ON FINITELY EQUIVALENT CONTINUA

JANUSZ J. CHARATONIK

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For positive integers \( m \) and \( n \), relations between (hereditary) \( m \)- and \( n \)-equivalence are studied, mostly for arc-like continua. Several structural and mapping problems concerning (hereditarily) finitely equivalent continua are formulated.

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A continuum means a compact connected metric space. For a positive integer \( n \), a continuum \( X \) is said to be \( n \)-equivalent provided that \( X \) contains exactly \( n \) topologically distinct subcontinua. A continuum \( X \) is said to be hereditarily \( n \)-equivalent provided that each nondegenerate subcontinuum of \( X \) is \( n \)-equivalent. If there exists a positive integer \( n \) such that \( X \) is \( n \)-equivalent, then \( X \) is said to be finitely equivalent. Thus, for \( n = 1 \), the concepts of “1-equivalent” and “hereditarily 1-equivalent” coincide, and they mean the same as “hereditarily equivalent” in the sense considered, for example, by Cook in [2].

Observe the following statement.

**Statement 1.** Each subcontinuum of an \( n \)-equivalent continuum is \( m \)-equivalent for some \( m \leq n \). Thus, each finitely equivalent continuum is hereditarily finitely equivalent.

Some structural results concerning finitely equivalent continua are obtained by Nadler Jr. and Pierce in [9]. They have shown that if a continuum \( X \) is (a) semi-locally connected at each of its noncut points, then it is finitely equivalent if and only if it is a graph; (b) aposyndetic at each of its noncut points and finitely equivalent, then it is a graph. Furthermore, in both cases (a) and (b), if \( X \) is \( n \)-equivalent, then each subcontinuum of \( X \) is a \( \theta_{n+1} \)-continuum. Recall that Nadler Jr. and Pierce in [9, page 209] posed the following problem.

**Problem 2.** Determine which graphs, or at least how many, are \( n \)-equivalent for each \( n \).

The arc and the pseudo-arc are the only known 1-equivalent continua. In [10] Whyburn has shown that each planar 1-equivalent continuum is tree-like, and planarity assumption has been deleted after 40 years by Cook [2] who proved tree-likeness of any 1-equivalent continuum. But it is still not known whether or not the arc and the pseudo-arc are the only ones among 1-equivalent continua.

In contrast to 1-equivalent case, 2-equivalent continua need not be hereditarily 2-equivalent, a simple closed curve is 2-equivalent while not hereditarily
2-equivalent. The 2-equivalent continua were studied by Mahavier in [5] who proved that if a 2-equivalent continuum contains an arc, then it is a simple triod, a simple closed curve or irreducible, and that the only locally connected 2-equivalent continua are a simple triod and a simple closed curve. It is also shown that if $X$ is a decomposable, not locally connected, 2-equivalent continuum containing an arc, then $X$ is arc-like and it is the closure of a topological ray $R$ such that the remainder $\text{cl}(R) \setminus R$ is an end continuum of $X$. Furthermore, two examples of 2-equivalent continua are presented in [5]: the first, [5, Example 1, page 246], is a decomposable continuum $X$ which is the closure of a ray $R$ such that the remainder $\text{cl}(R) \setminus R$ is homeomorphic to $X$; the second, [5, Example 2, page 247], is an arc-like hereditarily decomposable continuum containing no arc.

Looking for an example of a hereditarily 2-equivalent continuum note that the former example surely is not hereditarily 2-equivalent because it contains an arc. We analyze the latter one.

The continuum $M$ constructed in [5, Example 2, page 247] does not contain any arc, and it contains a continuum $N$ such that each subcontinuum of $M$ is homeomorphic to $M$ or to $N$, see [5, the paragraph following Lemma 3, page 249]. Further, by its construction, $N$ does contain continua homeomorphic to $M$ (see [5, the final part of the proof, page 251]). Therefore, the following statement is established.

**Theorem 3.** The continuum $M$ constructed in [5, Example 2, page 247] has the following properties:
(a) $M$ is an arc-like;
(b) $M$ is hereditarily decomposable;
(c) $M$ does not contain any arc;
(d) $M$ is hereditarily 2-equivalent.

In connection with the above theorem, the following problem can be posed.

**Problem 4.** Determine for what integers $n \geq 3$, there exists a continuum $M$ satisfying conditions (a), (b), and (c) of Theorem 3 and being hereditarily $n$-equivalent.

The following results are consequences of [1, Theorem, page 35].

**Theorem 5.** For each hereditarily $n$-equivalent continuum $X$, that does not contain any arc, there exists an $(n + 2)$-equivalent continuum $Y$ such that each of its subcontinua is homomorphic either to a subcontinuum of $X$ or to $Y$, or to an arc.

**Proof.** Indeed, a compactification $Y$ of a ray $R$ having the continuum $X$ as the remainder, that is, such that $X = \text{cl}(R) \setminus R$ is such a continuum.

Since if $M$ is arc-like and hereditarily decomposable, then so is any of compactifications $Y$ of a ray having the continuum $X$ as the remainder, we get the next result as a consequence of Theorem 5.
**Corollary 6.** If a continuum \( M \) satisfies conditions (a), (b), and (c) of Theorem 3 and is hereditarily \( n \)-equivalent, then any of compactifications of a ray having the continuum \( M \) as the remainder satisfies conditions (a) and (b) of Theorem 3 and is \((n + 2)\)-equivalent.

In [7], an uncountable family \( \mathcal{F} \) is constructed of compactifications of the ray with the remainder being the pseudo-arc.

**Statement 7.** Each member \( X \) of the (uncountable) family \( \mathcal{F} \) constructed in [7] is an arc-like 3-equivalent continuum. Any subcontinuum of \( X \) is homeomorphic to an arc, to a pseudo-arc, or to the whole \( X \).

A continuum \( X \) has the \( RNT \)-property (retractable onto near trees) provided that for each \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if a tree \( T \) is \( \delta \)-near to \( X \) with respect to the Hausdorff distance, then there is an \( \varepsilon \)-retraction of \( X \) onto \( T \), see [6, Definition 0]. It is shown in [6, Theorem 5] that if a continuum \( X \) is a compactification of the ray \( R \) and \( X \) has the RNT-property, then the remainder \( \text{cl}(R) \setminus R \subset X = \text{cl}(R) \) is the pseudo-arc. Therefore, Theorem 5 implies the following proposition.

**Proposition 8.** Each compactification \( X \) of the ray having the RNT-property is a 3-equivalent continuum. Each subcontinuum of \( X \) is homeomorphic to an arc, a pseudo-arc, or to the whole \( X \).

Observe that \( M \) of Theorem 3 being an arc-like is hereditarily unicoherent, and being hereditarily decomposable, it is a \( \lambda \)-dendroid (containing no arc). Another (perhaps the first) example of a \( \lambda \)-dendroid, in fact, an arc-like, containing no arc, has been constructed by Janiszewski in 1912, [3] but his description was rather intuitive than precise. It would be interesting to investigate if that old example of Janiszewski is or is not \( n \)-equivalent (hereditarily \( n \)-equivalent) for some \( n \).

The following problems can be considered as a program of a study in the area rather than particular questions.

**Problems 9.** For each positive integer \( n \), characterize continua which are (a) \( n \)-equivalent; (b) hereditarily \( n \)-equivalent.

**Problem 10.** Characterize continua which are finitely equivalent.

Sometimes a characterization of a class of spaces (or of spaces having a certain property) can be expressed in terms of containing some particular spaces. A classical illustration of this is a well-known characterization of nonplanar graphs by containing the two Kuratowski’s graphs: \( K_5 \) and \( K_{3,3} \), see, for example, [8, Theorem 9.36, page 159]. To be more precise, recall the following concept. Let \( \mathcal{A} \) be a class of spaces and let \( \mathcal{P} \) be a property. Then \( \mathcal{P} \) is said to be **finite (or countable) in the class** \( \mathcal{A} \) provided that there is a finite (or countable,
respectively) set \( \mathcal{F} \) of members of \( \mathcal{A} \) such that a member \( X \) has the property \( \mathcal{F} \) if and only if \( X \) contains a homeomorphic copy of some member of \( \mathcal{F} \). The result of [7] mentioned above in Statement 7 shows that this is not the way of characterizing 3-equivalent continua. Namely, the existence of the family \( \mathcal{F} \) shows the following theorem.

**Theorem 11.** The property of being 3-equivalent is neither finite nor countable in the class of (a) all continua; (b) arc-like continua.

A mapping \( f : X \to Y \) between continua \( X \) and \( Y \) is said to be

(i) *atomic* provided that for each subcontinuum \( K \) of \( X \), either \( f(K) \) is degenerate or \( f^{-1}(f(K)) = K \);

(ii) *monotone* provided that the inverse image of each subcontinuum of \( Y \) is connected;

(iii) *hereditarily monotone* provided that for each subcontinuum \( K \) of \( X \), the partial mapping \( f|K : K \to f(K) \) is monotone.

It is known that each atomic mapping is hereditarily monotone, see, for example, [4, (4.14), page 17]. Since each arcwise connected 2-equivalent continuum is either a simple closed curve or a simple triod, see [5, Theorem 2, page 244], each semilocally connected 3-equivalent continuum is either a simple 4-od [8, Definition 9.8, page 143] (i.e., a letter \( X \)) or a letter \( H \), see [9, page 209]. And since these continua are preserved under atomic mappings (as it is easy to see), we conclude that atomic mappings preserve the property of being 2-equivalent and being 3-equivalent for locally connected continua. However, this is not an interesting result, because each atomic mapping of an arcwise connected continuum onto a nondegenerate continuum is a homeomorphism, see [4, (6.3), page 51]. But the result cannot be extended to hereditarily monotone mappings, because a mapping that shrinks one arm of a simple triod to a point is hereditarily monotone and not atomic, and it maps a 2-equivalent continuum onto an arc that is 1-equivalent.

On the other hand, if \( X \) is the 2-equivalent continuum which is the closure of a ray \( R \) as described in [5, Example 1, page 246], then the mapping \( f : X \to [0,1] \), that shrinks the remainder \( \text{cl}(R) \setminus R \) to a point (and is a homeomorphism on \( R \)), is atomic and it maps 2-equivalent continuum \( X \) onto the 1-equivalent continuum \([0,1]\). Therefore, atomic mappings do not preserve the property of being a 2-equivalent continuum. In connection with these examples, the following question can be asked.

**Question 12.** Let a continuum \( X \) be \( n \)-equivalent and let a mapping \( f : X \to Y \) be an atomic surjection. Must then \( Y \) be \( m \)-equivalent for some \( m \leq n \)?

In general, we can pose the following problems.

**Problems 13.** What kinds of mappings between continua preserve the property of being: (a) \( n \)-equivalent? (b) hereditarily \( n \)-equivalent? (c) finitely equivalent?
REFERENCES


Janusz J. Charatonik: Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México DF, Mexico

E-mail address: jjc@math.unam.mx