_1,s,dy)P(s,y,t_2,A)\quad\text{for each } t_1\ne t_2\in T.\tag{3.3}$$

A transition measure $P(t_1,x_1,t_2,A)$ is called normalised if

$$P(t_1,x_1,t_2,X_{t_2})=1,\quad\text{for each } t_1\ne t_2\in T.\tag{3.4}$$

For each set $q = (t_0, t_1, \dots, t_{n+1})$ of pairwise distinct points in T, there is a measure in $X^s := \prod_{t\in s} X_t$ defined by the formula

$$\mu^q_{x_0}(E) = \int_E \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k), \quad E \in \mathfrak{U}^s := \prod_{t\in s} \mathfrak{U}_t, \tag{3.5}$$

where $s = q \setminus \{t_0\}$, variables $x_1, \dots, x_{n+1}$ are such that $(x_1, \dots, x_{n+1}) \in E$, and $x_0 \in X_{t_0}$ is fixed.

Let $E = E_1 \times X_{t_j} \times E_2$, where $E_1 \in \prod_{i=1}^{j-1} \mathfrak{U}_{t_i}$ and $E_2 \in \prod_{i=j+1}^{n+1} \mathfrak{U}_{t_i}$, then

$$\begin{aligned}\mu^q_{x_0}(E) &= \int_{E_1\times E_2} \left[\prod_{k=1}^{j-1} P(t_{k-1}, x_{k-1}, t_k, dx_k)\right] \\ &\quad \times \left[\int_{X_{t_j}} P(t_{j-1}, x_{j-1}, t_j, dx_j) \prod_{k=j+1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k)\right] \\ &= \mu^r_{x_0}(E_1 \times E_2),\end{aligned} \tag{3.6}$$

where $r = q \setminus \{t_j\}$. From (3.6), it follows that

$$[\mu^q_{x_0}]^{\pi^q_v} = \mu^v_{x_0}, \tag{3.7}$$

for each $v < q$ (i.e., $v \subset q$), where $\pi^q_v : X^s \to X^w$ is the natural projection, $s = q \setminus \{t_0\}$, $w = v \setminus \{t_0\}$. Therefore, due to conditions (3.4), (3.5), and (3.7), $\{\mu^q_{x_0}; \pi^q_v; \Upsilon_T\}$ is the consistent family of measures, which induce the quasimeasure $\tilde{\mu}_{x_0}$ on $(\tilde{X}_T, \tilde{\mathfrak{U}})$ such that $\tilde{\mu}_{x_0}(\pi_q^{-1}(E)) = \mu^q_{x_0}(E)$ for each $E \in \mathfrak{U}^s$, where Υ_T is the family of all finite subsets q in T such that $t_0 \in q \subset T$, $v \le q \in \Upsilon_T$, $\pi_q : \tilde{U}_T \to X^s$ is the natural projection, $s = q \setminus \{t_0\}$.

The quasimeasures given by (3.1), (3.2), (3.3), (3.4), (3.5), and (3.7) are called Markov quasimeasures.

Proposition 3.2. *If a normalized transition measure P satisfies the condition*

$$C := \sup_q \left[\sum_{k=1}^{n} \ln\left(\sup_x \|\nu_{x,t_{k-1},t_k}\|\right)\right] < \infty, \tag{3.8}$$

where $q = (t_0, t_1, \dots, t_n)$ with pairwise distinct points $t_0, \dots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov quasimeasure $\tilde{\mu}_{x_0}$ is bounded.

Proposition 3.3. *If*

$$C_x := \sup_q \left[\sum_{k=1}^{n} \ln\|\nu_{x,t_{k-1},t_k}\|\right] = \infty, \tag{3.9}$$

for each x, where $q = (t_0, t_1, \dots, t_n)$ with pairwise distinct points $t_0, \dots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov quasimeasure $\tilde{\mu}_{x_0}$ has the unbounded variation on each nonvoid set $E \in \mathfrak{U}^s$.

Proof. (1) If $E \in \tilde{\mathcal{U}}$, then $E \in \mathcal{U}^s$ for some set $q = (t_0, t_1, \dots, t_n)$ with pairwise distinct points $t_0, \dots, t_n \in T$ and $n \in \mathbb{N}$ and $s = q \setminus \{t_0\}$; consequently, $|\mu^q_{x_0}(E)| \le \prod_{k=1}^n \sup_x \|\nu_{x,t_{k-1},t_k}\| \le \exp(C)$ since $t_k \in T$ for each $k = 0, 1, \dots, n$.

(2) For each (t_1, t_2, x), there exists a compact set $\delta(t_1, t_2, x) \in \mathcal{U}_{t_2}$ such that $P(t_1, x_1, t_2, \delta(t_1, t_2, x)) > 1 + \epsilon(t_1, t_2, x_1, x)$, where $\epsilon(t_1, t_2, x) > 0$. In view of condition (3.9), for each $R > 0$ and x we choose q such that $\sum_{k=1}^n \epsilon(t_k, t_{k+1}, x_1, x) > R$. For chosen $u \ne u_1 \in T$ and $x \in X_u$, we represent the set $\delta(u, u_1, x)$ as a finite union of disjoint subsets γ_{j_1} such that for each γ_{j_1} and $u_2 \ne u_1$ there is a set δ_{j_1} satisfying $P(u_1, x_1, u_2, \delta_{j_1}) \ge 1 + \epsilon(u_1, u_2, x_1, x)$ for each $x \in \gamma_{j_1}$. Then by induction $\delta_{j_1,\dots,j_n} = \bigcup_{j_{n+1}=1}^{m_{n+1}} \gamma_{j_1,\dots,j_{n+1}}$ so that for $u_{n+2} \ne u_{n+1} \in T$ there is a set $\delta_{j_1,\dots,j_{n+1}}$ for which $P(u_{n+1}, x_{n+1}, u_{n+2}, \delta_{j_1,\dots,j_{n+1}}) \ge 1 + \epsilon(u_{n+1}, u_{n+2}, x_{n+1}, x)$ for each $x \in \gamma_{j_1,\dots,j_{n+1}}$. Put $\Gamma^{u,x_0}_{j_1,\dots,j_n} = \{x : x(u) = x_0, x(u_1) \in \gamma_{j_1}, \dots, x(u_n) \in \delta_{j_1,\dots,j_n}, x(u_{n+1}) \in \gamma_{j_1,\dots,j_n}\}$, and $\Gamma^{u,x_0} := \bigcup_{j_1,\dots,j_n} \Gamma^{u,x_0}_{j_1,\dots,j_n}$. Then,

$$\begin{aligned}\tilde{\mu}_{x_0}(\Gamma^{u,x_0}) &= \sum_{j_1,\dots,j_n} \int_{\delta_{j_1,\dots,j_n}} \int_{\gamma_{j_1,\dots,j_n}} \cdots \int_{\gamma_{j_1}} \prod_{k=1}^{n+1} P(u_{k-1}, x_{k-1}, u_k, dx_k) \\ &\ge \prod_{k=1}^{n} [1 + \epsilon(u_{k-1}, u_k, x_{k-1}, x_k)] > R.\end{aligned} \tag{3.10}$$

□

3.2. Evidently condition (3.8) of Proposition 3.2 is satisfied for the nonnegative normalized transition measure.

3.3. Let $X_t = X$ for each $t \in T$, $\tilde{X}_{t_0,x_0} := \{x \in \tilde{X}_T : x(t_0) = x_0\}$. We define a projection operator $\bar{\pi}_q : x \mapsto x_q$, where x_q is defined on $q = (t_0, \dots, t_{n+1})$ such that $x_q(t) = x(t)$ for each $t \in q$, that is, $x_q = x|_q$. For every $F : \tilde{X}_T \to \mathbf{C}$, there corresponds $(S_q F)(x) := F(x_q) = F_q(y_0, \dots, y_n)$, where $y_j = x(t_j)$. $F_q : X^q \to \mathbf{C}$. We put $\mathcal{F} := \{F \mid F : \tilde{X}_T \to \mathbf{C}, S_q F$ are $\mathcal{U}^q$-measurable$\}$. If $F \in \mathcal{F}$, $\tau = t_0 \in q$, then there exists an integral

$$J_q(F) = \int_{X^q} (S_q F)(x_0, \dots, x_n) \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k). \tag{3.11}$$

Definition 3.4. A function F is called integrable with respect to the Markov quasimeasure μ_{x_0} if the limit

$$\lim_q J_q(F) =: J(F) \tag{3.12}$$

along the generalized net by finite subsets q of T exists. This limit is called a "functional integral" with respect to the Markov quasimeasure

$$J(F) = \int_{\tilde{X}_{t_0,x_0}} F(x) \mu_{x_0}(dx). \tag{3.13}$$

Remark 3.5. Consider a complex-valued measure $P(t,A)$ on $(X,\mathcal{U})$ for each $t \in T := B(\mathbf{K},0,R)$ such that $A - x \in \mathcal{U}$ for each $A \in \mathcal{U}$ and $x \in X$, where $A \in \mathcal{U}$, X is a locally $\mathbf{K}$-convex space, and $\mathcal{U}$ is a σ-algebra of X. Suppose that P is a spatially homogeneous transition measure (see also Section 3.1), that is,

$$P(t_1,x_1,t_2,A) = P(t_2 - t_1, A - x_1), \tag{3.14}$$

for each $A \in \mathcal{U}$, $t_1 \neq t_2 \in T$, and every $x_1 \in X$, where $P(t,A)$ satisfies the condition

$$P(t_1 + t_2, A) = \int_X P(t_1, dy) P(t_2, A - y). \tag{3.15}$$

The transition measure $P(t_1,x_1,t_2,A)$ is called homogeneous. In particular, for $T = \mathbf{Z}_p$ we have

$$P(t+1,A) = \int_X P(t,dy) P(1, A - y). \tag{3.16}$$

If $P(t,A)$ is a continuous function by $t \in T$ for each fixed $A \in \mathcal{U}$, then (3.16) defines $P(t,A)$ for each $t \in T$, when $P(1,A)$ is given since $\mathbf{Z}$ is dense in $\mathbf{Z}_p$.

3.4. Notes and definitions. Let X be a locally $\mathbf{K}$-convex space and P satisfies conditions (3.1), (3.2), and (3.3). For x and $z \in \mathbf{Q}_p^n$, we denote by (z,x) the sum $\sum_{j=1}^n x_j z_j$, where $x = (x_j : j = 1,\dots,n)$, $x_j \in \mathbf{Q}_p$. We consider a character of X, $\chi_y : X \to \mathbf{C}$ given by

$$\chi_y(x) = \epsilon^{z^{-1}\{(e,y(x))\}_p}, \tag{3.17}$$

for each $\{(e,y(x))\}_p \neq 0$, $\chi_y(x) := 1$ for $\{(e,y(x))\}_p = 0$, where $\epsilon = 1^z$ is a root of unity, $z = p^{\operatorname{ord}(\{(e,y(x))\}_p)}$, $y \in X^*$, X^* denotes the topologically conjugated space of continuous $\mathbf{K}$-linear functionals on X, and the field $\mathbf{K}$ as the $\mathbf{Q}_p$-linear space is n-dimensional, that is, $\dim_{\mathbf{Q}_p} \mathbf{K} = n$, $\mathbf{K}$ as the Banach space over $\mathbf{Q}_p$ is isomorphic with $\mathbf{Q}_p^n$, $e = (1,\dots,1) \in \mathbf{Q}_p^n$ (see [8, 9, 15, 22]). Then,

$$\phi(t_1,x_1,t_2,y) := \int_X \chi_y(x) P(t_1,x_1,t_2,dx) \tag{3.18}$$

is the characteristic functional of the transition measure $P(t_1,x_1,t_2,dx)$ for each $t_1 \neq t_2 \in T = B(\mathbf{K},t_0,R)$ and each $x_1 \in X$. In the particular case of P satisfying conditions (3.14) and (3.15) with $t_0 = 0$, its characteristic functional is such that

$$\phi(t_1,x_1,t_2,y) = \psi(t_2 - t_1, y)\chi_y(x_1), \tag{3.19}$$

where

$$\psi(t,y) := \int_X \chi_y(x)P(t,dx), \qquad \psi(t_1+t_2,y) = \psi(t_1,y)\psi(t_2,y), \tag{3.20}$$

for each $t_1 \neq t_2 \in T$ and $y \in X^*$, $x_1 \in X$.

4. Non-Archimedean stochastic processes

4.1. Remark and definition. Let $(\Omega,\mathcal{F},\lambda)$ be a probability space. Points $\omega \in \Omega$ are called "elementary events" and values $\lambda(S)$ probabilities of events $S \in \mathcal{F}$. A measurable map $\xi : (\Omega,\mathcal{F}) \to (X,\mathcal{B})$ is called a random variable with values in X, where $\mathcal{B}$ is the σ-algebra of a locally $\mathbf{K}$-convex space X. The random variable ξ induces a normalized measure $\nu_\xi(A) := \lambda(\xi^{-1}(A))$ in X and a new probability space $(X,\mathcal{B},\nu_\xi)$. We take $X = C^0(T,H)$ (see Section 2.1) and the σ-algebra $\mathcal{B}$ which is the subalgebra of the Borel σ-algebra $Bf(X)$ of X, where H is a Banach space over $\mathbf{K}$, $T = B(\mathbf{K},t_0,R) =: B_R$, $0 < R < \infty$, $\mathbf{K}$ is the local field. A random variable $\xi : \omega \mapsto \xi(t,\omega)$ with values in $(X,\mathcal{B})$ is called a (non-Archimedean) stochastic process on T with values in H.

Events $S_1,\dots,S_n$ are called independent in total if $P(\prod_{k=1}^n S_k) = \prod_{k=1}^n P(S_k)$. Then σ-subalgebras $\mathcal{F}_k \subset \mathcal{F}$ are said to be independent if all collections of events $S_k \in \mathcal{F}_k$ are independent in total, where $k = 1,\dots,n$, $n \in \mathbb{N}$. To each collection of random variables ξ_y on $(\Omega,\mathcal{F})$ with $y \in \Upsilon$ is related the minimal σ-algebra $\mathcal{F}_\Upsilon \subset \mathcal{F}$ with respect to which all ξ_y are measurable, where Υ is a set. The collections $\{\xi_y : y \in \Upsilon_j\}$ are called independent if so are $\mathcal{F}_{\Upsilon_j}$, where $\Upsilon_j \subset \Upsilon$ for each $j = 1,\dots,n$, $n \in \mathbb{N}$.

Besides $X = C^0(T,H)$, it is possible to consider the product locally $\mathbf{K}$-convex spaces $X = H^T$.

Definition 4.1. Define a (non-Archimedean) stochastic process $w(t,\omega)$ with values in H as a stochastic process such that

(i) the differences $w(t_4,\omega) - w(t_3,\omega)$ and $w(t_2,\omega) - w(t_1,\omega)$ are independent for each chosen (t_1,t_2) and (t_3,t_4) with $t_1 \neq t_2$, $t_3 \neq t_4$, either t_1 or t_2 is not in the two-element set $\{t_3,t_4\}$, where $\omega \in \Omega$;

(ii) the random variable $w(t,\omega) - w(u,\omega)$ has a distribution $\mu^{F_{t,u}}$, where μ is a probability measure on $C^0(T,H)$, $\mu^g(A) := \mu(g^{-1}(A))$ for $g \in C^0(T,H)^*$, and each $A \in \mathcal{B}$, a continuous linear functional $F_{t,u}$ is given by the formula $F_{t,u}(w) := w(t,\omega) - w(u,\omega)$ for each $w \in L^q(\Omega,\mathcal{F},\lambda; C_0^0(T,H))$, where $1 \le q \le \infty$, $C_0^0(T,H) := \{f : f \in C^0(T,H), f(t_0) = 0\}$ is the closed subspace of $C_0^0(T,H)$;

(iii) we also put $w(0,\omega) = 0$, that is, we consider a Banach subspace $L^q(\Omega,\mathcal{F},\lambda;C_0^0(T,H))$ of $L^q(\Omega,\mathcal{F},\lambda;C^0(T,H))$, where $\Omega \neq \varnothing$.

This definition is justified by the following theorem.

Theorem 4.2. *There exists a family of pairwise inequivalent (non-Archimedean) stochastic processes on $C_0^0(T,H)$ of the cardinality $\mathcal{C}$, where $\mathcal{C} := \operatorname{card}(\mathbf{R})$.*

Proof. Since H is over the local field, then H has a projection π_0 on its Banach subspace H_0 of separable type over $\mathbf{K}$ (see its definition in [21]), that is, H_0 is isomorphic with $c_0(\alpha,\mathbf{K})$ with countable α. Therefore, a σ-additive measure μ_0 on $(H_0, Bf(H_0))$ induces a σ-additive measure μ on $(H, \pi_0^{-1}[Bf(H_0)])$, where $\pi_0^{-1}[Bf(H_0)] := \{\pi_0^{-1}(A) : A \in Bf(H_0)\}$. Therefore, it is sufficient to consider the case of H of separable type over $\mathbf{K}$.

If w is the real-valued nonnegative Haar measure on $\mathbf{K}$ with $w(B(\mathbf{K},0,1)) = 1$, then it does not have any atoms since it is defined on $Bf(\mathbf{K})$, each singleton $\{x\}$ is the Borel subset and $w(y+A) = w(A)$ for each $A \in Bf(\mathbf{K})$. Indeed, if w had some atom E, then it would be a singleton since $\mathbf{K}$ is the complete separable metric space, and for each disjoint w-measurable subsets A and S in E, either $w(A) = w(E) > 0$ with $w(S) = 0$, or $w(S) = w(E) > 0$ with $w(A) = 0$. But $\sum_{y\in\mathbf{K}} w(y+\{x\}) = \infty$ when $w(\{x\}) > 0$ for a singleton $\{x\}$ (see [2, Chapter VII]). Therefore, each measure $\mu_j(dx^j) = f_j(x^j)w(dx^j)$ on $\mathbf{K}$ does not have any atom since w does not have any atom, where $f_j \in L^1(\mathbf{K}, Bf(\mathbf{K}), w, \mathbf{R})$ (i.e., f_j is w-measurable and $\|f_j\|_{L^1} := \int_{\mathbf{K}} |f_j(x) \mid w(dx) < \infty)$ and $\mu_j(\mathbf{K}) = 1$. Hence, each measure μ on $C_0^0(T,H)$ does not have any atom when $\mu(dx) = \bigotimes_{j=1}^{\infty} \mu_j(dx^j)$, where $C_0^0(T,H)$ is isomorphic with $c_0(\omega_0,\mathbf{K})$, $x \in C_0^0(T,H)$, $x = (x^j : j \in \omega_0)$, $x^j \in \mathbf{K}$, $x = \sum_j x^j e_j$, e_j is the standard orthonormal base in $c_0(\omega_0,\mathbf{K})$, and ω_0 is the first countable ordinal, since $\mathbf{K}$ is the local field (see [8, 9, 21]).

Consider an operator $J \in L_1(c_0)$ in the Banach space $c_0 := c_0(\omega_0,\mathbf{K})$ such that $Je_i = v_i e_i$ with $v_i \neq 0$ for each i and a measure $\nu(dx) := f(x)w(dx)$, where $f : \mathbf{K} \to [0,1]$ is a function belonging to the space $L^1(\mathbf{K}, w, \mathbf{R})$ such that $\lim_{|x|\to\infty} f(x) = 0$ and $\nu(\mathbf{K}) = 1$, $\nu(S) > 0$ for each open subset S in $\mathbf{K}$, for example, when $f(x) > 0$ w-almost everywhere. In view of the Prohorov theorem, there exists the following σ-additive product measure.

Consider the product of measures (i) $\mu(dx) := \prod_{i=1}^{\infty} \nu_i(dx^i)$ on the σ-algebra of Borel subsets of c_0 since the Borel σ-algebras defined for the weak topology of c_0 and for the norm topology of c_0 coincide, where $\nu_i(dx^i) := f(x^i/v_i)\nu(dx^i/v_i)$ (see [2, 3, 8, 9]).

Let Z be a compact subset without isolated points in a local field $\mathbf{K}$, for example, $Z = B(\mathbf{K}, t_0, 1)$. Then the Banach space $C^0(Z,\mathbf{K})$ has the Amice polynomial orthonormal base $Q_m(x)$, where $x \in Z$, $m \in \mathbf{N}_o := \{0,1,2,\dots\}$ [1]. Each $f \in C^0$ has a decomposition $f(x) = \sum_m a_m(f)Q_m(x)$ such that $\lim_{m\to\infty} a_m = 0$, where $a_m \in \mathbf{K}$. These decompositions establish the isometric isomorphism $\theta : C^0(T,\mathbf{K}) \to c_0(\omega_0,\mathbf{K})$ such that $\|f\|_{C^0} = \max_m |a_m(f)| = \|\theta(f)\|_{c_0}$.

If $H = c_0(\omega_0,\mathbf{K})$, then the Banach space $C^0(T,H)$ is isomorphic with the tensor product $C^0(T,\mathbf{K}) \otimes H$ (see [21, Section 4.R]). If $J_i \in L_1(Y_i)$ is nondegenerate for each $i = 1,2$, that is, $\ker(J_i) = \{0\}$, then $J := J_1 \otimes J_2 \in L_1(Y_1 \otimes Y_2)$ is nondegenerate (see also [21, Theorem 4.33]). If u_i are roots of basic polynomials Q_m as in [1], then $Q_m(u_i) = 0$ for each $m > i$. The set $\{u_i : i\}$ is dense in T. Put $Y_1 = C^0(T,\mathbf{K})$ and $Y_2 = H$ and $J := J_1 \otimes J_2 \in L_1(Y_1 \otimes Y_2)$, where $J_1 Q_m := \alpha_m Q_m$ such that $\alpha_m \neq 0$ for each m and $\sum_i |\alpha_i| < \infty$. Take J_2 to be also nondegenerate.

Then J induces a product measure μ on $C^0(T,H)$ such that $\mu = \mu_1 \otimes \mu_2$, where μ_i are measures on Y_i induced by J_i due to formulas (3.17) and (3.18). Analogously, considering the following Banach subspace $C_0^0(T,H) := \{f \in C^0(T,H) : f(t_0) = 0\}$ and operators $J := J_1 \otimes J_2 \in L_1(C_0^0(T,\mathbf{K}) \otimes H)$, we get the measures μ on it also, where $t_0 \in T$ is a marked point.

For each finite number of points $(t_1,\dots,t_n) \subset T$ and $(z_1,\dots,z_n) \subset H$, there exists a closed subset $C^0(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n)) := \{f \in C^0(T,H) : f(t_i) = z_i;\ i = 1,\dots,n\}$ such that $C^0(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n)) = (z_1,\dots,z_n) + C^0(T,H;(t_1,\dots,t_n);(0,\dots,0))$, where $C^0(T,H;(t_1,\dots,t_n);(0,\dots,0))$ is the Banach subspace of finite codimension n in $C^0(T,H)$. Therefore,

We get that (ii) σ-algebras $F_{t_2,t_1}^{-1}(Bf(H))$ and $F_{t_4,t_3}^{-1}(Bf(H))$ are independent subalgebras in the Borel σ-algebra $Bf(C_0^0(T,H))$ when (t_1,t_2) and (t_3,t_4) satisfy Definition 4.1(i).

Put $P(t_1,x_1,t_2,A) := \mu(\{f : f(t_1) = x_1,\ f(t_2) \in A\})$ for each $t_1 \neq t_2 \in T$, $x_1 \in H$ and $A \in Bf(H)$. In view of (iii), we get that P satisfies conditions (3.1), (3.2), (3.3), and (3.4). By the construction above (and Proposition 3.2 also), the Markov quasimeasure $\tilde{\mu}_{x_0}$ induced by μ is bounded since μ is bounded, where $x_0 = 0$ for $C_0^0(T,H)$. Let Ω be a set of elementary events $\omega := \{f : f \in C_0^0(T,H),\ f(t_i) = x_i,\ i \in \Lambda_\omega\}$, where Λ_ω is a countable subset of $\mathbb{N}$, $x_i \in H$, $(t_i : i \in \Lambda_\omega)$ is a subset of T of pairwise distinct points. There exists the algebra $\tilde{\mathfrak{U}}$ of cylindrical subsets of $C_0^0(T,H)$ induced by projections $\pi_s : C_0^0(T,H) \to H^s$, where $H^s := \prod_{t\in s} H_t$, $s = (t_1,\dots,t_n)$ are finite subsets of T, $H_t = H$ for each $t \in T$. In view of the Kolmogorov theorem [4, 8, 9, 14], $\tilde{\mu}_{x_0}$ on $((C_0^0(T,H),\tau_w),\tilde{\mathfrak{U}})$ induces the probability measure λ on $(\Omega, Bf(\Omega))$, where τ_w is the weak topology in $C_0^0(T,H)$.

Therefore, using the product of measures, we get examples of such measures μ for which stochastic processes exist (see also [8, Theorem 3.23, Lemmas 2.3, 2.5, 2.8 and Section 3.30]). Hence, to each such measure on $C_0^0(T,H)$, there corresponds the stochastic process. Considering all operators $J := J_1 \otimes J_2 \in L_1(Y_1 \otimes Y_2)$ and the corresponding measures as above, we get $\mathscr{C}^{\aleph_0} = \mathscr{C}$ inequivalent measures by the Kakutani theorem II.4.1 [4] for each chosen f. □

4.2. We consider stochastic processes $E \in L^r(\Omega,\mathscr{F},\lambda;C^0(T,L_v(H)))$ such that $E = E(t,\omega)$, where $1 \le v \le \infty$, $1 \le r \le \infty$, $t \in T = B(\mathbf{K},t_0,R)$ and $\omega \in \Omega$ (see Section 2.5 and Definition 4.1).

Definition 4.3. For $L^r(\Omega,\mathscr{F},\lambda;C^0(T,L_v(H)))$, the non-Archimedean stochastic integral is defined by the equation

$$\mathscr{I}(E)(t,\omega) := (\hat{P}_w E)(t,\omega) = \sum_{j=0}^{\infty} E(t_j,\omega)[w(t_{j+1},\omega) - w(t_j,\omega)], \tag{4.1}$$

where $w = w(t,\omega)$, $t_j = \sigma_j(t)$ (see Section 2.1).

Proposition 4.4. *The non-Archimedean stochastic integral is the continuous* $\mathbf{K}$*-bilinear operator from* $L^r(\Omega,\mathcal{F},\lambda;C^0(T,L_v(H)))\otimes L^q(\Omega,\mathcal{F},\lambda;C_0^0(T,H))$ *into* $L^s(\Omega,\mathcal{F},\lambda;C^0(T,H))$*, where* $1/q+1/r=1/s$ *and* $1\le r,q,s\le\infty$*.*

Proof. It follows from Theorem 2.12, since $(\hat{P}_{aw+b\mathcal{Y}}E)=(a\hat{P}_wE)+(b\hat{P}_{\mathcal{Y}}E)$ and $(\hat{P}_w(aE+bV))=(a\hat{P}_wE)+b(\hat{P}_wV)$ for each $a,b\in\mathbf{K}$, each $w,\mathcal{Y}\in L^q(\Omega,\mathcal{F},\lambda;C_0^0(T,H))$ and each $E,V\in L^r(\Omega,\mathcal{F},\lambda;C^0(T,L_v(H)))$. □

4.3. Consider a function f from $T\times H$ into $Y=c_0(\beta,\mathbf{K})$ satisfying the following conditions:

(a) $f\in C^1(T\times H,Y)$,

(b) $(\bar{\Phi}^nf)(t,x;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n)\in C^0(T\times H^{n+1}\times\mathbf{K}^{\mathbf{n}},Y)$ for each $n\le m$,

(c) $(\bar{\Phi}^nf)(t,x;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n)=0$ for $n=m+1$,

(d) $f(t,x)-f(0,x)=(\hat{P}_tg)(t,x)$ with $g\in C^0(T\times H,Y)$, where $2\le m\in\mathbb{N}$, $f=f(t,x)$, $t\in T$, $x\in H$; $h_1,\dots,h_n\in H$, $\zeta_1,\dots,\zeta_n\in\mathbf{K}$; $\hat{P}_u$ is the antiderivation operator on $C^0(T,Y)$, $(\hat{P}_tg)(t,x)$ is defined for each fixed $x\in H$ by $t\in T$ such that $(\hat{P}_tg)(t,x)=\hat{P}_ug(u,x)|_{u=t}$ with $u\in T$ (see Section 2.1 and also about difference quotients $(\bar{\Phi}^nf)$ and spaces of functions of smoothness class C^n in [10, 15]).

Suppose that $a\in L^s(\Omega,\mathcal{F},\lambda;C^0(T,H))$, $w\in L^q(\Omega,\mathcal{F},\lambda;C_0^0(T,H))$ and $E\in L^r(\Omega,\mathcal{F},\lambda;C^0(T,L(H)))$, where $1/r+1/q=1/s$, $1\le r,q,s\le\infty$, $a=a(t,\omega)$, $E=E(t,\omega)$, $t\in T$, $\omega\in\Omega$. A stochastic process of the type

$$\xi(t,\omega)=\xi_0(\omega)+(\hat{P}_ua)(u,\omega)\big|_{u=t}+(\hat{P}_{w(u,\omega)}E)(u,\omega)\big|_{u=t} \tag{4.2}$$

is said to have a stochastic differential

$$d\xi(t,\omega)=a(t,\omega)dt+E(t,\omega)dw(t,\omega), \tag{4.3}$$

since $(\hat{P}_tg)'(t)=g(t)$ for each $g\in C^0(T,H)$, where $\xi_0\in L^s(\Omega,\mathcal{F},\lambda;H)$, $t_0,t\in T$, $w(t_0,\omega)=0$. In view of Lemma 2.2, Theorem 2.12, and Proposition 4.4, $\xi\in L^s(\Omega,\mathcal{F},\lambda;C^0(T,H))$.

Let $\hat{P}_{u^b,w^h}$ denote the antiderivation operator $\hat{P}_{(\xi_1,\dots,\xi_{b+h})}$ given by formula (2.4), where $\xi_1=u,\dots,\xi_b=u$, $\xi_{b+1}=w,\dots,\xi_{b+h}=w$. Henceforth, the notation

$$\begin{aligned}&\tilde{P}^n_{a,Ew}f(u,\xi(u,\omega))\\&:=\sum_{k=1}^{n}(k!)^{-1}\sum_{l=0}^{k}\binom{k}{l}\left(\hat{P}_{u^{k-l},w(u,\omega)^l}\left[\left(\frac{\partial^kf}{\partial x^k}\right)(u,\xi(u,\omega))\circ(a^{\otimes(k-l)}\otimes E^{\otimes l})\right]\right)\end{aligned} \tag{4.4}$$

is used for such operator when it exists (see the conditions 4.3 (a–d) above and (4.9), (4.13) below), where $n\in\mathbb{N}$ or $n=\infty$.

Theorem 4.5. *Let conditions (a), (b), (c), (d), (4.2), and (4.3) be satisfied, then*

$$f(t,\xi(t,\omega)) = f(t_0,\xi_0) + \hat{P}_u f'_t(u,\xi(u,\omega))\big|_{u=t} + \tilde{P}^m_{a,Ew} f(u,\xi(u,\omega))\big|_{u=t}. \tag{4.5}$$

Proof. Let $\{u_k : k = 0,1,\dots,n\}$ be a finite $|\pi|^l$ net in T, that is, for each $t \in T$ there exists k such that $|u_k - t| \le |\pi|^l$, where $n = n(k) \in \mathbb{N}$, $\pi \in \mathbf{K}$, $p^{-1} \le |\pi| < 1$, and $|\pi|$ is the generator of the valuation group of $\mathbf{K}$ since the ball T is compact. We choose $t = u_n$ and $t_0 = u_0$. Denote by $\eta(t)$ a stochastic process $f(t,\xi(t,\omega))$. Then by the Taylor formula (see [19, Theorem 29.4] and [8, Theorem A.5]), we have

$$\begin{aligned}
&f(t,\xi(t)) - f(u,\xi(u)) \\
&\quad = f'_t(u,\xi(u))(t-u) + f'_x(u,\xi(u))\cdot(\Delta\xi) + \frac{1}{2} f''_{t,t}(u,\xi(u))(t-u)^2 \\
&\qquad + f''_{t,x}(u,\xi(u))\cdot((t-u),\Delta\xi) + \frac{1}{2} f''_{x,x}(u,\xi(u))\cdot(\Delta\xi,\Delta\xi) \\
&\qquad + \{(\bar{\Phi}^2 f)(u,\xi(u);(t-u),(t-u);1,1) - \frac{1}{2} f''_{t,t}(u,\xi(u))(t-u)^2\} \\
&\qquad + \{(\bar{\Phi}^2 f)(u,\xi(u);(t-u),\Delta\xi;1,1) \\
&\qquad\quad + (\bar{\Phi}^2 f)(u,\xi(u);\Delta\xi,(t-u);1,1) - f''_{t,x}(u,\xi(u))\cdot(t-u,\Delta\xi)\} \\
&\qquad + \{(\bar{\Phi}^2 f)(u,\xi(u);\Delta\xi,\Delta\xi;1,1) - \frac{1}{2} f''_{x,x}(u,\xi(u))\cdot(\Delta\xi,\Delta\xi)\},
\end{aligned} \tag{4.6}$$

where $\Delta\xi = \xi(t) - \xi(u)$. For a brevity, we denote $\xi(t) = \xi(t,\omega)$ and $w(t) := w(t,\omega)$ for a chosen ω. If $t_n = \sigma_n(t)$ for each $n = 0,1,2,\dots$, then by formulas (4.2) and (2.4),

$$\xi(t_{n+1},\omega) - \xi(t_n,\omega) = a(t_n,\omega)(t_{n+1}-t_n) + E(t_n,\omega)(w(t_{n+1},\omega) - w(t_n,\omega)), \tag{4.7}$$

where $\{\sigma_n : n = 0,1,2,\dots\}$ is the approximation of the identity in T.

From condition (d), it follows that $(\partial f(t,x)/\partial t) = g(t,x) = (\hat{P}_t g)'_t$ and $\hat{P}_t(f'_t)(t,x) = f(t,x) - f(0,x)$, which also leads to the disappearance of terms $\partial^{m+b} f(t,x)/\partial t^b \partial x^m$ from formula (4.5) for each b and m such that $1 \le b$ and $2 \le m+b$. Now we approximate $f(t,x)$ by functions of the form $\sum_j \phi_j(t)\psi_j(x)$, so the problem reduces to the consideration of the functions $f(x)$ which are independent of t. Due to conditions (4.2) and (4.3), it is possible to put $\xi(t,\omega) = \xi_0(\omega) + a(\omega)(t-t_0) + E(\omega)[w(t) - w(t_0)]$. By the Taylor formula,

$$f(x) = f(x_0) + \sum_{n=1}^{m} (n!)^{-1} f^{(n)}(x_0)\cdot(x-x_0)^{\otimes n}, \tag{4.8}$$

for each $x, x_0 \in H$ since $\bar{\Phi}^{m+1} f = 0$. Let $t_k = \sigma_k(t)$ for each $k = 0,1,2,\dots$, then $\eta(t) - \eta(t_0) = \sum_{j=0}^{\infty}\{f(\xi_{j+1}) - f(\xi_j)\}$, where $\xi_j := \xi(t_j)$ since $\lim_{j\to\infty}\xi_j = \xi$. Then each term $f(\xi_{j+1}) - f(\xi_j)$ can be expressed by formula (4.8) due to

condition (b). On the other hand, $(\xi_{j+1}-\xi_j) = a(\omega)(t_{j+1}-t_j)+E(\omega)[w(t_{j+1})-w(t_j)]$ as the particular case of formula (4.7). From formulas (2.4), (4.6), (4.7) and (4.8) and Theorem 2.12, we get the statement of this theorem. □

Corollary 4.6. *If conditions (a), (d), (4.2) and (4.3) are satisfied, (b) is accomplished for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \|(\bar{\Phi}_x^n f)(t,x;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n)\|_{C^0(T\times(B(H,0,R_1))^{n+1}\times B(\mathbf{K}^{n+1},0,R_1),Y)} = 0 \quad \text{for each } 0<R_1<\infty, \tag{4.9}$$

then

$$f(t,\xi(t,\omega)) = f(t_0,\xi_0)+\hat{P}_u f_t'(u,\xi(u,\omega))\,|_{u=t} + (\tilde{P}_{a,Ew}^{\infty} f(u,x))\,|_{u=t}. \tag{4.10}$$

Proof. From the proof of Theorem 4.5, we get a function $f(x)$ for which

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} (n!)^{-1} f^{(n)}(x)\cdot(x-x_0)^{\otimes n} \tag{4.11}$$

due to condition (4.9). In view of Theorem 2.12,

$$\lim_{m\to\infty} \Bigg\| (m!)^{-1} \sum_{l=0}^{m} \binom{m}{l} \times (\hat{P}_{u^{m-l},w(u,\omega)^l}[(\partial^m f/\partial x^m)(u,\xi(u,\omega)) \circ (a^{\otimes(m-l)}\otimes E^{\otimes l})])|_{u=t} \Bigg\|_{L^s(\Omega,\mathcal{F},\lambda;C^0(T,Y))} = 0. \tag{4.12}$$

Approximating $f(x)$ by the Taylor formula up to terms $\bar{\Phi}^m f$ by finite sums and taking the limit while m tends to the infinity, we deduce formula (4.10) from formula (4.5), since for each chosen $\omega\in\Omega$, the functions $a(t,\omega)$ and $w(t,\omega)$ are bounded on the compact ball T. □

Theorem 4.7. *Let $f(u,x)\in C^\infty(T\times H,Y)$ and*

$$\lim_{n\to\infty} \max_{0\le l\le n} \|(\bar{\Phi}^n f)(t,x;h_1,\dots,h_n;\zeta_1,\dots,\zeta_n)\|_{C^0(T\times B(\mathbf{K},0,r)^l\times B(H,0,1)^{n-l}\times B(\mathbf{K},0,R_1)^{n-l},Y)} = 0 \tag{4.13}$$

for each $0<R_1<\infty$, where $h_j = e_1$ and $\zeta_j\in B(\mathbf{K},0,r)$ for variables corresponding to $t\in T = B(\mathbf{K},t_0,r)$ and $h_j\in B(H,0,1)$, $\zeta_j\in B(\mathbf{K},0,R_1)$ for variables

corresponding to $x \in H$, then

$$\begin{aligned} &f(t,\xi(t,\omega)) \\ &\quad = f(t_0,\xi_0) + \sum_{m+b\geq 1, 0\leq m\in\mathbf{Z}, 0\leq b\in\mathbf{Z}} ((m+b)!)^{-1} \sum_{l=0}^{m} \binom{m+b}{m}\binom{m}{l} \\ &\qquad \times (\hat{P}_{u^{b+m-l},w(u,\omega)^l}[(\partial^{(m+b)} f/\partial u^b \partial x^m)(u,\xi(u,\omega)) \\ &\qquad\qquad \circ (I^{\otimes b}\otimes a^{\otimes(m-l)}\otimes E^{\otimes l})])\big|_{u=t}. \end{aligned} \tag{4.14}$$

Proof. In view of the Taylor formula, we have (see [8, 9, 19])

$$\begin{aligned} f(t,x) &= f(t_0,x_0) \\ &\quad + \sum_{m+b=1}^{k} ((m+b)!)^{-1} \binom{m+b}{m} (\partial^{(m+b)} f/\partial u^b \partial x^m)(t_0,x_0)(t-t_0)^b \\ &\qquad \cdot (x-x_0)^{\otimes m} \\ &\quad + \sum_{m+b=k+1} \binom{k+1}{m} \Big[(\bar{\Phi}^{k+1} f)\big(t_0,x_0;(t-t_0)^{\otimes b},(x-x_0)^{\otimes m};1^{\otimes(k+1)}\big) \\ &\qquad\qquad - ((k+1)!)^{-1} (\partial^{(k+1)} f/\partial u^b \partial x^m)(t_0,x_0)(t-t_0)^b \\ &\qquad\qquad \cdot (x-x_0)^{\otimes m}\Big], \end{aligned} \tag{4.15}$$

for each $k \in \mathbb{N}$. In view of condition (4.13) and formulas (4.4), (2.4), and (4.7) we get formula (4.14) (see the proof of Theorem 4.5). □

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