ON CHAINS OF CENTERED VALUATIONS

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Received 5 June 2002

We study chains of centered valuations of a domain \( A \) and chains of centered valuations of \( A[\mathbf{X}] \) corresponding to valuations of \( A \). Finally, we make some applications to chains of valuations centered on the same ideal of \( A[\mathbf{X}] \) and extending the same valuation of \( A \).

2000 Mathematics Subject Classification: 13A18, 13F20.

1. Introduction and preliminary results. All rings will be commutative with unit element. The field of fractions of a domain \( R \) will be denoted by \( \text{Fr}(R) \). Let \( R \subseteq S \) be domains and take a prime ideal \( p \subseteq \text{Spec}(R) \). Then we write \( k(p) = \text{Fr}(R/p) \) and denote the transcendence degree of \( \text{Fr}(S) \) over \( \text{Fr}(R) \) by \( \text{trdeg}_R^S \). We will use the following notation: \( R[\mathbf{X}] = R[\mathbf{X}_1, \ldots, \mathbf{X}_n] \), \( p[\mathbf{X}] = p[\mathbf{X}_1, \ldots, \mathbf{X}_n] \), and \( \text{Fr}(R)(\mathbf{X}) = \text{Fr}(R)(\mathbf{X}_1, \ldots, \mathbf{X}_n) \).

Let \( P \in \text{Spec}(R[\mathbf{X}]) \), \( P \) lies over \( p \) if \( P \cap R = p \). Let \( A \) be a subring of a field \( K \), \( L/K \) a field extension, and \( v \) a valuation on \( K \). The subring \( A_v = \{ x \in K \mid v(x) \geq 0 \} \) is the valuation ring associated to \( v \), \( m(v) = \{ x \in K \mid v(x) > 0 \} \) is its maximal ideal, and \( k_v = A_v/m(v) \) is its residue field. The valuation \( v \) is positive on \( A \) if \( A \subseteq A_v \), and then \( v \) is a valuation on \( A \). The prime ideal \( m(v) \cap A \) is called the center of \( v \) on \( A \). The valuation \( v \) is called trivial if \( A_v = K \). If \( v' \) is a valuation on \( k_v \), then the set \( \{ x \in K \mid x \in A_{v'}, \frac{\mathbf{x}}{\mathbf{x}} \in A_{v'} \} \) is a valuation ring on \( K \). The valuation associated to this valuation ring is called the composite valuation and is denoted by \( v_1 = v'v \).

Let \( v, v' \) be valuations on \( K \). By definition, \( v \preceq v' \) if one of the following equivalent conditions is satisfied:

1. \( A_{v'} \subseteq A_v \),
2. \( m(v) \subseteq m(v') \),
3. \( v' = v''v \) for some valuation on \( k_v \);

\( v \) and \( v' \) are called equivalent if \( A_{v'} = A_v \); \( v < v' \) if \( v \preceq v' \) but not equivalent. A valuation \( w \) on \( L \) is an extension of \( v \) if \( A_v = A_w \cap K \). The valuation \( w \) on \( K(X) \) given by

\[ w \left( \sum_{i=0}^{n} a_i(X - a)^i \right) = \inf \{ v(a_i) \mid 0 \leq i \leq n \} \tag{1.1} \]

is called the canonical extension of \( v \) to \( K(X) \). We have that \( k_w = k_v(X) \).
The following classical results will be used in this paper; the proofs can be found in [2, Proposition 1.2], [4, Theorem 1.5], and [6, Propositions 1.1, 1.3, and 1.4].

**Proposition 1.1.** Let \( v \) be a valuation on \( K \) and \( w_0 < w_1 \) two valuations on \( L \) extending \( v \). If \( \text{trdeg}_{Kv}^{kw_1} \) is finite, then

\[
\text{trdeg}_{Kv}^{kw_1} < \text{trdeg}_{Kv}^{kw_0}.
\]  

(1.2)

**Proposition 1.2.** Let \( v_0 < v_1 \) be two valuations on \( K \) and \( w_1 \) a valuation on \( L \) extending \( v_1 \). Then there exists a valuation \( w_0 \) on \( L \) extending \( v_0 \), with \( w_0 < w_1 \).

**Proposition 1.3.** Let \( w \) be a valuation on \( L \) and \( v \) its restriction to \( K \). If \( \text{trdeg}_K^L \) is finite, then

\[
\text{trdeg}_K^L \leq \text{trdeg}_K^L.
\]  

(1.3)

**Proposition 1.4.** If \( p \subseteq q \) in \( \text{Spec}(A) \) and if \( v_0 \) is a valuation of \( K \) with center \( p \) on \( A \), then there exists a valuation \( v_1 \) of \( K \) with center \( q \) on \( A \) such that \( v_0 \leq v_1 \).

**Theorem 1.5.** Let \( f : A \to B \) be a homomorphism of domains. Then there exist an algebraic extension \( L' \) of \( \text{Fr}(B) \) and a valuation \( v \) on \( K \) with center \( \text{Ker}(f) \) on \( A \) such that

\[
A/\text{Ker}(f) \subseteq k_v \subseteq L.
\]  

(1.4)

In this paper, we will study chains of valuations of a polynomial ring \( A[X_1, \ldots, X_n] \) and of a field extension \( F \) of \( \text{Fr}(A) \). We give the length of chains of valuations which pass through a given valuation, and we characterize when a valuation is maximal or minimal in the following situations:

(a) all the valuations are centered on the same ideal,
(b) all the valuations extend the same valuation of \( \text{Fr}(A) \).

Then we study chains of centered valuations on a domain \( A \) and chains of centered valuations on \( A[X_1, \ldots, X_n] \) corresponding to valuations on \( A \). Finally, we give some applications to chains of valuations centered on the same ideal of \( A[X_1, \ldots, X_n] \) and extending the same valuation on \( A \).

2. **Valuations centered on the same ideal.** Throughout this section, \( K \) is the quotient field of an integral domain \( A \), \( L \) is a field extension of \( K \), and \( v \) is a valuation on \( A \).

**Proposition 2.1.** There exist \( n + 1 \) valuations \( w_0 < \cdots < w_n \) on \( A[n] \) extending \( v \) in such a way that, for each \( i \in \{0, \ldots, n\} \),

\[
\text{trdeg}_{Kv}^{kw_i} = n - i.
\]  

(2.1)
PROOF. Let $k = k_v$ and let $w_0$ be the canonical extension of $v$ to $K(X)$. It is well known that $k_{w_0} = k(X)$. Let $w$ be a valuation on $k(X)$, positive on $k[X]$ and with center $(X)$ on $k[X]$, and $w_1 = w w_0$ the composite valuation of $w$ and $w_0$. The valuation $w_0 < w_1$ as $w$ is not trivial, so $A_{w_1} \cap K \subseteq A_{w_0} \cap K = A_v$ and it is easy to see that

$$A_v \subseteq \{ z \in K(X) \mid z \in A_{w_0}, \overline{z} \in A_w \} \cap K. \quad (2.2)$$

Therefore, $w_1$ extends $v$. As $A[X] \subseteq A_{w_0}$ and $k[X] \subseteq A_w$, $X \in A_{w_0}$ and $\overline{X} = X \in A_w$, that is, $X \in A_{w_1}$, and $w_1$ is a valuation of $A[X]$. We have $\text{trdeg}_{k_v} k_{w_0} = 1$, and according to Proposition 1.1, $0 \leq \text{trdeg}_{k_v} k_{w_0} < \text{trdeg}_v k_{w_0} = 1$, so $\text{trdeg}_{k_v} k_{w_1} = 0$.

Let $n > 1$ and suppose that the property is true for $n - 1$. There exists $v_0 < v_1$, two valuations of $A[X_1]$ extending $v$, and for each $i \in \{0, 1\}$, $\text{trdeg}_{k_v} k_{w_i} = 1 - i$. There exists $n$ valuations $w_1 < \cdots < w_n$ of $A[n]$ extending $v_1$, and for each $i \in \{1, \ldots, n\}$, we have that $\text{trdeg}_{k_v} k_{w_i} = n - i$. According to Proposition 1.2, there exists a valuation $w_0$ of $K(X_1, \ldots, X_n)$ extending $v_0$ and $w_0 < w_1 < \cdots < w_n$. The valuation $w_0$ is a valuation of $A[n]$ because $A[n] \subseteq A_{w_1} \subset A_{w_0}$. For each $i \in \{1, \ldots, n\}$, $w_i$ extends $v$ and $\text{trdeg}_{k_v} k_{w_i} = \text{trdeg}_{k_v} k_{w_0} + \text{trdeg}_{k_v} k_{w_1} = n - i$, $w_0'$ extends $v$, and

$$\text{trdeg}_{k_v} k_{w_0} = \text{trdeg}_{k_v} k_{w_0'} + \text{trdeg}_{k_v} k_{v_0} = \text{trdeg}_{k_v} k_{w_0} + 1 > \text{trdeg}_{k_v} k_{w_1} = n - 1,$$

according to Proposition 1.3, $n - 1 < \text{trdeg}_{k_v} k_{w_0} \leq n$, that is, $\text{trdeg}_{k_v} k_{w_0} = n$. \hfill \Box

COROLLARY 2.2. If $\text{trdeg}_{k_v} k = n$, then there exist $n + 1$ valuations $w_0 < \cdots < w_n$ on $L$ extending $v$ such that $\text{trdeg}_{k_v} k_{w_i} = n - i$ for all $i \in \{0, \ldots, n\}$.

PROOF. Let $\{x_1, \ldots, x_n\}$ be a transcendence basis of $L$ over $K$, $v$ a valuation of $A_v$, and $A_v [x_1, \ldots, x_n] \equiv A_v [X_1, \ldots, X_n]$. According to Proposition 2.1, there exist $n + 1$ valuations $v_0 < \cdots < v_n$ on $K(x_1, \ldots, x_n)$ extending $v$ such that $\text{trdeg}_{k_v} k_{v_i} = n - i$ for each $i \in \{0, \ldots, n\}$. Let $w_n$ be a valuation of $L$ extending $v_n$. Applying Proposition 1.2, we obtain $n + 1$ valuations $w_0 < \cdots < w_n$ of $L$ such that for each $i \in \{0, \ldots, n\}$, $w_i$ prolongs $v_i$, then $w_i$ prolongs $v$, and

$$\text{trdeg}_{k_v} k_{w_i} = \text{trdeg}_{k_{k_{v_i}}} k_{w_i} + \text{trdeg}_{k_v} k_{v_i} = n - i.$$ \hfill \Box

LEMMA 2.3. Let $v_0$ be a valuation on $L$ with center $q$ on $A$. For each $k \in \mathbb{N}$ strictly smaller than $\text{trdeg}_{k(q)} k_{v_0}$, there exists a valuation $v_1$ of $L$ with center $q$ on $A$ such that $v_0 < v_1$ and $\text{trdeg}_{k(q)} k_{v_1} = k$. 

\hfill \Box
PROOF. Let \( \{z_1, \ldots, z_{k+1}\} \) be a family of elements of \( k_{v_0} \), algebraically independent over \( k(q) \). According to Theorem 1.5, there exist an algebraic extension \( L' \) of \( k(q)z_1, \ldots, z_k \) and a valuation \( \nu' \) of \( k_{v_0} \) with center \( (z_{k+1}) \) on \( (A/q)[z_1, \ldots, z_k] \), such that \( (A/q)[z_1, \ldots, z_k] \leq k_{\nu'} \subseteq L' \). Let \( \nu = \nu' \nu_0 \) be the composite valuation of \( \nu' \) with \( \nu_0 \), \( \nu_1 \) is a valuation of \( L \). For each \( b \in A, b \in A_{v_0} \) and \( \bar{b} \in A/q \subseteq A_{v_1} \), that is, \( b \in A_{v_1} \), and if \( a \in m(\nu_1) \cap A \), then \( \bar{a} \in (m(\nu_1)/m(\nu_0)) \cap (A/q) = m(\nu') \cap (A/q) = (0), \) that is, \( a \in q \) and \( m(\nu_1) \cap A \subseteq q \) or \( q = m(\nu_1) \cap A \subseteq m(\nu_1) \cap A \); therefore the center of \( \nu_1 \) on \( A \) is \( q \). As \( A_{\nu'} = A_{v_1}/m(\nu_0), m(\nu') = m(\nu_1)/m(\nu_0) \) and \( k_{\nu'} = A_{\nu'}/m(\nu') = A_{v_1}/m(\nu_1) \). Thus, \( \nu_0 < \nu_1 \) and \( \text{trdeg}_{k(q)}^{k_{\nu_1}} = \text{trdeg}_{k(q)}^{k_{\nu_0}} \) = \( \text{trdeg}_{k(q)}^{k_{\nu}} \).

\[ \text{THEOREM 2.4.} \text{ Let } w \text{ be a valuation on } L \text{ with center } q \text{ on } A. \text{ Then } \text{trdeg}_{k(q)}^{k_{w}} \text{ is the supremum of all natural numbers } \bar{k} \text{ for which there exists a chain of valuations } w = w_0 < \cdots < w_{k} \text{ on } L, \text{ with center } q \text{ on } A. \]

PROOF. Suppose that we have a chain of valuations \( w = w_0 < \cdots < w_{k} \) on \( L \), with center \( q \) on \( A \). If \( \text{trdeg}_{k(q)}^{k_{w}} \) is finite, then it follows from Proposition 1.1 that

\[ 0 \leq \text{trdeg}_{k(q)}^{k_{w_0}} < \cdots < \text{trdeg}_{k(q)}^{k_{w_1}} = \text{trdeg}_{k(q)}^{k_{w}}, \quad (2.5) \]

and consequently \( k \leq \text{trdeg}_{k(q)}^{k_{w}} \). This proves that \( \bar{k} \leq \text{trdeg}_{k(q)}^{k_{w}} \).

To prove the converse inequality, we consider two different cases:

(a) \( \text{trdeg}_{k(q)}^{k_{w}} = k_1 \in \mathbb{N} \) is finite. If \( k_1 = 0 \), then there is nothing to prove. Take \( k_1 > 0 \). By Lemma 2.3, there exists a valuation \( \nu_1 \) on \( L \) with center \( q \) on \( A \) such that \( w < \nu_1 \) and \( \text{trdeg}_{k(q)}^{k_{\nu_1}} = k_1 - 1 \). Using an easy induction argument, we find \( k_1 + 1 \) valuations with \( w = w_0 < \cdots < w_k \) on \( L \), all with center \( q \) on \( A \);

(b) \( \text{trdeg}_{k(q)}^{k_{w}} = \infty \). By Lemma 2.3, we can find, for every \( k \in \mathbb{N} \), a valuation \( \nu_1 \) on \( L \) with center \( q \) on \( A \) such that \( \text{trdeg}_{k(q)}^{k_{\nu_1}} = k \). It then follows from (a) that there exists a chain of valuations \( w = w_0 < \cdots < w_k \) on \( L \), all with center \( q \) on \( A \). We can do this for every \( k \in \mathbb{N} \), hence the supremum is infinite. \( \square \)

\[ \text{LEMMA 2.5.} \text{ Let } w \text{ be a valuation on } A[n] \text{ with center } q \text{ on } A. \]

(a) If \( \text{trdeg}_{k(q)}^{k_{w}} = \infty \), then for every \( k \in \mathbb{N} \), there exists a valuation \( \nu_1 \) on \( A[n] \) with center \( q \) on \( A \) such that \( w < \nu_1 \) and \( \text{trdeg}_{k(q)}^{k_{\nu_1}} = k \).

(b) If \( \text{trdeg}_{k(q)}^{k_{w}} = k \in \mathbb{N} \), then there exists a chain of valuations \( w = w_0 < \cdots < w_k \) on \( A[n] \), all with center \( q \) on \( A \).

PROOF. (a) Let \( Q \) be the center of \( w \) on \( A[n] \) and \( k_1 = \text{trdeg}_{A/q}^{A[n]/q} \). We know that \( k_1 = n - \text{ht}(Q/q[n]) \), where \( \text{ht}(Q/q[n]) \) means the height of the prime ideal \( Q/q[n] \), and there exists a chain \( Q = Q_0 < \cdots < Q_{k_1} \) of prime ideals of \( A[n] \), all lying over \( q \).
Assume first that \( k < k_1 \); then there exists \( i \in \{1, \ldots, k_1\} \) such that \( \text{trdeg}_{A[n]/Q_i}^{A[n]} = k \). Let \( w'' \) be a valuation on \( A[n] \) with center \( Q_i \) and \( w < w'' \) (see Proposition 1.4). According to Lemma 2.3, there exists a valuation \( w_1 \) on \( A[n] \) with center \( Q_i \) such that \( w'' \leq w_1 \) and \( \text{trdeg}_{A[n]/Q_i}^{k_w} = 0 \). Thus, \( w < w_1 \) and \( \text{trdeg}_{A[n]/Q_i}^{k_w} = \text{trdeg}_{A[n]/Q_i}^{A[n]} + \text{trdeg}_{A[n]/Q_i}^{A[n]/Q_i} = k \).

Now assume that \( k \geq k_1 \) and let \( \alpha = k - k_1 \). By Theorem 2.4, there exists a valuation \( w_1 \) on \( A[n] \) with center \( Q \) such that \( w < w_1 \) and \( \text{trdeg}_{A[n]/Q}^{A[n]} = \alpha \), hence

\[
\text{trdeg}_{A/q}^{k_w} = \text{trdeg}_{A[n]/Q}^{k_w} + \text{trdeg}_{A[n]/Q}^{A[n]/Q} = \alpha + k_1 = k.
\]

(2.6)

(b) Let \( Q \) be the center of \( w \) on \( A[n] \) and \( k_1 = \text{trdeg}_{A[n]/Q}^{k_w} \). According to Theorem 2.4, there exists a chain of valuations \( w = w_0 < \cdots < w_{k_1} \) on \( A[n] \), all with center \( Q \), such that \( \text{trdeg}_{A[n]/Q_i}^{k_w} = k_1 - i \) for each \( i \in \{1, \ldots, k_1\} \). Let \( \alpha = \text{trdeg}_{A[n]/Q}^{A[n]/Q_i} \); then there exists a chain \( Q = Q_0 \subset \cdots \subset Q_\alpha \) of prime ideals of \( A[n] \) lying over \( q \). According to Proposition 1.4, there exist \( \alpha + 1 \) valuations \( w_{k_1} < \cdots < w_{k_1 + \alpha} = w_k \) on \( A[n] \) such that \( w_{k_1 + j} \) has center \( Q_j \) on \( A[n] \) for each \( j \in \{0, \ldots, \alpha\} \). Therefore, \( w_{k_1 + j} \) has center \( q \) on \( A \), and the chain of valuations \( w_0 < \cdots < w_{k} \) meets the requirements.

**THEOREM 2.6.** Let \( w \) be a valuation on \( A[n] \) with center \( q \) on \( A \). Then \( \text{trdeg}_{A/q}^{k_w} \) is the supremum of all natural numbers \( K \) such that there exists a chain of valuations \( w = w_0 < \cdots < w_k \) on \( A[n] \) with center \( q \) on \( A \).

**PROOF.** Let \( w = w_0 < \cdots < w_k \) be a chain of valuations on \( A[n] \) with center \( q \) on \( A \). If \( \text{trdeg}_{A/q}^{k_w} \) is finite, then \( 0 \leq \text{trdeg}_{A/q}^{k_w} < \cdots < \text{trdeg}_{A/q}^{k_w} \), so \( k \leq \text{trdeg}_{A/q}^{k_w} \), and it follows that \( K < \text{trdeg}_{A/q}^{k_w} \).

Take \( k \leq \text{trdeg}_{A/q}^{k_w} \). We distinguish two cases:

1. \( \text{trdeg}_{A/q}^{k_w} \) is finite. It follows from Lemma 2.5(b) that there exists a chain of valuations \( w = w_0 < \cdots < w_k \) on \( A[n] \) with center \( q \) on \( A \);
2. \( \text{trdeg}_{A/q}^{k_w} \) is infinite. It follows from Lemma 2.5(a) that there exists a valuation \( w_1 \) on \( A[n] \) with center \( q \) on \( A \) such that \( w < w_1 \) and \( \text{trdeg}_{A/q}^{k_w} = k \). In both cases, we obtain the existence of a chain of valuations \( w = w_0 < \cdots < w_k \) on \( A[n] \), all with center \( q \) on \( A \).

**PROPOSITION 2.7.** Let \( w \) be a valuation on \( A[n] \) (resp., \( L \)) extending \( v \). Then \( \text{trdeg}_{A/q}^{k_w} \) is the supremum of the set of integers \( k \) such that there exists a chain of valuations \( w = w_0 < \cdots < w_k \) on \( A[n] \) (resp., \( L \)) extending \( v \).

**PROOF.** Let \( w' \) be a valuation on \( K(n) \) (resp., \( L \)). We first show that \( w' \) is a valuation on \( A[n] \) (resp., \( L \)) extending \( v \) if and only if \( w' \) is a valuation on \( A_v[n] \) (resp., \( L \)) with center \( m(v) \) on \( A_v \).

First, assume that \( w' \) is a valuation on \( A[n] \) (resp., \( L \)) extending \( v \). Then \( A_{w'} \cap K = A_v \) and \( w' \) is a valuation on \( A[n] \) (resp., \( L \)), hence \( w' \) is a valuation...
on $A_v[n]$ (resp., $L$) and

$$m(w') \cap A_v = m(w') \cap K \cap A_v = m(v) \cap A_v = m(v). \quad (2.7)$$

Conversely, $A[n] \subseteq A_v[n] \subseteq A_{w'}$, therefore $A_v \subseteq A_{w'} \cap K$. If $z \in A_{w'} \cap K$ and $z \notin A_v$, then $z^{-1} \in m(v) = m(w') \cap K$, a contradiction. Hence, $A_{w'} \cap K = A_v$ and $w'$ extends $v$.

To finish the proof, it suffices to apply Theorems 2.4 and 2.6 to $w$ and $m(v) \in \text{Spec}(A_v)$.

**Corollary 2.8.**

(a) Let $w$ be a valuation on $A[n]$ (resp., $L$) with center $q$ on $A$. Then $\text{ht}(m(w)/qA_w)$ is the supremum $\overline{k}$ of the integers $k$ for which there exists a chain of valuations $w_k < \cdots < w_0 = w$ on $A[n]$ (resp., $L$) with center $q$ on $A$.

(b) Let $w$ be a valuation on $A[n]$ (resp., $L$) extending $v$. Then $\text{ht}(m(w)/m(v)A_w)$ is the supremum $\overline{k}$ of the integers $k$ for which there exists a chain of valuations $w_k < \cdots < w_0 = w$ on $A[n]$ (resp., $L$) extending $v$.

**Proof.** (a) Let $w_k < \cdots < w_0 = w$ be a chain of valuations on $A[n]$ (resp., $L$) with center $q$ on $A$. We have $qA_w \subseteq m(w_k) \subseteq m(w_0) = m(w)$, and therefore $\text{ht}(m(w)/qA_w) \geq \overline{k}$.

Conversely, let $k_1 \in \mathbb{N}$ with $k_1 \leq \text{ht}(m(w)/qA_w)$. Then there exists a chain $P_{k_1} \subseteq \cdots \subseteq P_0 = m(w)$ in $\text{Spec}(A_w)$ such that $P_{k_1} \cap A = q$. The valuation rings $A_w = (A_{w'})_{p_0} \subseteq \cdots \subseteq (A_{w'})_{p_{k_1}}$ of $K(n)$ (resp., $L$) are all with center $q$ on $A$. For each $i \in \{0, \ldots, k_1\}$, let $w_i$ be the valuation on $K(n)$ (resp., $L$) associated to $(A_w')_{p_i}$. Then $w_{k_1} < \cdots < w_0 = w$ are valuations on $A(n)$ (resp., $L$), all with center $q$ on $A$.

(b) The proof follows immediately from (a) and the proof of Proposition 2.7.

**Corollary 2.9.**

(a) A valuation $w$ is a maximal (resp., minimal) element in the set of valuations on $A[n]$ (resp., $L$) extending $v$ if and only if $\text{trdeg}_{qA_w}^k w = 0$ (resp., $\text{ht}(m(w)/m(v)A_w) = 0$).

(b) A valuation $w$ is a maximal (resp., minimal) element in the set of valuations on $A[n]$ (resp., $L$) with center $q$ on $A$ if and only if $\text{trdeg}_{A/q}^k w = 0$ (resp., $\text{ht}(m(w)/qA_w) = 0$).

**Corollary 2.10.** Let $w$ be a valuation on $A[n]$ (resp., $L$).

(a) If $w$ has center $q$ on $A$, then the maximum length of a chain of valuations on $A[n]$ (resp., $L$) having center $q$ on $A$ and passing through $w$, is equal to

$$\text{ht}(m(w)/qA_w) + \text{trdeg}_{A/q}^k w. \quad (2.8)$$
(b) If \( w \) extends \( v \), then the maximum length of a chain of valuations on \( A[n] \) (resp., \( L \)) extending \( v \) is equal to
\[
\text{ht} \left( \frac{m(w)}{m(v)} A_w \right) + \text{trdeg}^{k_w}_{k_v}. \tag{2.9}
\]

**Proposition 2.11.** (a) Let \( s \) be the maximal value of \( \text{ht} \left( \frac{m(w)}{q A_w} \right) + \text{trdeg}^{k_w}_{k_v} \), where \( w \) runs through all valuations on \( A[n] \) (resp., \( K(n) \)) with center \( q \) on \( A \). Let \( t \) be the maximal value of \( \text{trdeg}^{k_v'}_{k_v} \), where \( v' \) runs through all valuations on \( K \) with center \( q \) on \( A \). Then \( s = n + t \).

(b) The value \( n \) is the maximal value of \( \text{ht} \left( \frac{m(w)}{m(v)} A_w \right) + \text{trdeg}^{k_w}_{k_v} \), where \( w \) runs through all valuations on \( A[n] \) (resp., \( K(n) \)) extending \( v \).

**Proof.** (a) Let \( w \) be a valuation on \( A[n] \) (resp., \( K(n) \)) with center \( q \) on \( A \) and let \((k_1, k_2) \in \mathbb{N}^2 \) be such that \( k_1 \leq \text{trdeg}^{k_w}_{k_v} \) and \( k_2 \leq \text{ht} \left( \frac{m(w)}{q A_w} \right) \). According to Theorems 2.4 and 2.6, there exists a chain of valuations \( w = w_0 < \cdots < w_{k_1} \) on \( A[n] \) (resp., \( K(n) \)) with center \( q \) on \( A \). By Corollary 2.8, there exists a chain of valuations \( w_{k_2} < \cdots < w_0 = w \) on \( A[n] \) (resp., \( K(n) \)) with center \( q \) on \( A \). Thus, we have a chain of valuations \( w_{k_2} < \cdots < w_0 = w < \cdots < w_{k_1} \) on \( A[n] \) (resp., \( K(n) \)) with center \( q \) on \( A \), and
\[
k_1 + k_2 \leq \text{trdeg}^{k_{w_{k_2}}}_{A/q} = \text{trdeg}^{k_{w_{k_2}}}_{k_v} + \text{trdeg}^{k_{w_{k_2}}}_{A/q} \leq \text{trdeg}^{K(n)}_{k_v} + t \leq n + t,
\tag{2.10}
\]
where \( w_{k_2} | K \) is the restriction of \( w_{k_2} \) to \( K \). Consequently, \( s \leq n + t \).

Conversely, take \( k \leq t \). Then there exists a valuation \( v' \) on \( K \) with center \( q \) on \( A \) such that \( k \leq \text{trdeg}^{k_v'}_{k_v} \). According to Proposition 2.1, there exists a valuation \( w_0 \) on \( A[n] \) extending \( v' \) with \( \text{trdeg}^{k_{w_0}}_{k_v'} = n \), and
\[
n + k \leq \text{trdeg}^{k_{w_0}}_{k_v'} + \text{trdeg}^{k_{v'}}_{A/q} = \text{trdeg}^{k_{w_0}}_{A/q} \leq s. \tag{2.11}
\]

(b) The proof follows immediately from (a) and the preceding results.

**Corollary 2.12.** If the transcendence degree of \( L \) on \( K \) is infinite, then there is no upper bound on \( \text{trdeg}^{k_w}_{A/q} + \text{ht} \left( \frac{m(w)}{q A_w} \right) \), with \( w \) running through all valuations on \( L \) with center \( q \) on \( A \).

**Proof.** The proof is immediate from the preceding proposition.

3. The symbol \( \delta((0), Q) \) in \( A[n] \). Throughout this section, \( A \) will be an integral domain, \((0) \neq q \) a prime ideal of \( A \), \( K \) the quotient field of \( A \), and \( n \) will be a nonnegative integer.

**Lemma 3.1.** Let \( Q \) be a superior of \( q \) in \( A[X] \) such that there exists \( a \in A \) with \( X - a \in Q \). Then, for each valuation \( v \) of \( K \) with center \( q \) on \( A \), there exists
a valuation \(w\) of \(K(X)\) with center \(Q\) on \(A[X]\), extending \(v\), and such that

\[
\text{trdeg}_{\text{Fr}(A[X]/Q)}^{kw} = \text{trdeg}_{\text{Fr}(A/q)}^{kv} + 1.
\] (3.1)

**Proof.** Let \(\delta\) be a strictly positive element of the value group \(G_v\) of \(v\). We define the valuation \(w\) on \(K(X)\) as follows: if

\[
f(X) = b_0 + \cdots + b_n(X-a)^n,
\] (3.2)

then

\[
w(f(X)) = \inf \{ v(a_i) + i\delta \mid i \in \{0, \ldots, n\} \}.
\] (3.3)

It is well known (see [1]) that \(\text{trdeg}_{kw} = 1\). We show that \(w\) is a valuation on \(A[X]\) with center \(Q\) on \(A[X]\). For \(f(X) = b_0 + \cdots + b_n(X-a)^n \in A[X]\), we have

\[
w(f(X)) = \inf \{ v(b_i) + i\delta \mid i \in \{0, \ldots, n\} \} \geq 0.
\] (3.4)

If \(f(X) \in m(w) \cap A[X]\), then \(b_0 \in m(v) \cap A = q\) and \(f(X) \in Q\).

Conversely, let \(g(X) = a_0 + \cdots + a_m(X-a)^m \in Q \subset A[X]\). For each \(i \in \{1, \ldots, m\}\), \(v(a_i) + i\delta > 0\) and \(a_0 \in m(v) \cap A = q\), hence

\[
w(g(X)) = \inf \{ v(a_i) + i\delta \mid i \in \{0, \ldots, m\} \} > 0,
\] (3.5)

that is, \(g(X) \in m(w) \cap A[X]\). Thus, we have

\[
\text{trdeg}_{\text{Fr}(A[X]/Q)}^{kw} = \text{trdeg}_{\text{Fr}(A/q)}^{kw} = \text{trdeg}_{k_v}^{kw} + \text{trdeg}_{\text{Fr}(A/q)}^{kv} = \text{trdeg}_{\text{Fr}(A/q)}^{kv} + 1.
\] (3.6)

**Lemma 3.2.** Let \(Q\) be a superior of \(q\) in \(A[X]\) and let \(v\) be a valuation on \(K\) with center \(q\) on \(A\). Then there exists a valuation \(w\) on \(K(X)\) with center \(Q\) on \(A[X]\) extending \(v\) such that \(\text{trdeg}_{\text{Fr}(A[X])}^{kw} = \text{trdeg}_{\text{Fr}(A/q)}^{kv} + 1\).

**Proof.** (a) Assume that \(A\) is integrally closed in the algebraic closure \(K'\) of \(K\). We have two different cases:

1. \(q\) is a maximal ideal of \(A\), \(Q' = Q/q[X]\) is generated by \(g(X) = X^n + \alpha_{n-1}X^{n-1} + \cdots + \alpha_0 \in (A/q)[X]\). Let \(a_i \in A\) be a representant of \(\alpha_i \in (A/q)\). Then

\[
f(X) = a_0 + \cdots + a_{n-1}(X-a)^{n-1} + X^n = \prod_{i=1}^{m} (X-r_i)^{\alpha_i} \in K[X].
\] (3.7)

Since \(r_i\) is integral over \(A\), so \(r_i \in A\), and

\[
g(X) = \prod_{i=1}^{m} (X-r_i)^{\alpha_i} \in Q'.
\] (3.8)
Then there exists $j \in \{1, \ldots, m\}$ such that $X - r_j \in Q$. We conclude by Lemma 3.2.

(2) Now, let $q$ be any prime ideal in $A$. Let $S = A - q$; we have $(S^{-1}Q)$ which is a superior to $qA_q$ in $A_q[X]$ and $A_q$ is integrally closed in $K'$. The valuation $v$ has center $qA_q$ on $A_q$, so there exists a valuation $w$ of $K[X]$ with center $S^{-1}Q$ on $A_q[X]$ extending $v$, with

$$\text{trdeg}_{K[X]/S^{-1}Q}^k v = \text{trdeg}_{A[X]/Q}^k v + 1, \quad (3.9)$$

$$\text{trdeg}_{A[X]/Q}^k w = \text{trdeg}_{A[X]/Q}^k v + 1.$$

(b) Let $A'$ be the integral closure of $A$ in the algebraic closure $K'$ of $K$. Let $v'$ be a valuation on $K'$ extending $v$. The integral closure of $A$ in $K'$ is the intersection of all the valuation rings on $K'$ that contain $A$, as $v$ is a valuation on $A$, so $v'$ is a valuation on $A'$. Let $q'$ be the center of $v'$ on $A'$, $q' \cap A = q$, and $q'[X] \cap A[X] = q[X]$. The closure $A'[X]$ is integral over $A[X]$, so there exists a prime ideal $Q'$ of $A'[X]$ such that $Q'$ is a superior of $q'$, and $Q'$ lies over $Q$. According to (a), there exists a valuation $w'$ of $K'[X]$ with center $Q'$ on $A'[X]$ extending $v'$ with

$$\text{trdeg}_{A'[X]/Q'}^k v' = \text{trdeg}_{A[X]/Q}^k v' + 1.$$ 

Let $w$ be the restriction of $w'$ to $K(X)$; $w$ is a valuation on $A[X]$,

$$m(w) \cap A[X] = m(w') \cap K(X) \cap A[X]$$

$$= m(w') \cap A'[X] \cap A[X] = Q' \cap A[X] = Q,$$ 

(3.10)

and $w$ prolongs $v$. Also

$$\text{trdeg}_{A'[X]/Q'}^k v' + \text{trdeg}_{A[X]/Q}^k v' = \text{trdeg}_{A[X]/Q}^k v + \text{trdeg}_{A[X]/Q}^k v'.$$ 

(3.11)

It follows from Proposition 1.3 that $\text{trdeg}_{A[X]/Q}^k v' \leq \text{trdeg}_{A[X]/Q}^k v = 0$, hence

$$\text{trdeg}_{A'[X]/Q'}^k v' = \text{trdeg}_{A[X]/Q}^k v,$$

$$\text{trdeg}_{A[X]/Q}^k w' = \text{trdeg}_{A[X]/Q}^k w,$$

$$\text{trdeg}_{A[X]/Q}^k w = \text{trdeg}_{A[X]/Q}^k v + 1.$$ 

(3.12)

REMARK 3.3. Let $Q$ be a prime ideal of $A[X]$ lying over $q$. Then, for each valuation $v$ of $K$ with center $q$ on $A$, there exists a valuation $w$ of $K(X)$ extending $v$ and with center $Q$ on $A[X]$ such that

$$\text{trdeg}_{A[X]/Q}^k w = \text{trdeg}_{A[X]/Q}^k v + \text{ht} (Q/X).$$ 

(3.13)

Indeed, Lemma 3.2 implies the case where $Q$ is a superior of $q$. If $Q = q[X]$, let $w$ be the canonical extension of $v$ to $K(X)$. It is well known (see Section 1) that $\text{trdeg}_{A[X]/Q}^k v = \text{trdeg}_{A[X]/Q}^k v$. 


**Theorem 3.4.** Let $Q$ be a prime ideal of $A[n]$ lying over $q$. Then, for each valuation $v$ of $K$ with center $q$ on $A$, there exists a valuation $w$ of $K(n)$ extending $v$ and with center $Q$ on $A[n]$ such that

$$\operatorname{trdeg}^{k_v}_{\text{Fr}(A[n]/Q)} = \operatorname{trdeg}^{k_v}_{\text{Fr}(A/q)} + \operatorname{ht}(Q/q[n]).$$

**(Proof.** One proceeds by induction on $n$. The case $n = 1$ follows from Remark 3.3. Assume that the statement is true for $n - 1$. Let $Q_1 = Q \cap A[X_1]$. Then there exists a valuation $w_1$ of $K(X_1)$ extending $v$, with center $Q_1$ on $A[X_1]$, and

$$\operatorname{trdeg}^{k_{w_1}}_{\text{Fr}(A[X_1]/Q_1)} = \operatorname{trdeg}^{k_v}_{\text{Fr}(A/q)} + \operatorname{ht}(Q_1/q[X_1]),$$

and there exists a valuation $w$ of $K(X_1)(X_2,\ldots,X_n) = K(n)$ extending $w_1$, with center $Q$ on $A[n]$, and

$$\operatorname{trdeg}^{k_w}_{\text{Fr}(A[n]/Q)} = \operatorname{trdeg}^{k_{w_1}}_{\text{Fr}(A[X_1]/Q_1)} + \operatorname{ht}(Q/Q_1[X_2,\ldots,X_n])$$
$$= \operatorname{trdeg}^{k_v}_{\text{Fr}(A/q)} + \operatorname{ht}(Q_1/q[X_1]) + \operatorname{ht}(Q/Q_1[X_2,\ldots,X_n]).$$

We conclude by remarking that

$$\operatorname{ht}(Q_1/q[X_1]) + \operatorname{ht}(Q/Q_1[X_2,\ldots,X_n]) = \operatorname{ht}(Q/q[X_1,\ldots,X_n]).$$

\[\square\]

**Notation 3.5.** Take $q_1 \subset q_2$ in $\text{Spec}(A)$. We will denote by $\delta(q_1,q_2)$ the maximal value $d$ for which there exists a valuation $v$ on $\text{Fr}(A/q_1)$ with center $q_2/q_1$ on $A/q_1$ such that $\operatorname{trdeg}^{k_v}_{\text{Fr}(A/q_2)} = d$.

Jaffard has shown in [2] that $\delta((0),q_2)$ is the greatest number $n$ such that there exists a chain of valuations $v_0 < \cdots < v_n$ on $A$ with center $q_2$ on $A$.

**Corollary 3.6.** Let $Q$ be a prime ideal of $A[n]$ lying over $q$. Then

$$\delta((0),Q) = \delta((0),q) + \operatorname{ht}(Q/q[n]).$$

**(Proof.** In the case where $Q = q[n]$, the result is well known (see [2]). Suppose that $Q \neq q[n]$. For each valuation $v$ of $K$ with center $q$ on $A$, there exists a valuation $w$ of $K(n)$ with center $Q$ on $A[n]$, extending $v$, with

$$\operatorname{trdeg}^{k_w}_{\text{Fr}(A[n]/Q)} = \operatorname{trdeg}^{k_v}_{\text{Fr}(A/q)} + \operatorname{ht}(Q/q[n]) \leq \delta((0),Q),$$

and consequently

$$\delta((0),Q) \geq \delta((0),q) + \operatorname{ht}(Q/q[n]).$$
Conversely, let \( w' \) be a valuation on \( K(n) \) with center \( Q \) on \( A[n] \) and let \( v' \) be its restriction to \( K \). The valuation \( v' \) has center \( q \) on \( A \) and

\[
\text{trdeg}_{\text{Fr}(A[n]/Q)} k_{w'} + \text{trdeg}_{\text{Fr}(A/q)} k_{v'} \leq n + \delta((0), q),
\]

(3.21)

and therefore

\[
\text{trdeg}_{\text{Fr}(A[n]/Q)} k_{w'} \leq \delta((0), q) + \text{ht}(Q/q[n]),
\]

\[
\delta((0), Q) \leq \delta((0), q) + \text{ht}(Q/q[n]).
\]

(3.22)

**Proposition 3.7.** Let \( q_1 \) be a prime ideal of \( A \) and \( Q_1 \subset Q_2 \) two prime ideals of \( A[n] \) lying over \( q_1 \). Then

\[
\delta(Q_1, Q_2) = \text{ht}(Q_2/q_1[n]) - \text{ht}(Q_1/q_1[n]) - 1.
\]

(3.23)

**Proof.** Let \( T = A - q_1 \). Then \( T^{-1}(A[n]/Q_1) \) is an \( \text{Fr}(A/q_1) \)-algebra of finite type, and therefore a Noetherian domain according to [3]. Then

\[
\delta(Q_1, Q_2) = \delta((0), Q_2/Q_1) = \delta((0), T^{-1}(Q_2/Q_1))
\]

\[
= \text{ht}(T^{-1}(Q_2/Q_1)) - 1 = \text{ht}(Q_2/Q_1) - 1
\]

(3.24)

We finish this section studying the case of trivial valuations and we assume that \( q = (0) \).

**Proposition 3.8.** Let \( Q \) be a prime ideal of \( A[n] \) lying over \((0)\). Then there exists a valuation \( w \) of \( K(n) \) with center \( Q \) on \( A[n] \) such that

\[
\text{trdeg}_{\text{Fr}(A[n]/Q)} k_w = \begin{cases} 
\text{ht}(Q) - 1 & \text{if } Q \neq 0, \\
0 & \text{if } Q = 0.
\end{cases}
\]

(3.25)

**Proof.** If \( Q = (0) \), then it suffices to take for \( w \) the trivial valuation on \( K(n) \). We suppose that \( Q \neq (0) \). If \( n = 1 \), then for each \( w \in A(Q) \), according to the preceding result, \( \text{trdeg}_{\text{Fr}(A[X]/Q)} k_w \leq \delta((0), Q) = 0 \), and therefore

\[
\text{trdeg}_{\text{Fr}(A[X]/Q)} k_w = \text{ht}(Q) - 1 = 0.
\]

(3.26)

Take \( n > 1 \), assume that the property holds for \( n - 1 \), and let \( Q_1 = Q \cap A[X_1] \). Then there exists a valuation \( w_1 \) of \( K(X_1) \) with center \( Q_1 \) on \( A[X_1] \) and \( \text{trdeg}_{\text{Fr}(A[X_1]/Q_1)} k_{w_1} = 0 \). If \( w_1 \) is the trivial valuation, then there exists a valuation \( w \) of \( K(X_1)(X_2, \ldots, X_n) = K(n) \) with center \( Q \) on \( A[X_1][X_2, \ldots, X_n] = A[n] \) and
and only if \( Q \) is maximal in \( A[n] \) and with center \( Q \) on \( A[n] \), and
\[
\trdeg_{\text{Fr}(A[n]/Q)}^{kw} = \ht(Q) - 1.
\]
If \( w_1 \) is not trivial, then \( Q_1 \neq (0) \) and it follows from Theorem 3.4 that there exists a valuation \( w \) on \( K(n) \) extending \( w_1 \) and with center \( Q \) on \( A[n] \), and
\[
\trdeg_{\text{Fr}(A[n]/Q)}^{kw} = \trdeg_{\text{Fr}(A[v_1]/Q_1)}^{kw} + \ht(Q/Q_1[X_2,\ldots,X_n])
\]
\[
= \ht(Q) - \ht(Q_1)
\]
\[
= \ht(Q) - 1.
\]

4. Valuations on \( A[n] \) centered on the same ideal and extending the same valuation. Let \( v \) be a valuation on \( K \) with center \( q \) on \( A \) and \( Q \) a prime ideal of \( A[n] \) lying over \( q \). We will use the following notation:

(a) \( A(Q) = \{ w \mid w \) is a valuation on \( A[n] \) with center \( Q \} \);
(b) \( A(v) = \{ w \mid w \) is a valuation on \( A[n] \) extending \( v \} \);
(c) \( A(v,Q) = \{ w \mid w \) is a valuation on \( A[n] \) with center \( Q \) extending \( v \} \).

**Lemma 4.1.** Let \( w \) be a valuation on \( K(n) \). Then \( w \) is maximal in \( A(v,Q) \) if and only if \( w \) is maximal in \( \{ w'' \mid w'' \) is a valuation on \( A_v[n] \) with center \( Q_1 = m(w) \cap A_v[n] \} \) and \( Q_1 \) is maximal in \( \{ Q' \mid Q' \in \text{Spec}(A_v[n]) \) with \( Q' \cap A[n] = Q \} \).

**Proof.** Suppose that \( w \) is maximal in \( A(v,Q) \). Assume that there exists \( Q_2 \in \text{Spec}(A_v[n]) \) such that \( Q_1 \subset Q_2 \) and \( Q_2 \cap A[n] = Q \). By Proposition 1.4, there exists a valuation \( w_2 \) on \( A_v[n] \) with center \( Q_2 \), and \( w < w_2 \) such that \( m(w_2) \cap A[n] = Q \) and \( A_v \subseteq A_{w_2} \cap K \subseteq A_w \cap K = A_v \). Thus, \( w_2 \) extends \( v \) with center \( Q \) on \( A[n] \), which is a contradiction.

Assume that \( w < w' \), with \( w' \) a valuation on \( A_v[n] \) with center \( Q_1 \). Then \( m(w') \cap A[n] = Q \) and \( w' \) extends \( v \), that is, \( w' \) is a valuation on \( A[n] \) with center \( Q \) and extending \( v \), which is impossible.

Conversely, if \( w < w' \) with \( w' \) a valuation on \( A[n] \) extending \( v \) and with center \( Q \), then \( m(w) \cap A_v[n] = Q_1 \subset m(w') \cap A_v[n] \), but \( m(w') \cap A_v[n] \cap A[n] = m(w') \cap A[n] = Q \), which is again a contradiction.

**Remark 4.2.** Take \( w \in A(v,Q) \). Let \( Q_1 = m(w) \cap A_v[n] \) and assume that \( Q_1 \) is maximal in \( \{ Q' \mid Q' \in \text{Spec}(A_v[n]) \) with \( Q' \cap A[n] = Q \} \). The ideal \( Q_1/m(v)[n] \) is maximal in \( \{ Q' \mid Q' \in \text{Spec}(k_v[n]) \) with \( Q' \cap (A/q)[n] = Q/q[n] \} \). According to [5], we have
\[
\ht(Q_1/m(v)[n]) - \ht(Q/q[n]) = \inf \{ \trdeg_{A/q}^{kw}, \trdeg_{A/q}^{A[n]/Q} \}. \tag{4.1}
\]

**Theorem 4.3.** For \( w \in A(v,Q) \), the following assertions are equivalent:
(a) \( w \) is maximal in \( A(v,Q) \);
(b) \( \inf(\trdeg_{kw}^{A_v}, \trdeg_{A[n]/Q}^{kw}) = 0 \).

**Proof.** First, suppose that \( w \) is maximal in \( A(v,Q) \).
**Case 1.** The transcendence degree of \( k(v) \) on \( k(q) \) is finite. Let \( Q_1 = m(w) \cap A_v[n] \). Then
\[
\text{trdeg}_{A[n]/Q}^k + n - \text{ht}(Q/q[n]) = \text{trdeg}_{A/q}^k + \text{trdeg}_{A_v[n]/Q_1}^A + n - \text{ht}(Q/q[n]) \tag{4.2}
\]
According to Lemma 4.1, \( \text{trdeg}_{A_v[n]/Q}^A = 0 \), and
\[
\text{trdeg}_{A[n]/Q}^A + n - \text{ht}(Q/q[n]) = n - \text{ht}(Q_1/m(v)[n]) + \text{trdeg}_{A/q}^k, \tag{4.3}
\]
and therefore
\[
\text{trdeg}_{A[n]/Q}^A = \text{trdeg}_{A/q}^k + \text{trdeg}_{A_v[n]/Q_1}^A + n - \text{ht}(Q/q[n]) = n - \text{ht}(Q_1/m(v)[n]) + \text{trdeg}_{A/q}^k, \tag{4.4}
\]
**Remark 4.2** implies that
\[
\text{trdeg}_{A[n]/Q}^k = \text{trdeg}_{A/q}^k - \inf \left( \text{trdeg}_{A/q}^k, \text{trdeg}_{A_v[n]/Q}^A \right) \tag{4.5}
\]
If \( \text{trdeg}_{A[n]/Q}^k \neq 0 \), then \( \text{trdeg}_{A[n]/Q}^k = \text{trdeg}_{A/q}^k + \text{ht}(Q/q[n]) - n \), and therefore
\[
\text{trdeg}_{A/q}^k = \text{trdeg}_{A/q}^k + \text{trdeg}_{A_v[n]/Q_1}^A = \text{trdeg}_{A/q}^k, \tag{4.6}
\]
that is, \( \text{trdeg}_{A/q}^k = 0 \).

**Case 2.** The transcendence degree of \( k(v) \) on \( k(q) \) is infinite. We will show that \( \text{trdeg}_{k_v}^k = 0 \). According to Lemma 4.1 and **Remark 4.2**, \( \text{ht}(Q_1/m(v)[n]) - \text{ht}(Q/q[n]) = \text{trdeg}_{A[n]/Q}^A = n - \text{ht}(Q/q[n]) \), \( \tag{4.7} \)
and therefore \( \text{ht}(Q_1/m(v)[n]) = n \). Thus,
\[
\text{trdeg}_{A_v[n]/Q_1}^A + \text{trdeg}_{A_v[n]/m(v)}^A = \text{trdeg}_{A_v[n]/Q}^A, \tag{4.8}
\]
so
\[
\text{trdeg}_{k_v}^k = \text{trdeg}_{A_v[n]/Q}^A = n - \text{ht}(Q_1/m(v)[n]) = 0. \tag{4.9}
\]
Conversely, assume that \( \inf(\text{trdeg}_{k_v}^k, \text{trdeg}_{A[n]/Q}^A) = 0 \). It follows from Corollary 2.9 that \( w \) is maximal in \( A(v,Q) \). \( \square \)

**Proposition 4.4.** If \( w_0 \in A(v,Q) \) is not maximal in \( A(v,Q) \), then there exists \( w_1 \) in \( A(v,Q) \) such that \( w_0 < w_1 \) and \( \text{trdeg}_{A[n]/Q_1}^k = \text{trdeg}_{A[n]/Q}^k - 1 \).
Proof. Let $Q_0 = m(w_0) \cap A_v[n]$. According to Lemma 4.1, $\text{trdeg}_{A_v[n]}/Q_0^{k_0} \neq 0$ or $Q_0$ is not maximal in \{ $Q' \mid Q' \in \text{Spec}(A_v[n])$ with $Q' \cap A[n] = Q$ \}.

If $\text{trdeg}_{A_v[n]}/Q_0^{k_0} \neq 0$, then it follows from Lemma 2.3 that there exists a valuation $w_1$ on $A_v[n]$ with center $Q_0$ such that $w_0 < w_1$ and $\text{trdeg}_{A_v[n]/Q_0^{k_0}}^{k_0} = \text{trdeg}_{A_v[n]/Q_0}^{k_0} - 1$. The valuation $w_1$ extends $v$ since $A_v \subseteq A_{w_1} \cap K \subseteq A_{w_0} \cap K = A_v$.

If $\text{trdeg}_{A_v[n]/Q_0^{k_0}}^{k_0} = 0$, let $Q_1$ be a prime ideal of $A_v[n]$ lying over $Q_0$ with $Q_0 \subset Q_1$ and $\text{ht}(Q_1/Q_0) = 1$. According to Proposition 1.4 and Theorem 2.4, there exists a valuation $w_1$ on $A_v[n]$ with center $Q_1$ such that $w_0 < w_1$ and $\text{trdeg}_{A_v[n]/Q_1}^{k_1} = \text{trdeg}_{A_v[n]/Q_0}^{k_0}$; $w_1$ is a valuation on $A[n]$ with center $Q$. Then

\[
\text{trdeg}_{A[n]/Q}^{k_0} = \text{trdeg}_{A[n]/Q}^{k_1} = \text{ht}(Q/q[n]) - \text{ht}(Q_0/m(v)[n]) + \text{trdeg}_{A/q}^{k_0},
\]

and therefore

\[
\text{trdeg}_{A[n]/Q}^{k_0} - \text{trdeg}_{A[n]/Q}^{k_1} = \text{ht}(Q_1/m(v)[n]) - \text{ht}(Q_0/m(v)[n]) = \text{ht}(Q_1/Q_0) = 1.
\] (4.11)

Theorem 4.5. For $w \in A(v,Q)$, let $d(w,v,Q) = \text{Sup}\{ k \mid k \in \mathbb{N}, \exists k+1 \text{ valuations } w_0 < \cdots < w_k \text{ in } A(v,Q) \}$. Then

\[
d(w,v,Q) = \inf \left( \text{trdeg}_{A[n]/Q}^{k_w}, \text{trdeg}_{A[n]/Q}^{k_w} \right).
\] (4.12)

Proof. Let $k = d(w,v,Q)$. According to Proposition 4.4, there exist $w_0 < \cdots < w_k$ in $A(v,Q)$, with $\text{trdeg}_{A[n]/Q}^{k_0} = \text{trdeg}_{A[n]/Q}^{k_1} - i$ for each $i \in \{0,\ldots,k\}$ and $w_k$ maximal in $A(v,Q)$, that is,

\[
\text{trdeg}_{A[n]/Q}^{k_0} = \text{trdeg}_{A[n]/Q}^{k_1} - \inf \left( \text{trdeg}_{A[n]/Q}^{k_0}, \text{trdeg}_{A[n]/Q}^{k_0} \right) = \text{trdeg}_{A[n]/Q}^{k_0} - k,
\] (4.13)

and therefore

\[
k = \text{trdeg}_{A[n]/Q}^{k_w} - \text{trdeg}_{A[n]/Q}^{k_v} + \inf \left( \text{trdeg}_{A[n]/Q}^{k_0}, \text{trdeg}_{A[n]/Q}^{k_0} \right)
\] (4.14)

and therefore

\[
s = \sup \{ k \mid \exists k+1 \text{ valuations } w_0 < \cdots < w_0 = w \text{ on } A(v,Q) \}
\] (4.15)

Proposition 4.6. For $w \in A(v,Q)$,

\[
h = \inf \left( \text{ht}(m(w)/ QA_w), \text{ht}(m(w)/ m(v) A_w) \right).
\]
**Proof.** Let $k \in \mathbb{N}$ and let $w_k < \cdots < w_0 = w$ be valuations on $A(v, Q)$. As $A(v, Q) \subseteq A(v) \cap A(Q)$, $k \leq \text{inf}(ht(m(w)/QA_w), ht(m(w)/m(v)A_w))$, and consequently $s \leq h$.

The converse inequality is trivial if $s = \infty$. So let $s < \infty$ and assume $s < h$. Take $w_k < \cdots < w_0 = w$ in $A(v, Q)$. Then

\[
ht(m(w)/QA_w) = \text{ht}(m(w)/m(w_k)) + \text{ht}(m(w_k)/QA_{w_k}) = k + \text{ht}(m(w_k)/QA_{w_k}),
\]

and therefore $\text{ht}(m(w_k)/QA_{w_k}) \neq 0$, and there exists $w'_{k+1} \in A(Q)$ with $w'_{k+1} < w_k$. As

\[
\text{ht}(m(w)/m(v)A_w) = \text{ht}(m(w)/m(w_k)) + \text{ht}(m(w_k)/m(v)A_{w_k}) = k + \text{ht}(m(w_k)/m(v)A_{w_k}),
\]

we have that $\text{ht}(m(w_k)/m(v)A_{w_k}) \neq 0$, and therefore there exists $w''_{k+1} \in A(v)$ with $w''_{k+1} < w_k$. Thus, $A_{w_k} \subseteq A_{w'_{k+1}}, A_{w_k} \subseteq A_{w''_{k+1}}, A_{w'_{k+1}} = (A_{w_k})_{P'}$, and $A_{w''_{k+1}} = (A_{w_k})_{P''}$ with $P'$ and $P''$ two prime ideals of $A_{w_k}$. As $P'$ and $P''$ are comparable, $A_{w'_{k+1}} \subseteq A_{w''_{k+1}}$, or $A_{w''_{k+1}} \subseteq A_{w'_{k+1}}$. If $w''_{k+1} \subseteq w'_{k+1}$, $m(w'_{k+1}) \subseteq m(w''_{k+1})$ and $m(v) = m(w'_{k+1}) \cap A_v \subseteq m(w''_{k+1}) \cap A_v$, that is, $m(w'_{k+1}) \cap A_v = m(v)$ and $w'_{k+1}$ extends $v$, it will be a contradiction since $w'_{k+1} \in A(v, Q)$.

If $w'_{k+1} \subseteq w''_{k+1}$, then $m(w''_{k+1}) \subseteq m(w'_{k+1}) \subset m(v)$, and then $m(w''_{k+1}) \cap A[n] = Q$ and $w''_{k+1} \in A(v, Q)$, which is again a contradiction.

We conclude that $s = h$ and finish the proof. \qed

**Corollary 4.7.** For $w \in A(v, Q)$, the following assertions are equivalent:
(a) $w$ is minimal in $A(v, Q)$;
(b) $\text{inf}(ht(m(w)/QA_w), ht(m(w)/m(v)A_w)) = 0$;
(c) $w$ is minimal in $A(Q)$ or in $A(v)$.

**Notation 4.8.** For each valuation $w$ on $A(v, Q)$, let $l(w)$ be the maximal length of a chain of valuations on $A(v, Q)$ passing through $w$. The maximum value of $l(w)$, where $w$ runs through the set $A(v, Q)$, will be denoted by $d(A(v, Q))$.

**Theorem 4.9.** For each valuation $w$ on $A(v, Q)$,
(a) $l(w) = \text{inf}(ht(m(w)/QA_w), ht(m(w)/m(v)A_w)) + \text{inf}(\text{trdeg}^k_{w})$,
(b) $d(A(v, Q)) = \text{trdeg}^k_{w} + \text{ht}(Q/q[n])$.

**Proof.** (a) The proof follows immediately from Theorem 4.5 and Proposition 4.6.
(b) Let $w' \in A(v, Q)$. Then

\[
\text{trdeg}^k_{A[n]/Q} + n - \text{ht}(Q/q[n]) = \text{trdeg}^k_{w'} = \text{trdeg}^k_{w'} + \text{trdeg}^k_{v},
\]

(4.18)
thus
\[
\text{trdeg}_{A[n]/Q}^{k_v} = \text{trdeg}_{k(q)}^{k_v} + \text{ht}(Q/q[n]) + \left(\text{trdeg}_{k(q)}^{k_{w'}} - n\right)
\]
\[
\leq \text{trdeg}_{k(q)}^{k_v} + \text{ht}(Q/q[n]).
\]
(4.19)

Therefore,
\[
d(A(v, Q)) \leq \text{trdeg}_{k(q)}^{k_v} + \text{ht}(Q/q[n]).
\]

**Theorem 3.4** implies the existence of \( w \in A(v, Q) \) satisfying \( \text{trdeg}_{A[n]/Q}^{k_w} = \text{trdeg}_{k(q)}^{k_v} + \text{ht}(Q/q[n]) \), and the converse inequality follows.

\[\square\]

**REFERENCES**


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