PROPERTIES OF CERTAIN \( p \)-VALENTLY CONVEX FUNCTIONS

DINGGONG YANG and SHIGEYOSHI OWA

Received 10 September 2002

A subclass \( \mathcal{C}_p(\lambda,\mu) \) \((p \in \mathbb{N}, 0 < \lambda < 1, -\lambda \leq \mu < 1)\) of \( p \)-valently convex functions in the open unit disk \( \mathbb{U} \) is introduced. The object of the present paper is to discuss some interesting properties of functions belonging to the class \( \mathcal{C}_p(\lambda,\mu) \).

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let \( \mathcal{A}_p \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})
\]

which are analytic in the open unit disk \( \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \). A function \( f(z) \) in \( \mathcal{A}_p \) is said to be \( p \)-valently convex of order \( \alpha \) if it satisfies

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > p\alpha \quad (z \in \mathbb{U})
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\). We denote by \( \mathcal{H}_p(\alpha) \) the subclass of \( \mathcal{A}_p \) consisting of functions which are \( p \)-valently convex of order \( \alpha \) in \( \mathbb{U} \). In particular, we denote \( \mathcal{H}_1(0) = \mathcal{K} \).

A function \( f(z) \in \mathcal{A}_1 \) is said to be uniformly convex in \( \mathbb{U} \) if \( f(z) \) is in the class \( \mathcal{K} \) and has the property that the image arc \( f(\gamma) \) is convex for every circular arc \( \gamma \) contained in \( \mathbb{U} \) with center at \( t \in \mathbb{U} \). We also denote by \( \mathcal{W}K \) the subclass of \( \mathcal{A}_1 \) consisting of all uniformly convex functions in \( \mathbb{U} \). Goodman [2] has introduced the class \( \mathcal{W}K \) and given that \( f(z) \in \mathcal{A}_1 \) belongs to the class \( \mathcal{W}K \) if and only if

\[
\text{Re}\left\{1 + \frac{(z-t)f''(z)}{f'(z)}\right\} \geq 0 \quad ((z,t) \in \mathbb{U} \times \mathbb{U}).
\]

Ma and Minda [3] and Rønning [5] have showed a more applicable characterization for \( \mathcal{W}K \). We state this as the following theorem.

**Theorem 1.1.** Let \( f(z) \in \mathcal{A}_1 \). Then \( f(z) \in \mathcal{W}K \) if and only if

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\]
In view of Theorem 1.1, Owa [4] considered a subclass \( \mathcal{W}(\mu) \) \((-1 < \mu < 1)\) of \( \mathcal{A}_1 \). A function \( f(z) \in \mathcal{A}_1 \) is said to be a member of the class \( \mathcal{W}(\mu) \) \((-1 < \mu < 1)\) if and only if
\[
\Re \left\{ \frac{1 + zf''(z)}{f'(z)} \right\} - \mu > \left| \frac{Zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\]

(1.5)

In this paper, we investigate the following subclass of \( \mathcal{A}_p \).

**Definition 1.2.** A function \( f(z) \in \mathcal{A}_p \) is said to be a member of the class \( \mathcal{C}_p(\lambda, \mu) \) if
\[
\Re \left\{ \frac{1 + zf''(z)}{f'(z)} \right\} - p\mu > \lambda \quad (z \in \mathbb{U})
\]
for some \( \lambda (0 < \lambda < 1) \) and \( \mu (-\lambda \leq \mu < 1) \).

Let \( f(z) \) and \( g(z) \) be analytic in \( \mathbb{U} \). Then we say that \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{U} \), written \( f(z) \prec g(z) \), if there exists an analytic function \( w(z) \) in \( \mathbb{U} \) such that \( |w(z)| \leq |z| \) and \( f(z) = g(w(z)) \). If \( g(z) \) is univalent in \( \mathbb{U} \), then the subordination \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(\mathbb{U}) \subset g(\mathbb{U}) \).

2. Subordination properties. Our first result for properties of functions \( f(z) \in \mathcal{A}_p \) is contained in the following theorem.

**Theorem 2.1.** A function \( f(z) \in \mathcal{C}_p(\lambda, \mu) \) if and only if
\[
1 + \frac{zf''(z)}{f'(z)} < h(z)
\]
with
\[
h(z) = p + \frac{p(1-\mu)}{2\sin^2 \beta} \left\{ \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2\beta/\pi} + \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2\beta/\pi} - 2 \right\} \quad (\beta = \arccos \lambda).
\]

(2.2)

**Proof.** Let \( 1 + zf''(z)/f'(z) = w \) and \( w = u + iv \). Then inequality (1.6) can be written as
\[
u - p\mu > \lambda \sqrt{(u-p)^2 + v^2}.
\]

(2.3)

By computing, we find that inequality (2.3) is equivalent to
\[
\left( u + \frac{p(\lambda^2 - \mu)}{1 - \lambda^2} \right)^2 - \frac{\lambda^2}{1 - \lambda^2} v^2 > \left( \frac{p\lambda(1-\mu)}{1 - \lambda^2} \right)^2,
\]
\[
u > \frac{p(\lambda + \mu)}{1 + \lambda}.
\]

(2.4)

(2.5)

Thus the domain of the values of \( 1 + zf''(z)/f'(z) \) for \( z \in \mathbb{U} \) is contained in
\[
\mathbb{D} = \{ w = u + iv : u \text{ and } v \text{ satisfy (2.4) and (2.5)} \}.
\]

(2.6)
In order to prove our theorem, it suffices to show that the function $h(z)$ given by (2.2) maps $U$ conformally onto the domain $D$.

Consider the transformations

$$w_1 = \frac{1 - \lambda^2}{p(1 - \mu)} w + \frac{\lambda^2 - \mu}{1 - \mu},$$

(2.7)

$$t = \frac{1}{2} \left( w_2^{\pi/\beta} + w_2^{-\pi/\beta} \right),$$

(2.7)

where $\beta = \arccos \lambda$ and $w_2 = w_1 + \sqrt{w_1^2 - 1}$ is the inverse function of

$$w_1 = \frac{w_2 + 1}{2}.$$  

(2.8)

It is easy to verify that composite function $t = t(w)$ maps $D^+$ defined by

$$D^+ = \{ w = u + iv : u and v satisfy (2.4), (2.5), and v > 0 \}$$

(2.9)

conformally onto the upper-half plane $\text{Im}(t) > 0$ so that $w = p$ corresponds to $t = 1$ and $w = p(\lambda + \mu)/(1 + \lambda)$ to $t = -1$. With the help of the symmetry principle, this function $t = t(w)$ maps $D$ conformally onto the domain

$$G = \{ t : |\arg(t + 1)| < \pi \}.$$  

(2.10)

Since

$$t = 2 \left( \frac{1 + z}{1 - z} \right)^2 - 1$$

(2.11)

maps $U$ onto $G$, we see that

$$w = p + \frac{p(1 - \mu)}{2(1 - \lambda^2)} \left\{ \left( t + \sqrt{t^2 - 1} \right)^{\beta/\pi} + \left( t + \sqrt{t^2 - 1} \right)^{-\beta/\pi} - 2 \right\}$$

$$= p + \frac{p(1 - \mu)}{2\sin^2 \beta} \left\{ \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2(\beta/\pi)} + \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2(\beta/\pi)} - 2 \right\}$$

(2.12)

maps $U$ onto $D$ with $h(0) = p$. Hence the proof of the theorem is completed.

Theorem 2.1 gives the following corollaries.

**Corollary 2.2.** If $f(z) \in \mathcal{C}_p(\lambda, \mu)$, then $f(z) \in \mathcal{H}_p((\lambda + \mu)/(1 + \lambda))$ and the order $(\lambda + \mu)/(1 + \lambda)$ is sharp with the extremal function

$$f_0(z) = p \int_0^z \left( t_2^{p-1} \exp \int_0^{t_2} \frac{h(t_1) - p}{t_1} dt_1 \right) dt_2,$$

(2.13)

where $h(z)$ is given by (2.2).
**Proof.** Using (2.5) in the proof of Theorem 2.1 and noting that
\[ \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \text{Re}(h(z)) - p\frac{\lambda + \mu}{1 + \lambda} \] (2.14)
as \( z = \text{Re}(z) \to -1 \), we have the corollary.

**Corollary 2.3.** If \( f(z) \in C_p(\lambda, \mu) \) and \(-\lambda < \mu < \lambda < 1\), then
\[ \left| \arg\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| < \arctan\left(\frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}}\right) \quad (z \in \mathbb{U}). \] (2.15)
The bound in (2.15) is sharp with the extremal function \( f_0(z) \) given by (2.13).

**Proof.** Let the function \( h(z) \) be defined by (2.4). Then \( h(\mathbb{U}) = \mathbb{D} \) and an easy calculation yields that
\[ \min\{\theta : |\arg(h(z))| < \theta(z \in \mathbb{U})\} = \arctan\left(\frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}}\right) \quad (2.16) \]
for \(-\lambda < \mu < \lambda < 1\). Therefore, the corollary follows immediately from Theorem 2.1.

Next we derive the following theorem.

**Theorem 2.4.** Let \( f(z) \in C_p(\lambda, \mu) \) and let \( h(z) \) be defined by (2.2). Then
\[ \frac{f'(z)}{pz^{p-1}} < \exp \int_0^z \frac{h(t) - p}{t} dt, \] (2.17)
\[ \left| \frac{f'(z)}{pz^{p-1}} \right| \leq \exp \int_0^1 \frac{h(\rho) - p}{\rho} d\rho \quad (z \in \mathbb{U}). \] (2.18)
The bound in (2.18) is sharp with the extremal function \( f_0(z) \) given by (2.13).

**Proof.** Since the function \( h(z) - p \) is univalent and starlike (with respect to the origin), by Theorem 2.1 and the result due to Suffridge [6, Theorem 3], we have
\[ \log\left(\frac{f'(z)}{pz^{p-1}}\right) = \int_0^z \left(\frac{f''(t)}{f'(t)} - \frac{p-1}{t}\right) dt < \int_0^z \frac{h(t) - p}{t} dt, \] (2.19)
which implies the subordination (2.17).

Furthermore, noting that the univalent function \( h(z) \) maps the disk \(|z| < \rho\) \((0 < \rho \leq 1)\) onto the domain which is convex and symmetric with respect to the real axis, we deduce that
\[ \text{Re}\int_0^z \frac{h(t) - p}{t} dt = \int_0^1 \text{Re}\left\{ \frac{h(\rho z) - p}{\rho} \right\} d\rho \leq \int_0^1 \frac{h(\rho) - p}{\rho} d\rho \] (2.20)
for \( z \in \mathbb{U} \). Thus inequality (2.18) follows from (2.19) and (2.20).
Remark 2.5. If we let \( \beta = \pi/4 \) and \( x = ((1 + \sqrt{\rho})/(1 - \sqrt{\rho}))^{1/2} \) \((0 \leq \rho < 1)\), then

\[
\int_0^1 \left\{ \left( \frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}} \right)^{2(\beta/\pi)} + \left( \frac{1 - \sqrt{\rho}}{1 + \sqrt{\rho}} \right)^{2(\beta/\pi)} - 2 \right\} \frac{d\rho}{\rho} = 8 \int_1^{+\infty} \left( \frac{x}{x^2 + 1} - \frac{1}{x + 1} \right) dx = 4\log 2.
\]

(2.21)

Thus, as the special case of Theorem 2.4, we have that if \( f(z) \in \mathscr{C}_p(1/\sqrt{\mathbb{Z}}, \mu) \) \((-1/\sqrt{\mathbb{Z}} \leq \mu < 1)\), then

\[
\left| \frac{f'(z)}{pz^{p-1}} \right| \leq 16p(1-\mu) \quad (z \in \mathbb{U})
\]

(2.22)

and the result is sharp.

3. Coefficient inequalities

Theorem 3.1. If

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}
\]

belongs to \( \mathscr{C}_p(\lambda, \mu) \), then

\[
|a_{p+1}| \leq \frac{8p^2(1-\mu)}{p + 1} \left( \frac{\beta}{\pi \sin \beta} \right)^2 \quad (\beta = \arccos \lambda).
\]

(3.2)

Proof. It can be easily verified that

\[
1 + \frac{zf''(z)}{f'(z)} = p + \left( 1 + \frac{1}{p} \right) a_{p+1} z + \cdots,
\]

\[
h(z) = p + \frac{p(1-\mu)}{2 \sin^2 \beta} \left( \frac{8\beta}{\pi} + \frac{8\beta}{\pi} \left( \frac{2\beta}{\pi} - 1 \right) \right) z + \cdots
\]

(3.3)

\[
= p + 8p(1-\mu) \left( \frac{\beta}{\pi \sin \beta} \right)^2 z + \cdots,
\]

where \( h(z) \) is given by (2.2). Since

\[
f(z) = z^p + a_{p+1} z^{p+1} + \cdots \in \mathscr{C}_p(\lambda, \mu),
\]

(3.4)

it follows from (3.3) and Theorem 2.1 that

\[
\frac{\pi^2}{8p(1-\mu)} \left( \frac{\sin \beta}{\beta} \right)^2 \left( 1 + \frac{zf''(z)}{f'(z)} - p \right)
\]

\[
= \frac{p + 1}{8p^2(1-\mu)} \left( \frac{\pi \sin \beta}{\beta} \right)^2 a_{p+1} z + \cdots
\]

(3.5)

\[
< \frac{\pi^2}{8p(1-\mu)} \left( \frac{\sin \beta}{\beta} \right)^2 (h(z) - p).
\]
It is well known that if

\[ f(z) = \sum_{n=1}^{\infty} a_n z^n < g(z) \]  

for \( g(z) \in \mathcal{K} \), then (cf. Duren [1])

\[ |a_n| \leq 1 \quad (n = 1, 2, 3, \ldots). \]  

(3.7)

Noting that

\[ \frac{\pi^2}{8p(1-\mu)} \left( \frac{\sin \beta}{\beta} \right)^2 (h(z) - p) \in \mathcal{K}, \]  

(3.8)

we get (3.2). Also the bound in (3.2) is sharp for the function \( f_0(z) \) given by (2.13).

\[ \square \]

**REFERENCES**


Dinggong Yang: Department of Mathematics, Suzhou University, Suzhou, Jiangsu 215006, China

Shigeyoshi Owa: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: owa@math.kindai.ac.jp
Submit your manuscripts at http://www.hindawi.com