ON SAKAGUCHI FUNCTIONS

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Let $S_s(\alpha)$ ($0 \leq \alpha < 1/2$) be the class of functions $f(z) = z + \dots$ which are analytic in the unit disk and satisfy there $\operatorname{Re}\{zf'(z)/(f(z) - f(-z))\} > \alpha$. In the present paper, we find the sharp lower bound on $\operatorname{Re}\{(f(z) - f(-z))/z\}$ and investigate two subclasses $S_0(\alpha)$ and $T_0(\alpha)$ of $S_s(\alpha)$. We derive sharp distortion inequalities and some properties of the partial sums for functions in the classes $S_0(\alpha)$ and $T_0(\alpha)$.

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1. Introduction. Let A be the class of functions $f(z) = z + \dots$ which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by S the subclass of A consisting of functions which are univalent in E . A function $f(z) \in A$ is said to be in the class $S_s(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha \quad (z \in E) \quad (1.1)$$

for some α ($0 \leq \alpha < 1/2$). Further, a function $f(z) \in A$ is said to be in the class $T_s(\alpha)$ if and only if $zf'(z) \in S_s(\alpha)$ for some α ($0 \leq \alpha < 1/2$). We denote $S_s(0) = S_s$. Sakaguchi [3] introduced the class S_s and proved that $S_s \subset C(\subset S)$, where C is the class of close-to-convex functions. The class S_s has been studied by several authors (e.g., Stankiewicz [4] and Wu [6]).

For the class $S_s(\alpha)$, Owa et al. [2] have proved the following theorem.

THEOREM 1.1. *If $f(z) \in S_s(\alpha)$ with $1/4 \leq \alpha < 1/2$, then*

$$\operatorname{Re}\left\{\frac{f(z) - f(-z)}{z}\right\} > \frac{2}{3 - 4\alpha} \quad (z \in E). \quad (1.2)$$

The following two lemmas were shown by Cho et al. in [1].

LEMMA 1.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} \leq 1-2\alpha \quad (1.3)$$

for some α ($0 \leq \alpha < 1/2$), then $f(z) \in S_s(\alpha)$.

LEMMA 1.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{4(n-1)^2 |a_{2n-2}| + (2n-1)(2n-1-2\alpha) |a_{2n-1}|\} \leq 1-2\alpha \quad (1.4)$$

for some α ($0 \leq \alpha < 1/2$), then $f(z) \in T_s(\alpha)$.

In view of Lemmas 1.2 and 1.3, Cho et al. [1] introduced the subclass $S_0(\alpha)$ of $S_s(\alpha)$ consisting of functions which satisfy inequality (1.3) in Lemma 1.2, and the subclass $T_0(\alpha)$ of $T_s(\alpha)$ consisting of functions which satisfy inequality (1.4) in Lemma 1.3. It is easy to see that $T_0(\alpha) \subset S_0(\alpha)$ for $0 \leq \alpha < 1/2$.

The following theorems (Theorems 1.4, 1.5, and 1.6) are the main results of [1].

THEOREM 1.4. *If $f(z) \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, then*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2(1-\alpha)} \quad (z \in E). \quad (1.5)$$

THEOREM 1.5. *If $f(z) \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, then*

$$\begin{aligned} |z| - \frac{1-2\alpha}{2} |z|^2 - \frac{1-2\alpha}{3-2\alpha} |z|^3 &\leq |f(z)| \\ &\leq |z| + \frac{1-2\alpha}{2} |z|^2 + \frac{1-2\alpha}{3-2\alpha} |z|^3, \\ 1 - (1-2\alpha) |z| - \frac{3(1-2\alpha)}{3-2\alpha} |z|^2 &\leq |f'(z)| \\ &\leq 1 + (1-2\alpha) |z| + \frac{3(1-2\alpha)}{3-2\alpha} |z|^2, \end{aligned} \quad (1.6)$$

for $z \in E$.

THEOREM 1.6. *If $f(z) \in T_0(\alpha)$ with $0 \leq \alpha < 1/2$, then*

$$\begin{aligned} |z| - \frac{1-2\alpha}{4} |z|^2 - \frac{1-2\alpha}{3(3-2\alpha)} |z|^3 &\leq |f(z)| \\ &\leq |z| + \frac{1-2\alpha}{4} |z|^2 + \frac{1-2\alpha}{3(3-2\alpha)} |z|^3, \\ 1 - \frac{1-2\alpha}{2} |z| - \frac{1-2\alpha}{3-2\alpha} |z|^2 &\leq |f'(z)| \\ &\leq 1 + \frac{1-2\alpha}{2} |z| + \frac{1-2\alpha}{3-2\alpha} |z|^2, \end{aligned} \quad (1.7)$$

for $z \in E$.

In the present paper, we improve the above results and find the sharp bounds.

2. Main results

THEOREM 2.1. *If $f(z) \in S_s(\alpha)$ ($0 \leq \alpha < 1/2$), then*

$$\operatorname{Re} \left\{ \frac{f(z) - f(-z)}{z} \right\} > 4^\alpha \quad (z \in E). \quad (2.1)$$

The result is sharp.

PROOF. For $f(z) \in S_s(\alpha)$, it follows from (1.1) that the function

$$g(z) = \frac{f(z) - f(-z)}{2} \quad (2.2)$$

is an odd starlike function of order $2\alpha \in [0, 1)$, that is, $g(-z) = -g(z)$ and

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 2\alpha \quad (z \in E). \quad (2.3)$$

Since every odd function in S is the square-root transform of some function in S , there exists an $h(z) \in S$ such that $g(z) = \sqrt{h(z^2)}$. Further, $h(z)$ is also starlike of order 2α , or equivalently

$$\frac{zh'(z)}{h(z)} - 1 \prec 2(1 - 2\alpha) \frac{z}{1 - z}, \quad (2.4)$$

where the symbol $\prec$ stands for subordination.

Applying a result due to Suffridge [5, Theorem 3] to (2.4), we have

$$\int_0^z \left(\frac{h'(t)}{h(t)} - \frac{1}{t} \right) dt \prec 2(1 - 2\alpha) \int_0^z \frac{dt}{1 - t}, \quad (2.5)$$

which yields that

$$\frac{h(z)}{z} \prec \frac{1}{(1 - z)^{2(1 - 2\alpha)}}. \quad (2.6)$$

Thus, there exists an analytic function $w(z)$ in E such that $|w(z)| \leq |z|$ and

$$\frac{h(z)}{z} = \left(\frac{1}{1 - w(z)} \right)^{2(1 - 2\alpha)} \quad (z \in E), \quad (2.7)$$

and hence

$$\frac{g(z)}{z} = \left(\frac{1}{1 - w(z^2)} \right)^{1 - 2\alpha} \quad (z \in E). \quad (2.8)$$

With the help of the elementary inequality $\operatorname{Re}(t^\beta) \geq (\operatorname{Re} t)^\beta$ ($0 < \beta \leq 1$ and $\operatorname{Re} t > 0$), it follows from (2.8) that

$$\begin{aligned} \operatorname{Re} \frac{g(z)}{z} &\geq \left\{ \operatorname{Re} \left(\frac{1}{1 - w(z^2)} \right) \right\}^{1-2\alpha} \geq \left(\frac{1}{1 + |w(z^2)|} \right)^{1-2\alpha} \\ &\geq \left(\frac{1}{1 + |z|^2} \right)^{1-2\alpha} > \frac{1}{2^{1-2\alpha}} \quad (z \in E) \end{aligned} \quad (2.9)$$

for $0 \leq \alpha < 1/2$. This proves (2.1).

To establish the sharpness, we consider the function

$$f_0(z) = \frac{z}{(1 + z^2)^{1-2\alpha}}. \quad (2.10)$$

It is easy to verify that $f_0(z) \in S_s(\alpha)$ and

$$\operatorname{Re} \left\{ \frac{f_0(z) - f_0(-z)}{z} \right\} = 2 \operatorname{Re} \left\{ \frac{1}{(1 + z^2)^{1-2\alpha}} \right\} \rightarrow 4^\alpha \quad (2.11)$$

as $z = \operatorname{Re} z \rightarrow 1$. The proof of the theorem is now complete. $\square$

REMARK 2.2. This theorem improves and extends Theorem 1.1.

As an immediate consequence of Theorem 2.1, we have the following corollary.

COROLLARY 2.3. *If $f(z) \in S_s(\alpha)$ with $0 \leq \alpha < 1/2$ and if $f(z)$ is odd, then*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2^{1-2\alpha}} \quad (z \in E). \quad (2.12)$$

The result is sharp with the extremal function $f_0(z)$ given by (2.10). In particular, if $f(z) \in S_s$ and $f(z)$ is odd, then

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2} \quad (z \in E) \quad (2.13)$$

and the result is sharp.

THEOREM 2.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, then for $z \in E$,*

(a) *the following inequalities hold true:*

$$|z| - \frac{1-2\alpha}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{2}|z|^2. \quad (2.14)$$

Equalities are attained by the function

$$f(z) = z - \frac{1-2\alpha}{2}z^2 \in S_0(\alpha), \quad (2.15)$$

(b) the following inequalities hold true:

$$\begin{aligned} 1 - 2|a_2||z| - \frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2 \\ \leq |f'(z)| \leq 1 + 2|a_2||z| + \frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2. \end{aligned} \quad (2.16)$$

Equalities are attained, for example, by

$$f(z) = z - az^2 - \frac{1-2\alpha-2a}{3-2\alpha}z^3 \in S_0(\alpha) \quad (2.17)$$

or

$$f(z) = z + az^2 + \frac{1-2\alpha-2a}{3-2\alpha}z^3 \in S_0(\alpha), \quad (2.18)$$

where $0 \leq a \leq (1-2\alpha)/2$.

PROOF. (a) Writing inequality (1.3) in Lemma 1.2 as

$$\sum_{n=2}^{\infty} [n - (1 + (-1)^{n+1})\alpha] |a_n| \leq 1 - 2\alpha, \quad (2.19)$$

we arrive at $\sum_{n=2}^{\infty} |a_n| \leq (1-2\alpha)/2$. Therefore, it follows from $f(z) \in S_0(\alpha)$ that for $z \in E$,

$$\begin{aligned} |f(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - \frac{1-2\alpha}{2}|z|^2 \geq 0, \\ |f(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{1-2\alpha}{2}|z|^2. \end{aligned} \quad (2.20)$$

(b) Since $n - (1 + (-1)^{n+1})\alpha \geq n(1-2\alpha/3)$ for $n \geq 3$, it follows from (2.19) that

$$2|a_2| + \left(1 - \frac{2\alpha}{3}\right) \sum_{n=3}^{\infty} n|a_n| \leq 1 - 2\alpha \quad (2.21)$$

for $f(z) \in S_0(\alpha)$. Hence, we deduce that for $z \in E$,

$$\begin{aligned} |f'(z)| &\geq 1 - 2|a_2||z| - |z|^2 \sum_{n=3}^{\infty} n|a_n| \\ &\geq 1 - 2|a_2||z| - \frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2 > 0, \\ |f'(z)| &\leq 1 + 2|a_2||z| + \frac{3(1-2\alpha-2|a_2|)}{3-2\alpha}|z|^2. \end{aligned} \quad (2.22)$$

□

REMARK 2.5. Note that $|a_2| \leq (1 - 2\alpha)/2$. [Theorem 2.4](#) is an improvement of [Theorem 1.5](#).

THEOREM 2.6. *If $f(z) \in T_0(\alpha)$ with $0 \leq \alpha < 1/2$, then for $z \in E$, the following inequalities hold true:*

$$|z| - \frac{1-2\alpha}{4}|z|^2 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{4}|z|^2, \quad (2.23)$$

$$1 - \frac{1-2\alpha}{2}|z| \leq |f'(z)| \leq 1 + \frac{1-2\alpha}{2}|z|. \quad (2.24)$$

The results are sharp with the extremal function $f(z) = z - ((1 - 2\alpha)/4)z^2 \in T_0(\alpha)$.

PROOF. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_0(\alpha)$. Writing inequality (1.4) in [Lemma 1.3](#) as

$$\sum_{n=2}^{\infty} n[n - (1 + (-1)^{n+1})\alpha] |a_n| \leq 1 - 2\alpha, \quad (2.25)$$

we get $\sum_{n=2}^{\infty} |a_n| \leq (1 - 2\alpha)/4$. From this, we easily have (2.23).

Noting that $f(z) \in T_0(\alpha)$ if and only if $zf'(z) \in S_0(\alpha)$, (2.24) follows directly from [Theorem 2.4\(a\)](#). $\square$

REMARK 2.7. [Theorem 2.6](#) improves [Theorem 1.6](#).

THEOREM 2.8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, and let*

$$s_1(z) = z, \quad s_m(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (m \geq 2). \quad (2.26)$$

Then for $z \in E$, the following inequalities hold true:

$$\operatorname{Re} \frac{f(z)}{s_m(z)} > \frac{m + (1 + (-1)^{m+1})\alpha}{m + 1 - (1 + (-1)^m)\alpha}, \quad (2.27)$$

$$\operatorname{Re} \frac{s_m(z)}{f(z)} > \frac{m + 1 - (1 + (-1)^m)\alpha}{m + 2 - (3 + (-1)^m)\alpha}. \quad (2.28)$$

The results are sharp for each $m \geq 1$.

PROOF. Let

$$\lambda_n = \frac{n - (1 + (-1)^{n+1})\alpha}{1 - 2\alpha} \quad (n \geq 2). \quad (2.29)$$

Then $\lambda_{n+1} > \lambda_n > 1$ ($n \geq 2$) and it follows from (2.19) that

$$\sum_{n=2}^m |a_n| + \lambda_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \lambda_n |a_n| \leq 1 \quad (m \geq 2). \quad (2.30)$$

If we let

$$p_1(z) = \lambda_{m+1} \left\{ \frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{\lambda_{m+1}} \right) \right\}, \quad (2.31)$$

then

$$p_1(z) = 1 + \frac{\lambda_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^m a_n z^{n-1}} \quad (m \geq 2) \quad (2.32)$$

and from (2.30), we have

$$\begin{aligned} \left| \frac{p_1(z) - 1}{p_1(z) + 1} \right| &= \left| \frac{\lambda_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^m a_n z^{n-1} + \lambda_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{\lambda_{m+1} \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - \lambda_{m+1} \sum_{n=m+1}^{\infty} |a_n|} \leq 1 \quad (z \in E). \end{aligned} \quad (2.33)$$

Thus, we conclude that $\operatorname{Re} p_1(z) > 0$ ($z \in E$), that is,

$$\operatorname{Re} \frac{f(z)}{s_m(z)} > 1 - \frac{1}{\lambda_{m+1}} \quad (z \in E). \quad (2.34)$$

This proves (2.27) for $m \geq 2$.

If we take

$$f(z) = z - \frac{1 - 2\alpha}{m+1 - (1 + (-1)^m)\alpha} z^{m+1} \in S_0(\alpha), \quad (2.35)$$

then $s_m(z) = z$ and

$$\frac{f(z)}{s_m(z)} \rightarrow \frac{m + (1 + (-1)^{m+1})\alpha}{m+1 - (1 + (-1)^m)\alpha} \quad (2.36)$$

as $z \rightarrow 1$. Hence, the bound in (2.27) is best possible for each $m \geq 2$.

Similarly, if we put

$$p_2(z) = (1 + \lambda_{m+1}) \left(\frac{s_m(z)}{f(z)} - \frac{\lambda_{m+1}}{1 + \lambda_{m+1}} \right), \quad (2.37)$$

then by (2.30), we deduce that

$$\begin{aligned} \left| \frac{p_2(z) - 1}{p_2(z) + 1} \right| &= \left| \frac{-(1 + \lambda_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^m a_n z^{n-1} - (\lambda_{m+1} - 1) \sum_{n=m+1}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{(1 + \lambda_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - (\lambda_{m+1} - 1) \sum_{n=m+1}^{\infty} |a_n|} \\ &\leq 1 \end{aligned} \quad (2.38)$$

for $z \in E$. From this, we easily have (2.28) for $m \geq 2$. Further, the bound in (2.28) is best possible for the function $f(z)$ given by (2.35).

For $m = 1$, replacing (2.30) by

$$\lambda_2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \lambda_n |a_n| \leq 1 \quad (2.39)$$

and proceeding as the above, the proof of the theorem is completed. $\square$

Letting $m = 1$ in Theorem 2.8, we have the following corollary.

COROLLARY 2.9. *Let $f(z) \in S_0(\alpha)$ with $0 \leq \alpha < 1/2$, then for $z \in E$, the following inequalities hold true:*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1+2\alpha}{2}, \quad (2.40)$$

$$\operatorname{Re} \frac{z}{f(z)} > \frac{2}{3-2\alpha}. \quad (2.41)$$

The results are sharp.

REMARK 2.10. Inequality (2.40) in Corollary 2.9 is an improvement of Theorem 1.4 for $0 < \alpha < 1/2$ and agrees with Theorem 1.4 for $\alpha = 0$.

Using inequality (2.25) instead of (2.19), the following theorem can be proved on the lines of the proof of Theorem 2.8. We omitted the details of its proof.

THEOREM 2.11. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_0(\alpha)$ with $0 \leq \alpha < 1/2$, and let*

$$\begin{aligned} s_1(z) &= z, & s_m(z) &= z + \sum_{n=2}^{\infty} a_n z^n \quad (m \geq 2), \\ \mu_{m+1} &= \frac{(m+1)[m+1 - (1+(-1)^m)\alpha]}{1-2\alpha} \quad (m \geq 1). \end{aligned} \quad (2.42)$$

Then for $z \in E$, the following inequalities hold true:

$$\operatorname{Re} \frac{f(z)}{s_m(z)} > \frac{\mu_{m+1}-1}{\mu_{m+1}}, \quad (2.43)$$

$$\operatorname{Re} \frac{s_m(z)}{f(z)} > \frac{\mu_{m+1}}{1+\mu_{m+1}}. \quad (2.44)$$

The results are sharp for each $m \geq 1$, with the extremal function

$$f(z) = z - \frac{z^{m+1}}{\mu_{m+1}} \in T_0(\alpha). \quad (2.45)$$

Theorem 2.11, with $m = 1$, leads to the following corollary.

COROLLARY 2.12. *Let $f(z) \in T_0(\alpha)$ with $0 \leq \alpha < 1/2$. Then for $z \in E$,*

$$\operatorname{Re} \frac{f(z)}{z} > \frac{3+2\alpha}{4}, \quad \operatorname{Re} \frac{z}{f(z)} > \frac{4}{5-2\alpha}. \quad (2.46)$$

The results are sharp.

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REFERENCES

- [1] N. E. Cho, O. S. Kwon, and S. Owa, *Certain subclasses of Sakaguchi functions*, South-east Asian Bull. Math. **17** (1993), no. 2, 121-126.
- [2] S. Owa, Z. Wu, and F. Y. Ren, *A note on certain subclass of Sakaguchi functions*, Bull. Soc. Roy. Sci. Liège **57** (1988), no. 3, 143-149.
- [3] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72-75.
- [4] J. Stankiewicz, *Some remarks on functions starlike with respect to symmetric points*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **19** (1965), 53-59.
- [5] T. J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775-777.
- [6] Z. Wu, *On classes of Sakaguchi functions and Hadamard products*, Sci. Sinica Ser. A **30** (1987), no. 2, 128-135.

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