FUZZY SUPER IRRESOLUTE FUNCTIONS

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The concept of fuzzy super irresolute function was considered and studied by Šostak's (1985). A comparison between this type and other existing ones is established. Several characterizations, properties, and their effect on some fuzzy topological spaces are studied. Also, a new class of fuzzy topological spaces under the terminology fuzzy $S^*$-closed spaces is introduced and investigated.

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1. Introduction and preliminaries. Śostak [10], introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang fuzzy topology [1], in the sense that not only the objects are fuzzified, but also the axiomatics. In [11, 12], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al. [2, 3] have redefined the same concept. In [8], Ramadan gave a similar definition, namely "smooth topological space." It has been developed in many directions [4, 5, 6, 7, 13].

In the present note, some counterexamples and characterizations of fuzzy super irresolute functions are examined. It is seen that fuzzy super irresolute function implies each of fuzzy irresolute [9] and fuzzy continuity [10], but not conversely. Also, properties preserved by fuzzy super irresolute functions are examined. Finally, we define a fuzzy $S^*$-closed space in fuzzy topological spaces in Šostak sense and characterize such a space from different angles. Our aim is to compare the introduced type of fuzzy covering property with the existing ones.

Throughout this note, let $X$ be a nonempty set, $I = [0,1]$, and $I_s = (0,1]$. For $\alpha \in I$, $\alpha(x) = \alpha$ for all $x \in X$. The following definition and results which will be needed.

**Definition 1.1** [10]. A function $\tau : I^X \rightarrow I$ is called a fuzzy topology on $X$ if it satisfies the following conditions:

1. $\tau(0) = \tau(1) = 1$,
2. $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$,
3. $\tau(\bigvee_{i \in I} \mu_i) \geq \bigwedge_{i \in I} \tau(\mu_i)$ for any $\{\mu_i\}_{i \in I} \subset I^X$.

The pair $(X, \tau)$ is called a fuzzy topological space (FTS).
Remark 1.2. Let \((X, \tau)\) be an FTS. Then, for each \(\alpha \in I\), \(\tau_\alpha = \{\mu \in I^X : \tau(\mu) \geq r\}\) is a Chang’s fuzzy topology on \(X\).

Theorem 1.3 [3]. Let \((X, \tau)\) be an FTS. Then, for each \(r \in I_\circ\) and \(\lambda \in I^X\), an operator \(C_\tau : I^X \times I_\circ \rightarrow I^X\) is defined as follows:

\[
C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X : \lambda \leq \mu, \tau(1 - \mu) \geq r\}.
\]

For \(\lambda, \mu \in I^X\) and \(r, s \in I_\circ\), the operator \(C_\tau\) satisfies the following conditions:

1. \(C_\tau(0, r) = 0\), \(\lambda \leq C_\tau(\lambda, r)\),
2. \(C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)\),
3. \(C_\tau(\lambda, r) \leq C_\tau(\lambda, s)\) if \(r \leq s\),
4. \(C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)\).

Theorem 1.4 [9]. Let \((X, \tau)\) be an FTS. Then, for each \(r \in I_\circ\) and \(\lambda \in I^X\), an operator \(I_\tau : I^X \times I_\circ \rightarrow I^X\) is defined as follows:

\[
I_\tau(\lambda, r) = \bigvee \{\mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r\}.
\]

For \(\lambda, \mu \in I^X\) and \(r, s \in I_\circ\), the operator \(I_\tau\) satisfies the following conditions:

1. \(I_\tau(1 - \lambda, r) = 1 - C_\tau(\lambda, r)\),
2. \(I_\tau(1, r) = 1\), \(\lambda \geq I_\tau(\lambda, r)\),
3. \(I_\tau(\lambda, r) \land I_\tau(\mu, r) = I_\tau(\lambda \land \mu, r)\),
4. \(I_\tau(\lambda, r) \leq I_\tau(\lambda, s)\) if \(r \leq s\),
5. \(I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)\).

Definition 1.5 [9]. Let \((X, \tau)\) be an FTS. Then, for each \(r \in I_\circ\) and \(\lambda \in I^X\), the following statements hold:

1. \(\lambda\) is called \(r\)-fuzzy semi-open (\(r\)-FSO) if there exists \(\nu \in I^X\) with \(\tau(\nu) \geq r\) such that \(\nu \leq \lambda \leq C_\tau(\nu, r)\); equivalently, \(\lambda \leq C_\tau(I_\tau(\lambda, r), r)\);
2. \(\lambda\) is called \(r\)-fuzzy semiclosed (\(r\)-FSC) if there exists \(\nu \in I^X\) with \(\tau(1 - \nu) \geq r\) such that \(I_\tau(\nu, r) \leq \lambda \leq \nu\); equivalently, \(I_\tau(C_\tau(\lambda, r), r) \leq \lambda\);
3. \(\lambda\) is called \(r\)-fuzzy semiclopen (\(r\)-FSCO) if \(\lambda\) is \(r\)-FSO and \(r\)-FSC;
4. \(\lambda\) is called \(r\)-fuzzy regular open (\(r\)-FRO) if \(\lambda = I_\tau(C_\tau(\lambda, r), r)\);
5. the \(r\)-fuzzy semi-interior of \(\lambda\), denoted \(SI_\tau(\lambda, r)\), is defined by \(SI_\tau(\lambda, r) = \bigvee\{\nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-FSO}\}\);
6. the \(r\)-fuzzy semiclosure of \(\lambda\), denoted \(SC_\tau(\lambda, r)\), is defined by \(SC_\tau(\lambda, r) = \bigwedge\{\nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-FSC}\}\).

Theorem 1.6 [9]. Let \((X, \tau)\) be an FTS. For \(\lambda \in I^X\) and \(r \in I_\circ\), the following statements are valid:

1. \(\lambda\) is \(r\)-FSO if and only if \(\lambda = SI_\tau(\lambda, r)\), and \(\lambda\) is \(r\)-FSC if and only if \(\lambda = SC_\tau(\lambda, r)\);
2. \(I_\tau(\lambda, r) \leq SI_\tau(\lambda, r) \leq \lambda \leq SC_\tau(\lambda, r) \leq C_\tau(\lambda, r)\).
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(3) $\text{SC}_\tau(\text{SC}_\tau(\lambda, r), r) = \text{SC}_\tau(\lambda, r)$;
(4) $C_\tau(\text{SC}_\tau(\lambda, r), r) = \text{SC}_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$;
(5) $\text{SI}_\tau(1 - \lambda, r) = 1 - \text{SC}_\tau(\lambda, r)$.

**Lemma 1.7.** For any fuzzy set $\lambda$ in an FTS $(X, \tau)$ and $r \in I_*$, if $\tau(\lambda) \geq r$, then $I_\tau(C_\tau(\lambda, r), r) = \text{SC}_\tau(\lambda, r)$.

**Proof.** Since $\text{SC}_\tau(\lambda, r)$ is $r$-FSC, $I_\tau(C_\tau(\text{SC}_\tau(\lambda, r), r), r) \leq \text{SC}_\tau(\lambda, r)$ and hence, by Theorem 1.6(4), $I_\tau(C_\tau(\lambda, r), r) \leq \text{SC}_\tau(\lambda, r)$. To prove the opposite inclusion, since $\tau(\lambda) \geq r$, $r \in I_*$, we have $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$ so that $1 - \lambda \geq 1 - I_\tau(C_\tau(\lambda, r), r) = C_\tau(I_\tau(1 - \lambda, r), r)$. But $C_\tau(I_\tau(1 - \lambda, r), r)$ is $r$-FSO. Hence $C_\tau(I_\tau(1 - \lambda, r), r) \leq \text{SI}_\tau(1 - \lambda, r)$ and so $\text{SC}_\tau(\lambda, r) \leq I_\tau(C_\tau(\lambda, r), r)$.

**Definition 1.8.** Let $(X, \tau)$ and $(Y, \eta)$ be FTSs and let $f : X \to Y$ be a function which is called

(1) fuzzy continuous (FC) if and only if $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$, [10],
(2) fuzzy open if and only if $\tau(\lambda) \leq \eta(f(\lambda))$ for each $\lambda \in I^X$, [10],
(3) fuzzy semicontinuous (FSC) if and only if $f^{-1}(\mu)$ is $r$-FSO set of $X$ for each $\eta(\mu) \geq r$, $r \in I_* [9],$
(4) fuzzy irresolute (FI) if and only if $f^{-1}(\mu)$ is $r$-FSO set of $X$ for each $\mu$ is $r$-FSO set of $Y$, $r \in I_*$ [9].

2. Fuzzy super irresolute functions

**Definition 2.1.** Let $(X, \tau)$ and $(Y, \eta)$ be FTSs and let $f : X \to Y$ be a function which is called

(1) fuzzy super irresolute (F-super I) if and only if $\tau(f^{-1}(\mu)) \geq r$ for each $\mu$ is $r$-FSO set of $Y$, $r \in I_*$,
(2) fuzzy completely continuous (FCC) if and only if $f^{-1}(\mu)$ is $r$-FRO set of $X$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, $r \in I_* [9],$
(3) fuzzy completely irresolute (FCI) if and only if $f^{-1}(\mu)$ is $r$-FRO set of $X$ for each $r$-FSO set $\mu \in I^Y$ and $r \in I_*$.

**Remark 2.2.** One can show the connection between these types and other existing ones by the following diagram:

\[
\begin{array}{ccc}
\text{FCI} & \longrightarrow & \text{F-super I} \\
\downarrow & & \downarrow \\
\text{FCC} & \longrightarrow & \text{FI} \\
\downarrow & & \downarrow \\
\text{FC} & \longrightarrow & \text{FSC.}
\end{array}
\] (2.1)

The converse of the previous implications need not be true in general as shown in the following counterexample.
COUNTEREXAMPLE 2.3. Let \( \mu_1, \mu_2, \) and \( \mu_3 \) be fuzzy subsets of \( X = \{a,b,c\} \) defined as follows:

\[
\begin{align*}
\mu_1(a) &= 0.9, & \mu_1(b) &= 0.0, & \mu_1(c) &= 0.1, \\
\mu_2(a) &= 0.9, & \mu_2(b) &= 0.7, & \mu_2(c) &= 0.2, \\
\mu_3(a) &= 0.9, & \mu_3(b) &= 0.3, & \mu_3(c) &= 0.2.
\end{align*}
\]

(2.2)

Then \( \tau, \eta : I^X \to I, \) defined as

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, \\
\frac{1}{2}, & \text{if } \lambda = \mu_1, \\
\frac{1}{3}, & \text{if } \lambda = \mu_2, \\
0, & \text{otherwise},
\end{cases}
\eta(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, \\
\frac{1}{3}, & \text{if } \lambda = \mu_1, \mu_2, \\
\frac{1}{2}, & \text{if } \lambda = \mu_3, \\
0, & \text{otherwise},
\end{cases}
\]

(2.3)

are fuzzy topologies on \( X. \) Then,

1. the identity function \( \text{id}_X : (X, \tau) \to (X, \eta) \) is FI but not F-super I because \( \mu_3 \) is \( 1/3 \)-FSO in \( (X, \eta) \) and \( \tau(f^{-1}(\mu_3)) = \tau(\mu_3) = 0; \)

2. the identity function \( \text{id}_X : (X, \tau) \to (X, \tau) \) is FC but not F-super I function.

DEFINITION 2.4. An FTS \( (X, \tau) \) is said to be fuzzy extremally disconnected if and only if \( \tau(C_{\tau}(\lambda, r)) \geq r \) for every \( \tau(\lambda) \geq r \) for each \( \lambda \in I^X \) and \( r \in I \).

THEOREM 2.5. For a function \( f : X \to Y, \) the following statements are true:

1. if \( X \) is fuzzy extremally disconnected and \( f \) is FI, then \( f \) is F-super I;

2. if \( Y \) is fuzzy extremally disconnected and \( f \) is FCI (resp., FC), then \( f \) is F-super I;

3. if both \( X \) and \( Y \) are fuzzy extremally disconnected, then the concepts F-super I, FCI, FI, FCC, FSC, and FC are equivalent.

PROOF. The proof is obvious. \( \square \)

THEOREM 2.6. Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be FTSs. Let \( f : X \to Y \) be a function. The following statements are equivalent:

1. a map \( f \) is F-super I;

2. for each \( r \)-FSC \( \mu \in I^Y, \) \( \tau(1 - f^{-1}(\mu)) \geq r, \) \( r \in I; \)

3. for each \( \lambda \in I^X \) and \( r \in I, \) \( f(C_{\tau_1}(\lambda, r)) \leq SC_{\tau_2}(f(\lambda), r); \)

4. for each \( \mu \in I^Y \) and \( r \in I, \) \( C_{\tau_1}(f^{-1}(\mu), r) \leq f^{-1}(SC_{\tau_2}(\mu, r)); \)

5. for each \( \mu \in I^Y \) and \( r \in I, \) \( f^{-1}(SI_{\tau_2}(\mu, r)) \leq I_{\tau_1}(f^{-1}(\mu), r). \)

PROOF. (1) \( \Leftrightarrow \) (2). It is easily proved from Theorem 1.4 and from \( f^{-1}(1 - \mu) = 1 - f^{-1}(\mu). \)

(2) \( \Rightarrow \) (3). Suppose there exist \( \lambda \in I^X \) and \( r \in I, \) such that

\[
f(C_{\tau_1}(\lambda, r)) \notin SC_{\tau_2}(f(\lambda), r).
\]

(2.4)
There exist \( y \in Y \) and \( t \in I \), such that
\[
f(C_{T_1}(\lambda, r))(y) > t > SC_{T_2}(f(\lambda), r)(y).
\] (2.5)

If \( f^{-1}(\{y\}) = \emptyset \), it is a contradiction because \( f(C_{T_1}(\lambda, r))(y) = 0 \).
If \( f^{-1}(\{y\}) \neq \emptyset \), there exists \( x \in f^{-1}(\{y\}) \) such that
\[
f(C_{T_1}(\lambda, r))(y) \geq C_{T_1}(\lambda, r)(x) > t > SC_{T_2}(f(\lambda), r)(f(x)).
\] (2.6)

Since \( SC_{T_2}(f(\lambda), r)(f(x)) < t \), there exists \( r \)-FSC \( \mu \in I^Y \) with \( f(\lambda) \leq \mu \) such that
\[
SC_{T_2}(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.
\] (2.7)

Moreover, \( f(\lambda) \leq \mu \) implies \( \lambda \leq f^{-1}(\mu) \). From (2), \( \tau(1 - f^{-1}(\mu)) \geq r \). Thus, \( C_{T_1}(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t \), which is a contradiction to (2.6).

(3) \( \Rightarrow \) (4). For all \( \mu \in I^Y \), \( r \in I^\circ \), put \( \lambda = f^{-1}(\mu) \). From (3), we have
\[
f(C_{T_1}(f^{-1}(\mu), r)) \leq SC_{T_2}(f(f^{-1}(\mu)), r) \leq SC_{T_2}(\mu, r),
\] (2.8)

which implies that
\[
C_{T_1}(f^{-1}(\mu), r) \leq f^{-1}(f(C_{T_1}(f^{-1}(\mu), r))) \leq f^{-1}(SC_{T_2}(\mu, r)).
\] (2.9)

(4) \( \Rightarrow \) (5). It is easily proved from Theorem 1.4(1).
(5) \( \Rightarrow \) (1). Let \( \mu \) be \( r \)-FSO set of \( Y \). From Theorem 1.6(1), \( \mu = SI_{T_2}(\mu, r) \). By (5),
\[
f^{-1}(\mu) \leq I_{T_1}(f^{-1}(\mu), r).
\] (2.10)

On the other hand, by Theorem 1.4(2),
\[
f^{-1}(\mu) \geq I_{T_1}(f^{-1}(\mu), r).
\] (2.11)

Thus, \( f^{-1}(\mu) = I_{T_1}(f^{-1}(\mu), r) \), that is, \( \tau(f^{-1}(\mu)) \geq r \). \( \square \)

3. Properties preserved by F-super I functions

**Definition 3.1.** Let \( (X, \tau) \) be an FTS and \( r \in I^\circ \). Then
(1) \( X \) is called \( r \)-fuzzy compact (resp., \( r \)-fuzzy almost compact and \( r \)-fuzzy nearly compact) if and only if for each family \( \{\lambda_i \in I^X : \tau(\lambda_i) \geq r, i \in \Gamma\} \) such that \( \bigvee_{i \in \Gamma} \lambda_i = 1 \), there exists a finite index set \( \Gamma_i \subset \Gamma \) such that \( \bigvee_{i \in \Gamma_i} \lambda_i = 1 \) (resp., \( \bigvee_{i \in \Gamma_i} C_T(\lambda_i, r) = 1 \) and \( \bigvee_{i \in \Gamma_i} I_T(C_T(\lambda_i, r), r) = 1 \));
(2) \( X \) is called \( r \)-fuzzy semicompact (resp., \( r \)-fuzzy S-closed) if and only if for each family \( \{\lambda_i \in I^X : \lambda_i \leq C_T(I_T(\lambda_i, r), r), i \in \Gamma\} \) such that \( \bigvee_{i \in \Gamma} \lambda_i = 1 \), there exists a finite index set \( \Gamma_i \subset \Gamma \) such that \( \bigvee_{i \in \Gamma_i} \lambda_i = 1 \) (resp., \( \bigvee_{i \in \Gamma_i} C_T(\lambda_i, r) = 1 \)).
**Theorem 3.2.** Every surjective F-super I image of \( r \)-fuzzy compact space is \( r \)-fuzzy semicompact, \( r \in I^* \).

**Proof.** Let \((X,\tau)\) be \( r \)-fuzzy compact, \( r \in I^* \), and let \( f : (X,\tau) \rightarrow (Y,\eta) \) be F-super I surjective function. If \( \{\lambda_i \in I^Y : \lambda_i \leq C_\eta(I_\eta(\lambda_i,r),r), \ i \in \Gamma \} \) with \( \bigvee_{i \in \Gamma} \lambda_i = 1 \), then \( \bigvee_{i \in \Gamma} f^{-1}(\lambda_i) = 1 \). Since \( f \) is F-super I, \( \tau(f^{-1}(\lambda_i)) \geq r \). Since \( X \) is \( r \)-fuzzy compact, there exists a finite subset \( \Gamma_0 \subset \Gamma \) with \( \bigvee_{i \in \Gamma_0} f^{-1}(\lambda_i) = 1 \). From the surjectivity of \( f \), we deduce

\[
1 = f(1) = \left( \bigvee_{i \in \Gamma_0} f^{-1}(\lambda_i) \right) = \bigvee_{i \in \Gamma_0} f^{-1}(\lambda_i) = \bigvee_{i \in \Gamma_0} \lambda_i.
\]  
(3.1)

So, \( Y \) is \( r \)-fuzzy semicompact.

**Corollary 3.3.** Every surjective F-super I image of \( r \)-fuzzy compact space is \( r \)-fuzzy \( S \)-closed, \( r \in I^* \).

**Theorem 3.4.** Every surjective F-super I image of \( r \)-fuzzy almost compact space is \( r \)-fuzzy \( S \)-closed, \( r \in I^* \).

**Proof.** The proof is similar to that of Theorem 3.2.

**Corollary 3.5.** \( r \)-fuzzy semicompactness and \( r \)-fuzzy \( S \)-closedness are preserved under an F-super I surjection function, \( r \in I^* \).

**Proof.** The Corollary is a direct consequence of Theorems 3.2 and 3.4.

**Theorem 3.6.** Let \( f : X \rightarrow Y \) be FSC and F-super I surjective function. If \( X \) is \( r \)-fuzzy nearly compact, then \( Y \) is \( r \)-fuzzy \( S \)-closed, \( r \in I^* \).

**Proof.** Let \((X,\tau)\) be \( r \)-fuzzy nearly compact, and let \( r \in I^* \), \( f : (X,\tau) \rightarrow (Y,\eta) \) be FSC and F-super I surjective function. If \( \{\lambda_i \in I^Y : \lambda_i \leq C_\eta(I_\eta(\lambda_i,r),r), \ i \in \Gamma \} \) with \( \bigvee_{i \in \Gamma} \lambda_i = 1 \), then \( \bigvee_{i \in \Gamma} f^{-1}(\lambda_i) = 1 \). Since \( f \) is F-super I, \( \tau(f^{-1}(\lambda_i)) \geq r \). Since \( X \) is \( r \)-fuzzy nearly compact, there exists a finite subset \( \Gamma_0 \subset \Gamma \) with \( \bigvee_{i \in \Gamma_0} I_\tau(C_\tau(f^{-1}(\lambda_i),r),r) = 1 \). From the surjectivity of \( f \), we deduce

\[
1 = f(1) = \left( \bigvee_{i \in \Gamma_0} I_\tau(C_\tau(f^{-1}(\lambda_i),r),r) \right)
\]

\[
= \bigvee_{i \in \Gamma_0} f(I_\tau(C_\tau(f^{-1}(\lambda_i),r),r))
\]

\[
\leq \bigvee_{i \in \Gamma_0} f(f^{-1}(C_\eta(\lambda_i,r))) \quad \text{(since} \ f \ \text{is FSC [9]).}
\]

Thus \( \bigvee_{i \in \Gamma_0} C_\eta(\lambda_i,r) = 1 \). Hence \( Y \) is \( r \)-fuzzy \( S \)-closed.
4. Fuzzy $S^*$-closed spaces: characterizations and comparisons

**Definition 4.1.** Let $(X, \tau)$ be an FTS and $r \in I_\ast$. Then $X$ is called $r$-fuzzy $S^*$-closed if and only if for each family $\{\lambda_i \in I^X : \lambda_i \leq C_r(I_\tau(\lambda_i, r), r), \ i \in \Gamma\}$ such that $\bigvee_{i \in \Gamma} \lambda_i = 1$, there exists a finite index set $\Gamma_\ast \subseteq \Gamma$ such that

$$\bigvee_{i \in \Gamma_\ast} \text{SC}_\tau(\lambda_i, r) = 1. \quad (4.1)$$

**Theorem 4.2.** For an FTS $(X, \tau)$, $r \in I_\ast$, the following statements are equivalent:

1. $X$ is $r$-fuzzy $S^*$-closed;
2. for every family $\{\lambda_i \in I^X : \lambda_i$ is $r$-FSCO, $i \in \Gamma\}$ such that $\bigvee_{i \in \Gamma} \lambda_i = 1$, there exists a finite index set $\Gamma_\ast \subseteq \Gamma$ such that $\bigvee_{i \in \Gamma_\ast} \lambda_i = 1$;
3. every family of $r$-FSCO sets having the finite intersection property has nonnull intersection;
4. for every family $\{\lambda_i \in I^X : \lambda_i$ is $r$-FSC, $i \in \Gamma\}$ such that $\bigwedge_{i \in \Gamma} \lambda_i = 1$, there exists a finite index set $\Gamma_\ast \subseteq \Gamma$ such that $\bigwedge_{i \in \Gamma_\ast} \text{SI}_\tau(\lambda_i, r) = 1$.

**Proof.** (1)$\Rightarrow$(2). The proof is obvious.

(2)$\Rightarrow$(3). Let $\{\lambda_i\}_{i \in \Gamma}$ be a family of $r$-FSCO sets having the finite intersection property. If possible, let $\bigwedge_{i \in \Gamma} \lambda_i = 0$. Then $\bigvee_{i \in \Gamma} (1 - \lambda_i) = 1$, where each $(1 - \lambda_i)$ is $r$-FSCO. By (2), there exists a finite subset $\Gamma_\ast$ of $\Gamma$ such that $\bigvee_{i \in \Gamma_\ast} (1 - \lambda_i) = 1$, that is, $\bigwedge_{i \in \Gamma_\ast} \lambda_i = 0$, which is a contradiction.

(3)$\Rightarrow$(1). Suppose that $\{\lambda_i : i \in \Gamma\}$ is a family of $r$-FSO sets of $X$ with $\bigvee_{i \in \Gamma} \lambda_i = 1$, and it has no finite subfamily $\{\lambda_{i_1}, \ldots, \lambda_{i_n}\}$ such that $\bigvee_{j=1}^n \text{SC}_\tau(\lambda_{i_j}, r) = 1$. Then $\bigwedge_{i=1}^n (1 - \text{SC}_\tau(\lambda_{i_j}, r)) \neq 0$. Thus, $\{1 - \text{SC}_\tau(\lambda_i, r) : i \in \Gamma\}$ is a family of $r$-FSCO sets having the finite intersection property. By (3), $\bigwedge_{i \in \Gamma} (1 - \text{SC}_\tau(\lambda_i, r)) \neq 0$, and hence, $\bigvee_{i \in \Gamma} \lambda_i \neq 1$, which is a contradiction.

(1)$\Rightarrow$(4). If $\{\lambda_i : i \in \Gamma\}$ is a family of nonnull $r$-FSC sets in $X$, $r \in I_\ast$, with $\bigwedge_{i \in \Gamma} \lambda_i = 0$, then $\{1 - \lambda_i : i \in \Gamma\}$ is $r$-FSO sets in $X$ with $\bigvee_{i \in \Gamma} (1 - \lambda_i) = 1$. By (1), there is a finite subset $\Gamma_\ast \subseteq \Gamma$ such that

$$\bigwedge_{i \in \Gamma_\ast} \text{SI}_\tau(\lambda_i, r) = 0. \quad (4.2)$$

that is, $\bigwedge_{i \in \Gamma_\ast} \text{SI}_\tau(\lambda_i, r) = 0$.

(4)$\Rightarrow$(1). For any $\{\lambda_i \in I^X : \lambda_i$ is $r$-FSO, $i \in \Gamma\}$ such that $\bigvee_{i \in \Gamma} \lambda_i = 1$, $\{1 - \lambda_i, i \in \Gamma\}$ is a family of $r$-FSC sets such that $\bigwedge_{i \in \Gamma} (1 - \lambda_i) = 0$. We can assume, without loss of generality, that each $1 - \lambda_i \neq 0$. By (4), there is a finite subset $\Gamma_\ast \subseteq \Gamma$ such that $\bigwedge_{i \in \Gamma_\ast} \text{SI}_\tau(1 - \lambda_i, r) = 0$, that is, $\bigvee_{i \in \Gamma_\ast} \text{SC}_\tau(\lambda_i, r) = 1$, which proves the $r$-fuzzy $S^*$-closedness of $X$. □

**Theorem 4.3.** Let $(X, \tau)$ be an FTS and $r \in I_\ast$. If $X$ is $r$-fuzzy semicompact, then $X$ is $r$-fuzzy $S^*$-closed as well.
**Proof.** Since for every \( \lambda \in I^X \) and \( r \in I_\circ \) we have \( \lambda \leq SC_\tau(\lambda, r) \), this immediately follows from the definitions.

**Theorem 4.4.** Let \((X, \tau)\) be an FTS and \( r \in I_\circ \). If \( X \) is \( r \)-fuzzy \( S^* \)-closed, then \( X \) is \( r \)-fuzzy \( S \)-closed as well.

**Proof.** Since for every \( \lambda \in I^X \) and \( r \in I_\circ \) we have \( SC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \), this immediately follows from the definitions.

That the converse is false is evident from the following counterexample.

**Counterexample 4.5.** Let \( \mathbb{N} \) denote the set of natural numbers with the fuzzy topology \( \tau : I^\mathbb{N} \rightarrow I \) defined as

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, 1, \\
\frac{1}{3}, & \text{if } \lambda = \mu, \nu, \\
\frac{1}{2}, & \text{if } \lambda = \mu \vee \nu, \\
0, & \text{otherwise,}
\end{cases}
\]

(4.3)

where \( \mu(1) = 1, \mu(i) = 0 \) (for \( i = 2, 3, 4, \ldots \)), and \( \nu(2) = 1, \mu(j) = 0 \) (for \( j = 1, 3, 4, \ldots \)). Let \( \rho^1_i \) and \( \rho^2_i \) (for \( i = 3, 4, 5, \ldots \)) be the fuzzy sets in \( I^\mathbb{N} \) given by

\[
\rho^1_i(x) = \begin{cases} 
1, & \text{for } x = 1 \text{ and } i, \\
0, & \text{otherwise,}
\end{cases}
\]

(4.4)

\[ \rho^2_i(x) = \begin{cases} 
1, & \text{for } x = 2 \text{ and } i, \\
0, & \text{otherwise.}
\end{cases} \]

Then \( \mathcal{U} = \{ \rho^1_i, \rho^2_i : i = 3, 4, 5, \ldots \} \) are \( 1/3 \)-FSCO sets with \( \bigvee_{\rho \in \mathcal{U}} \rho = 1 \) having no finite subcover. Hence \((\mathbb{N}, \tau)\) is not \( 1/3 \)-fuzzy \( S^* \)-closed, but it is easily seen that \((\mathbb{N}, \tau)\) is \( 1/3 \)-fuzzy \( S \)-closed.

**Theorem 4.6.** For any fuzzy extremally disconnected FTS \((X, \tau)\) and \( r \in I_\circ \), \( X \) is \( r \)-fuzzy \( S^* \)-closed if and only if \( X \) is \( r \)-fuzzy \( S \)-closed.

**Proof.**

**Necessity.** It follows from the proof of Theorem 4.4.

**Sufficiency.** We are going to prove that if \((X, \tau)\) is any fuzzy extremally disconnected FTS, then \( C_\tau(\lambda, r) = SC_\tau(\lambda, r) \) for every \( r \)-FSO set \( \lambda \) in \((X, \tau)\) and \( r \in I_\circ \). Then our result follows from Definitions 3.1(2) and 4.1.

We always have \( SC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \) for every \( \lambda \in I^X \) and \( r \in I_\circ \). So, we have to prove that with our hypothesis we have \( C_\tau(\lambda, r) \leq SC_\tau(\lambda, r) \) for every \( \lambda \in I^X \) and \( r \in I_\circ \).

If \( \lambda \) is \( r \)-FSO in \((X, \tau)\), then there exists \( \nu \in I^X \) with \( \tau(\nu) \geq r \) such that \( \nu \leq \lambda \leq C_\tau(\nu, r) \). So, \( C_\tau(\lambda, r) = C_\tau(\nu, r) \), where \( \tau(\nu) \geq r \). Because \((X, \tau)\) is
fuzzy extremally disconnected, we have that
\[ C_\tau(\lambda, r) = C_\tau(\nu, r) = I_\tau(C_\tau(\nu, r), r) = I_\tau(C_\tau(\lambda, r), r). \] (4.5)

By Lemma 1.7, we have \( C_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r) \leq SC_\tau(\lambda, r) \).

**Remark 4.7.** From Theorems 4.3 and 4.4, we have that \( r \)-fuzzy semicompactness implies \( r \)-fuzzy \( S \)-closedness, \( r \in I_\circ \).

**Remark 4.8.** Obviously, for \( r \in I_\circ \), \( r \)-fuzzy \( S \)-closed space is \( r \)-fuzzy almost compact. Hence \( r \)-fuzzy compact space need not be \( r \)-fuzzy \( S^* \)-closed. That an \( r \)-fuzzy \( S^* \)-closed space is not necessarily \( r \)-fuzzy compact is shown by the following counterexample.

**Counterexample 4.9.** Let \( X \) be any nonempty set and let \( \tau : I^X \to I \) be defined as
\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, 1, \\
1/2, & \text{if } \lambda = \alpha, \text{ for } 1/2 < \alpha < 1, \\
0, & \text{otherwise}. 
\end{cases} \] (4.6)

Then \((X, \tau)\) is an FTS which is not \(1/2\)-fuzzy compact. Now for any \( \alpha \in I^X \) with \( \tau(\alpha) \geq 1/2 \), \( C_\tau(\alpha, 1/2) = 1 \) and hence \( I_\tau(C_\tau(\alpha, 1/2), 1/2) = 1 \), for all \( \alpha \in (1/2, 1] \). Since, by Lemma 1.7, \( SC_\tau(\alpha, 1/2) = I_\tau(C_\tau(\alpha, 1/2), 1/2) = 1 \), we have for any \( r \)-FSO set \( \lambda \), \( SC_\tau(\lambda, 1/2) = 1 \). Hence \( X \) is \( r \)-fuzzy \( S^* \)-closed.

However, we have the following theorem.

**Theorem 4.10.** For \( r \in I_\circ \), every \( r \)-fuzzy \( S^* \)-closed space is \( r \)-fuzzy nearly compact, \( r \in I_\circ \).

**Proof.** If \( X \) is not \( r \)-fuzzy nearly compact, then there exists \( \{\lambda_i \in I^X, i \in \Gamma\} \) with \( \tau(\lambda_i) \geq r \) and \( \bigvee_{i \in \Gamma} \lambda_i = 1 \) such that for any finite subset \( \Gamma_0 \subset \Gamma \),
\[
\bigvee_{i \in \Gamma_0} I_\tau(C_\tau(\lambda_i, r), r) \neq 1, \] (4.7)
that is,
\[
\bigvee_{i \in \Gamma_0} SC_\tau(\lambda_i, r) \neq 1 \] (4.8)
(by Lemma 1.7). Thus, \( X \) is not \( r \)-fuzzy \( S^* \)-closed.

In order to investigate for the condition under which \( r \)-fuzzy \( S^* \)-closed space is \( r \)-fuzzy compact, we set the following definition.
**Definition 4.11.** An FTS \((X, \tau)\) is called \(r\)-fuzzy \(S\)-regular if and only if for each \(r\)-FSO set \(\mu \in I^X, r \in I_*\),

\[
\mu = \bigvee \{ \rho \in I^X | \rho \text{ is } r\text{-FSO}, \ SC_\tau(\rho, r) \leq \mu \}. \tag{4.9}
\]

An FTS \((X, \tau)\) is called fuzzy \(S\)-regular if and only if it is \(r\)-fuzzy \(S\)-regular for each \(r \in I_*\).

**Theorem 4.12.** If an FTS \((X, \tau)\) is \(r\)-fuzzy \(S\)-regular and \(r\)-fuzzy \(S^*\)-closed, \(r \in I_*\), then it is \(r\)-fuzzy compact.

**Proof.** Let \(\{ \lambda_i \in I^X | \tau(\lambda_i) \geq r, i \in \Gamma \}\) be a family such that \(\bigvee_{i \in \Gamma} \lambda_i = 1\). Since \((X, \tau)\) is \(r\)-fuzzy \(S\)-regular, for each \(\tau(\lambda_i) \geq r\), \(\lambda_i\) is \(r\)-FSO,

\[
\lambda_i = \bigvee_{i_k \in K_i} \{ \lambda_{i_k} \mid \lambda_{i_k} \text{ is } r\text{-FSO}, \ SC_\tau(\lambda_{i_k}, r) \leq \lambda_i \}. \tag{4.10}
\]

Hence \(\bigvee_{i \in \Gamma} (\bigvee_{i_k \in K_i} \lambda_{i_k}) = 1\). Since \((X, \tau)\) is \(r\)-fuzzy \(S^*\)-closed, there exists a finite index \(J \times K_J\) such that

\[
1 = \bigvee_{j \in J} \left( \bigvee_{j_k \in K_J} SC_\tau(\lambda_{j_k}, r) \right). \tag{4.11}
\]

For each \(j \in J\), since

\[
\bigvee_{j_k \in K_J} SC_\tau(\lambda_{j_k}, r) \leq \lambda_j, \tag{4.12}
\]

we have \(\bigvee_{j \in J} \lambda_j = 1\). Hence \((X, \tau)\) is \(r\)-fuzzy compact.

It is evident that every FI function is FSC. That the converse is not always true is shown in [9]. Again, it is proved in [9] that \(f : X \to Y\) is FI if and only if \(f^{-1}(\mu)\) is \(r\)-FSC for every \(r\)-FSC set \(\mu\) in \(Y\) and \(r \in I_*\). Now we have the following theorem.

**Theorem 4.13.** The FI image of \(r\)-fuzzy \(S^*\)-closed space is \(r\)-fuzzy \(S^*\)-closed, \(r \in I_*\).

**Theorem 4.14.** If \(f : (X, \tau) \to (Y, \eta)\) is FI surjective and \(X\) is \(r\)-fuzzy \(S^*\)-closed, then \(Y\) is \(r\)-fuzzy \(S\)-closed, \(r \in I_*\).

**Proof.** If \(\{ \lambda_i \in I^Y : \lambda_i \text{ is } r\text{-FSO}, i \in \Gamma \}\) is a family such that \(\bigvee_{i \in \Gamma} \lambda_i = 1\), then \(\bigvee_{i \in \Gamma} f^{-1}(\lambda_i) = 1\). Since \(f\) is FI, then, for each \(i \in \Gamma\), \(f^{-1}(\lambda_i)\) is \(r\)-FSO set of \(X\). By \(r\)-fuzzy \(S^*\)-closedness of \(X\), there is a finite subset \(\Gamma_s \subset \Gamma\) such that
\[ V_{i \in \Gamma} SC_{\tau}(f^{-1}(\lambda_i, r)) = 1. \] Now,

\[ 1 = f(1) = f \left( \bigvee_{i \in \Gamma} SC_{\tau}(f^{-1}(\lambda_i, r)) \right) \]
\[ \leq f \left( \bigvee_{i \in \Gamma} C_{\tau}(f^{-1}(\lambda_i, r)) \right) \]
\[ \leq \bigvee_{i \in \Gamma} C_{\eta}(\lambda_i, r), \]

which implies that \( Y \) is \( r \)-fuzzy \( S \)-closed.

**Theorem 4.15.** If \( f : (X, \tau) \to (Y, \eta) \) is CI surjective and \( X \) is \( r \)-fuzzy nearly compact, then \( Y \) is \( r \)-fuzzy semicompact, \( r \in I^* \).

**Proof.** The proof is similar to that of Theorem 4.14.

**Definition 4.16.** Let \((X, \tau)\) and \((Y, \eta)\) be FTSs. A function \( f : (X, \tau) \to (Y, \eta) \) is called semiweakly continuous if and only if

\[ f^{-1}(\lambda) \leq SI_{\tau}(f^{-1}(SC_{\eta}(\lambda)), r), \]

(4.14)

for each \( r \)-FSO set \( \lambda \) in \((Y, \eta)\), \( r \in I^* \).

**Theorem 4.17.** Let \((X, \tau)\) and \((Y, \eta)\) be FTSs and let \( f : (X, \tau) \to (Y, \eta) \) be a semiweakly continuous function. If \( X \) is \( r \)-fuzzy semicompact, then \( Y \) is \( r \)-fuzzy \( S^* \)-closed, \( r \in I^* \).

**Proof.** If \( \{\lambda_i \in I^*: \lambda_i \text{ is } r\text{-FSO}, i \in \Gamma\} \) is a family such that \( \bigvee_{i \in \Gamma} \lambda_i = 1 \). From the semiweak continuity of \( f \), we have \( f^{-1}(\lambda_i) \leq SI_{\tau}(f^{-1}(SC_{\eta}(\lambda_i, r)), r) \). So, \( SI_{\tau}(f^{-1}(SC_{\eta}(\lambda_i, r)), r) \) is a family of \( r \)-FSO sets in \((X, \tau)\) with

\[ \bigvee_{i \in \Gamma} SI_{\tau}(f^{-1}(SC_{\eta}(\lambda_i, r)), r) = 1. \]

(4.15)

By the semicompactness of \( X \), there exists a finite subset \( \Gamma_0 \subset \Gamma \) such that \( \bigvee_{i \in \Gamma_0} SI_{\tau}(f^{-1}(SC_{\eta}(\lambda_i, r)), r) = 1 \). So,

\[ 1 = f(1) = f \left( \bigvee_{i \in \Gamma_0} SI_{\tau}(f^{-1}(SC_{\eta}(\lambda_i, r)), r) \right) \]
\[ \leq \bigvee_{i \in \Gamma_0} f f^{-1}(SC_{\eta}(\lambda_i), r) \]
\[ \leq \bigvee_{i \in \Gamma_0} SC_{\eta}(\lambda_i, r) \]

(4.16)

Hence, \( \bigvee_{i \in \Gamma} SC_{\eta}(\lambda_i, r) = 1 \) and \( Y \) is \( r \)-fuzzy \( S^* \)-closed.
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