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ON CHAINS OF CENTERED VALUATIONS

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We study chains of centered valuations of a domain A and chains of centered valuations of $A[X_1,\dots,X_n]$ corresponding to valuations of A. Finally, we make some applications to chains of valuations centered on the same ideal of $A[X_1,\dots,X_n]$ and extending the same valuation of A.

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1. Introduction and preliminary results. All rings will be commutative with unit element. The field of fractions of a domain R will be denoted by $\mathrm{Fr}(R)$. Let $R \subseteq S$ be domains and take a prime ideal $p \in \mathrm{Spec}(R)$. Then we write $k(p) = \mathrm{Fr}(R/p)$ and denote the transcendence degree of $\mathrm{Fr}(S)$ over $\mathrm{Fr}(R)$ by $\mathrm{tr\,deg}_R^S$. We will use the following notation: $R[n] = R[X_1,\dots,X_n]$, $p[n] = p[X_1,\dots,X_n]$, and $\mathrm{Fr}(R)(n) = \mathrm{Fr}(R)(X_1,\dots,X_n)$. Let $P \in \mathrm{Spec}(R[n])$, P lies over p if $P \cap R = p$. We call $P \in \mathrm{Spec}(R[1])$ a superior of p if P lies over p and $P \neq p[1]$. Let A be a subring of a field K, L/K a field extension, and v a valuation on K. The subring $A_v = \{x \in K \mid v(x) \geq 0\}$ is the valuation ring associated to v, $m(v) = \{x \in K \mid v(x) > 0\}$ is its maximal ideal, and $k_v = A_v/m(v)$ is its residue field. The valuation v is positive on A if $A \subseteq A_v$, and then v is a valuation on A. The prime ideal $m(v) \cap A$ is called the center of v on A. The valuation v is called trivial if $A_v = K$. If v' is a valuation on k_v, then the set $\{x \in K \mid x \in A_v,\ \overline{x} \in A_{v'}\}$ is a valuation ring on K. The valuation associated to this valuation ring is called the composite valuation and is denoted by $v_1 = v'v$.

Let v, v' be valuations on K. By definition, $v \leq v'$ if one of the following equivalent conditions is satisfied:

(1) $A_{v'} \subseteq A_v$,

(2) $m(v) \subseteq m(v')$,

(3) $v' = v''v$ for some valuation on k_v;

v and v' are called equivalent if $A_{v'} = A_v$; $v < v'$ if $v \leq v'$ but not equivalent. A valuation w on L is an extension of v if $A_v = A_w \cap K$. The valuation w on $K(X)$ given by

$$w\left(\sum_{i=0}^{n} a_i (X-a)^i\right) = \inf\{v(a_i) \mid 0 \leq i \leq n\} \tag{1.1}$$

is called the canonical extension of v to $K(X)$. We have that $k_w = k_v(X)$.

The following classical results will be used in this paper; the proofs can be found in [2, Proposition 1.2], [4, Theorem 1.5], and [6, Propositions 1.1, 1.3, and 1.4].

Proposition 1.1. *Let v be a valuation on K and $w_0 < w_1$ two valuations on L extending v. If $\operatorname{tr\,deg}_{k_v}^{k_{w_0}}$ is finite, then*

$$\operatorname{tr\,deg}_{k_v}^{k_{w_1}} < \operatorname{tr\,deg}_{k_v}^{k_{w_0}}. \tag{1.2}$$

Proposition 1.2. *Let $v_0 < v_1$ be two valuations on K and w_1 a valuation on L extending v_1. Then there exists a valuation w_0 on L extending v_0, with $w_0 < w_1$.*

Proposition 1.3. *Let w be a valuation on L and v its restriction to K. If $\operatorname{tr\,deg}_K^L$ is finite, then*

$$\operatorname{tr\,deg}_{k_v}^{k_w} \le \operatorname{tr\,deg}_K^L. \tag{1.3}$$

Proposition 1.4. *If $p \subseteq q$ in $\operatorname{Spec}(A)$ and if v_0 is a valuation of K with center p on A, then there exists a valuation v_1 of K with center q on A such that $v_0 \le v_1$.*

Theorem 1.5. *Let $f : A \to B$ be a homomorphism of domains. Then there exist an algebraic extension L' of $\operatorname{Fr}(B)$ and a valuation v on K with center $\operatorname{Ker}(f)$ on A such that*

$$A/\operatorname{Ker}(f) \subseteq k_v \subseteq L. \tag{1.4}$$

In this paper, we will study chains of valuations of a polynomial ring $A[X_1, \dots, X_n]$ and of a field extension F of $\operatorname{Fr}(A)$. We give the length of chains of valuations which pass through a given valuation, and we characterize when a valuation is maximal or minimal in the following situations:

(a) all the valuations are centered on the same ideal,

(b) all the valuations extend the same valuation of $\operatorname{Fr}(A)$.

Then we study chains of centered valuations on a domain A and chains of centered valuations on $A[X_1,\dots,X_n]$ corresponding to valuations on A. Finally, we give some applications to chains of valuations centered on the same ideal of $A[X_1,\dots,X_n]$ and extending the same valuation on A.

2. Valuations centered on the same ideal. Throughout this section, K is the quotient field of an integral domain A, L is a field extension of K, and v is a valuation on A.

Proposition 2.1. *There exist $n+1$ valuations $w_0 < \cdots < w_n$ on $A[n]$ extending v in such a way that, for each $i \in \{0,\dots,n\}$,*

$$\operatorname{tr\,deg}_{k_v}^{k_{w_i}} = n - i. \tag{2.1}$$

Proof. Let $k = k_v$ and let w_0 be the canonical extension of v to $K(X)$. It is well known that $k_{w_0} = k(X)$. Let w be a valuation on $k(X)$, positive on $k[X]$ and with center (X) on $k[X]$, and $w_1 = ww_0$ the composite valuation of w and w_0. The valuation $w_0 < w_1$ as w is not trivial, so $A_{w_1} \cap K \subseteq A_{w_0} \cap K = A_v$ and it is easy to see that

$$A_v \subseteq \{z \in K(X) \mid z \in A_{w_0},\ \overline{z} \in A_w\} \cap K. \tag{2.2}$$

Therefore, w_1 extends v. As $A[X] \subseteq A_{w_0}$ and $k[X] \subseteq A_w$, $X \in A_{w_0}$ and $\overline{X} = X \in A_w$, that is, $X \in A_{w_1}$, and w_1 is a valuation of $A[X]$. We have $\operatorname{tr\,deg}_{k_v}^{k_{w_0}} = 1$, and according to Proposition 1.1, $0 \le \operatorname{tr\,deg}_{k_v}^{k_{w_1}} < \operatorname{tr\,deg}_{k_v}^{k_{w_0}} = 1$, so $\operatorname{tr\,deg}_{k_v}^{k_{w_1}} = 0$.

Let $n > 1$ and suppose that the property is true for $n-1$. There exists $v_0 < v_1$, two valuations of $A[X_1]$ extending v, and for each $i \in \{0,1\}$, $\operatorname{tr\,deg}_{k_v}^{k_{v_i}} = 1 - i$. There exists n valuations $w_1 < \cdots < w_n$ of $A[n]$ extending v_1, and for each $i \in \{1,\dots,n\}$, we have that $\operatorname{tr\,deg}_{k_{v_1}}^{k_{w_i}} = n-i$. According to Proposition 1.2, there exists a valuation w_0' of $K(X_1,\dots,X_n)$ extending v_0 and $w_0' < w_1 < \cdots < w_n$. The valuation w_0' is a valuation of $A[n]$ because $A[n] \subseteq A_{w_1} \subset A_{w_0'}$. For each $i \in \{1,\dots,n\}$, w_i extends v and $\operatorname{tr\,deg}_{k_v}^{k_{w_i}} = \operatorname{tr\,deg}_{k_{v_1}}^{k_{w_i}} + \operatorname{tr\,deg}_{k_v}^{k_{v_1}} = n - i$, w_0' extends v, and

$$\begin{aligned} \operatorname{tr\,deg}_{k_v}^{k_{w_0'}} &= \operatorname{tr\,deg}_{k_{v_0}}^{k_{w_0'}} + \operatorname{tr\,deg}_{k_v}^{k_{v_0}} \\ &= \operatorname{tr\,deg}_{k_{v_0}}^{k_{w_0'}} + 1 > \operatorname{tr\,deg}_{k_v}^{k_{w_1}} = n-1, \end{aligned} \tag{2.3}$$

according to Proposition 1.3, $n-1 < \operatorname{tr\,deg}_{k_v}^{k_{w_0'}} \le n$, that is, $\operatorname{tr\,deg}_{k_v}^{k_{w_0'}} = n$. □

Corollary 2.2. *If* $\operatorname{tr\,deg}_K^L = n$*, then there exist* $n+1$ *valuations* $w_0 < \cdots < w_n$ *on* L *extending* v *such that* $\operatorname{tr\,deg}_{k_v}^{k_{w_i}} = n - i$ *for all* $i \in \{0,\dots,n\}$.

Proof. Let $\{x_1,\dots,x_n\}$ be a transcendence basis of L over K, v a valuation of A_v, and $A_v[x_1,\dots,x_n] \cong A_v[X_1,\dots,X_n]$. According to Proposition 2.1, there exist $n+1$ valuations $v_0 < \cdots < v_n$ on $K(x_1,\dots,x_n)$ extending v such that $\operatorname{tr\,deg}_{k_v}^{k_{v_i}} = n-i$ for each $i \in \{0,\dots,n\}$. Let w_n be a valuation of L extending v_n. Applying Proposition 1.2, we obtain $n+1$ valuations $w_0 < \cdots < w_n$ of L such that for each $i \in \{0,\dots,n\}$, w_i prolongs v_i, then w_i prolongs v, and

$$\operatorname{tr\,deg}_{k_v}^{k_{w_i}} = \operatorname{tr\,deg}_{k_{v_i}}^{k_{w_i}} + \operatorname{tr\,deg}_{k_v}^{k_{v_i}} = n - i. \tag{2.4}$$

□

Lemma 2.3. *Let* v_0 *be a valuation on* L *with center* q *on* A*. For each* $k \in \mathbb{N}$ *strictly smaller than* $\operatorname{tr\,deg}_{k(q)}^{k_{v_0}}$*, there exists a valuation* v_1 *of* L *with center* q *on* A *such that* $v_0 < v_1$ *and* $\operatorname{tr\,deg}_{k(q)}^{k_{v_1}} = k$.

Proof. Let $\{z_1,\ldots,z_{k+1}\}$ be a family of elements of k_{v_0}, algebraically independent over $k(q)$. According to Theorem 1.5, there exist an algebraic extension L' of $k(q)(z_1,\ldots,z_k)$ and a valuation v' of k_{v_0} with center (z_{k+1}) on $(A/q)[z_1,\ldots,z_{k+1}]$, such that $(A/q)[z_1,\ldots,z_k] \subseteq k_{v'} \subseteq L'$. Let $v_1 = v'v_0$ be the composite valuation of v' with v_0, v_1 is a valuation of L. For each $b \in A$, $b \in A_{v_0}$ and $\overline{b} \in A/q \subseteq A_{v'}$, that is, $b \in A_{v_1}$, and if $a \in m(v_1) \cap A$, then $\overline{a} \in (m(v_1)/m(v_0)) \cap (A/q) = m(v') \cap (A/q) = (0)$, that is, $a \in q$ and $m(v_1) \cap A \subseteq q$ or $q = m(v_0) \cap A \subseteq m(v_1) \cap A$; therefore the center of v_1 on A is q. As $A_{v'} = A_{v_1}/m(v_0)$, $m(v') = m(v_1)/m(v_0)$ and $k_{v'} = A_{v'}/m(v') = A_{v_1}/m(v_1)$. Thus, $v_0 < v_1$ and $\operatorname{trdeg}_{k(q)}^{k_{v_1}} = \operatorname{trdeg}_{k(q)}^{k_{v'}} = \operatorname{trdeg}_{k(q)}^{k(q)(z_1,\ldots,z_k)} = k$. □

Theorem 2.4. *Let w be a valuation on L with center q on A. Then $\operatorname{trdeg}_{k(q)}^{k_w}$ is the supremum of all natural numbers $\overline{k}$ for which there exists a chain of valuations $w = w_0 < \cdots < w_{\overline{k}}$ on L, with center q on A.*

Proof. Suppose that we have a chain of valuations $w = w_0 < \cdots < w_k$ on L, with center q on A. If $\operatorname{trdeg}_{k(q)}^{k_w}$ is finite, then it follows from Proposition 1.1 that

$$0 \le \operatorname{trdeg}_{k(q)}^{k_{w_k}} < \cdots < \operatorname{trdeg}_{k(q)}^{k_{w_0}} = \operatorname{trdeg}_{k(q)}^{k_w}, \tag{2.5}$$

and consequently $k \le \operatorname{trdeg}_{k(q)}^{k_w}$. This proves that $\overline{k} \le \operatorname{trdeg}_{k(q)}^{k_w}$.

To prove the converse inequality, we consider two different cases:

(a) $\operatorname{trdeg}_{k(q)}^{k_w} = k_1 \in \mathbb{N}$ is finite. If $k_1 = 0$, then there is nothing to prove. Take $k_1 > 0$. By Lemma 2.3, there exists a valuation w_1 on L with center q on A such that $w < w_1$ and $\operatorname{trdeg}_{k(q)}^{k_{w_1}} = k_1 - 1$. Using an easy induction argument, we find $k_1 + 1$ valuations with $w = w_0 < \cdots < w_k$ on L, all with center q on A;

(b) $\operatorname{trdeg}_{k(q)}^{k_w} = \infty$. By Lemma 2.3, we can find, for every $k \in \mathbb{N}$, a valuation w_1 on L with center q on A such that $\operatorname{trdeg}_{k(q)}^{k_{w_1}} = k$. It then follows from (a) that there exists a chain of valuations $w = w_0 < \cdots < w_k$ of L, all with center q on A. We can do this for every $k \in \mathbb{N}$, hence the supremum is infinite. □

Lemma 2.5. *Let w be a valuation on $A[n]$ with center q on A.*

(a) *If $\operatorname{trdeg}_{k(q)}^{k_w} = \infty$, then for every $k \in \mathbb{N}$, there exists a valuation w_1 on $A[n]$ with center q on A such that $w < w_1$ and $\operatorname{trdeg}_{k(q)}^{k_{w_1}} = k$.*

(b) *If $\operatorname{trdeg}_{k(q)}^{k_w} = k \in \mathbb{N}$, then there exists a chain of valuations $w = w_0 < \cdots < w_k$ on $A[n]$, all with center q on A.*

Proof. (a) Let Q be the center of w on $A[n]$ and $k_1 = \operatorname{trdeg}_{A/q}^{A[n]/Q}$. We know that $k_1 = n - \operatorname{ht}(Q/q[n])$, where $\operatorname{ht}(Q/q[n])$ means the height of the prime ideal $Q/q[n]$, and there exists a chain $Q = Q_0 \subset \cdots \subset Q_{k_1}$ of prime ideals of $A[n]$, all lying over q.

Assume first that $k<k_1$; then there exists $i\in\{1,\dots,k_1\}$ such that $\operatorname{tr deg}_{A/q}^{A[n]/Q_i} = k$. Let w'' be a valuation on $A[n]$ with center Q_i and $w < w''$ (see Proposition 1.4). According to Lemma 2.3, there exists a valuation w_1 on $A[n]$ with center Q_i such that $w'' \le w_1$ and $\operatorname{tr deg}_{A[n]/Q_i}^{k_{w_1}} = 0$. Thus, $w < w_1$ and $\operatorname{tr deg}_{k(q)}^{k_{w_1}} = \operatorname{tr deg}_{A[n]/Q_i}^{k_{w_1}} + \operatorname{tr deg}_{A/q}^{A[n]/Q_i} = k$.

Now assume that $k \ge k_1$ and let $\alpha = k-k_1$. By Theorem 2.4, there exists a valuation w_1 on $A[n]$ with center Q such that $w < w_1$ and $\operatorname{tr deg}_{A[n]/Q}^{k_{w_1}} = \alpha$, hence

$$\operatorname{tr deg}_{A/q}^{k_{w_1}} = \operatorname{tr deg}_{A[n]/Q}^{k_{w_1}} + \operatorname{tr deg}_{A/q}^{A[n]/Q} = \alpha + k_1 = k. \tag{2.6}$$

(b) Let Q be the center of w on $A[n]$ and $k_1 = \operatorname{tr deg}_{A[n]/Q}^{k_w}$. According to Theorem 2.4, there exists a chain of valuations $w = w_0 < \cdots < w_{k_1}$ on $A[n]$, all with center Q, such that $\operatorname{tr deg}_{A[n]/Q}^{k_{w_i}} = k_1 - i$ for each $i \in \{0,\dots,k_1\}$. Let $\alpha = \operatorname{tr deg}_{A/q}^{A[n]/Q}$; then there exists a chain $Q = Q_0 \subset \cdots \subset Q_\alpha$ of prime ideals of $A[n]$ lying over q. According to Proposition 1.4, there exist $\alpha+1$ valuations $w_{k_1} < \cdots < w_{k_1+\alpha} = w_k$ on $A[n]$ such that w_{k_1+j} has center Q_j on $A[n]$ for each $j \in \{0,\dots,\alpha\}$. Therefore, w_{k_1+j} has center q on A, and the chain of valuations $w_0 < \cdots < w_k$ meets the requirements. □

Theorem 2.6. *Let w be a valuation on $A[n]$ with center q on A. Then $\operatorname{tr deg}_{A/q}^{k_w}$ is the supremum of all natural numbers $\overline{k}$ such that there exists a chain of valuations $w = w_0 < \cdots < w_{\overline{k}}$ on $A[n]$ with center q on A.*

Proof. Let $w = w_0 < \cdots < w_k$ be a chain of valuations on $A[n]$ with center q on A. If $\operatorname{tr deg}_{A/q}^{k_w}$ is finite, then $0 \le \operatorname{tr deg}_{A/q}^{k_{w_k}} < \cdots < \operatorname{tr deg}_{A/q}^{k_{w_0}}$, so $k \le \operatorname{tr deg}_{A/q}^{k_w}$, and it follows that $\overline{k} < \operatorname{tr deg}_{A/q}^{k_w}$.

Take $k \le \operatorname{tr deg}_{A/q}^{k_w}$. We distinguish two cases:

(1) $\operatorname{tr deg}_{A/q}^{k_w}$ is finite. It follows from Lemma 2.5(b) that there exists a chain of valuations $w = w_0 < \cdots < w_k$ on $A[n]$ with center q on A;

(2) $\operatorname{tr deg}_{A/q}^{k_w}$ is infinite. It follows from Lemma 2.5(a) that there exists a valuation w_1 on $A[n]$ with center q on A such that $w < w_1$ and $\operatorname{tr deg}_{A/q}^{k_{w_1}} = k$.

In both cases, we obtain the existence of a chain of valuations $w = w_0 < \cdots < w_k$ on $A[n]$, all with center q on A. □

Proposition 2.7. *Let w be a valuation on $A[n]$ (resp., L) extending v. Then $\operatorname{tr deg}_{k_v}^{k_w}$ is the supremum of the set of integers k such that there exists a chain of valuations $w = w_0 < \cdots < w_k$ on $A[n]$ (resp., L) extending v.*

Proof. Let w' be a valuation on $K(n)$ (resp., L). We first show that w' is a valuation on $A[n]$ (resp., L) extending v if and only if w' is a valuation on $A_v[n]$ (resp., L) with center $m(v)$ on A_v.

First, assume that w' is a valuation on $A[n]$ (resp., L) extending v. Then $A_{w'} \cap K = A_v$ and w' is a valuation on $A[n]$ (resp., L), hence w' is a valuation on $A_v[n]$ (resp., L) and

$$m(w')\cap A_v = m(w')\cap K\cap A_v = m(v)\cap A_v = m(v). \tag{2.7}$$

Conversely, $A[n]\subseteq A_v[n]\subseteq A_{w'}$, therefore $A_v\subseteq A_{w'}\cap K$. If $z\in A_{w'}\cap K$ and $z\notin A_v$, then $z^{-1}\in m(v)=m(w')\cap K$, a contradiction. Hence, $A_{w'}\cap K=A_v$ and w' extends v.

To finish the proof, it suffices to apply Theorems 2.4 and 2.6 to w and $m(v)\in \operatorname{Spec}(A_v)$. □

Corollary 2.8. (a) *Let w be a valuation on $A[n]$ (resp., L) with center q on A. Then $\operatorname{ht}(m(w)/qA_w)$ is the supremum $\overline{k}$ of the integers k for which there exists a chain of valuations $w_k<\cdots<w_0=w$ on $A[n]$ (resp., L) with center q on A.*

(b) *Let w be a valuation on $A[n]$ (resp., L) extending v. Then $\operatorname{ht}(m(w)/m(v)A_w)$ is the supremum $\overline{k}$ of the integers k for which there exists a chain of valuations $w_k<\cdots<w_0=w$ on $A[n]$ (resp., L) extending v.*

Proof. (a) Let $w_k<\cdots<w_0=w$ be a chain of valuations on $A[n]$ (resp., L) with center q on A. We have $qA_w\subseteq m(w_k)\subset\cdots\subset m(w_0)=m(w)$, and therefore $\operatorname{ht}(m(w)/qA_w)\geq\overline{k}$.

Conversely, let $k_1\in\mathbb{N}$ with $k_1\leq\operatorname{ht}(m(w)/qA_w)$. Then there exists a chain $P_{k_1}\subset\cdots\subset P_0=m(w)$ in $\operatorname{Spec}(A_w)$ such that $P_{k_1}\cap A=q$. The valuation rings $A_w=(A_w)_{P_0}\subset\cdots\subset(A_w)_{P_{k_1}}$ of $K(n)$ (resp., L) are all with center q on A. For each $i\in\{0,\dots,k_1\}$, let w_i be the valuation on $K(n)$ (resp., L) associated to $(A_w)_{P_i}$. Then $w_{k_1}<\cdots<w_0=w$ are valuations on $A(n)$ (resp., L), all with center q on A.

(b) The proof follows immediately from (a) and the proof of Proposition 2.7. □

Corollary 2.9. (a) *A valuation w is a maximal (resp., minimal) element in the set of valuations on $A[n]$ (resp., L) extending v if and only if $\operatorname{tr\,deg}_{k_v}^{k_w}=0$ (resp., $\operatorname{ht}(m(w)/m(v)A_w)=0$).*

(b) *A valuation w is a maximal (resp., minimal) element in the set of valuations on $A[n]$ (resp., L) with center q on A if and only if $\operatorname{tr\,deg}_{A/q}^{k_w}=0$ (resp., $\operatorname{ht}(m(w)/qA_w)=0$).*

Corollary 2.10. *Let w be a valuation on $A[n]$ (resp., L).*

(a) *If w has center q on A, then the maximum length of a chain of valuations on $A[n]$ (resp., L) having center q on A and passing through w, is equal to*

$$\operatorname{ht}(m(w)/qA_w)+\operatorname{tr\,deg}_{A/q}^{k_w}. \tag{2.8}$$

(b) *If w extends v, then the maximum length of a chain of valuations on $A[n]$ (resp., L) extending v is equal to*

$$\operatorname{ht}(m(w)/m(v)A_w) + \operatorname{tr}\deg_{k_v}^{k_w}. \tag{2.9}$$

Proposition 2.11. (a) *Let s be the maximal value of $\operatorname{ht}(m(w)/qA_w) + \operatorname{tr}\deg_{A/q}^{k_w}$, where w runs through all valuations on $A[n]$ (resp., $K(n)$) with center q on A. Let t be the maximal value of $\operatorname{tr}\deg_{A/q}^{k_{v'}}$, where v' runs through all valuations on K with center q on A. Then $s = n + t$.*

(b) *The value n is the maximal value of $ht(m(w)/m(v)A_w) + \operatorname{tr}\deg_{k_v}^{k_w}$, where w runs through all valuations on $A[n]$ (resp., $K(n)$) extending v.*

Proof. (a) Let w be a valuation on $A[n]$ (resp., $K(n)$) with center q on A and let $(k_1, k_2) \in \mathbb{N}^2$ be such that $k_1 \le \operatorname{tr}\deg_{A/q}^{k_w}$ and $k_2 \le \operatorname{ht}(m(w)/qA_w)$. According to Theorems 2.4 and 2.6, there exists a chain of valuations $w = w_0 < \cdots < w_{k_1}$ on $A[n]$ (resp., $K(n)$) with center q on A. By Corollary 2.8, there exists a chain of valuations $w_{k_2} < \cdots < w_0 = w$ on $A[n]$ (resp., $K(n)$) with center q on A. Thus, we have a chain of valuations $w_{k_2} < \cdots < w_0 = w < \cdots < w_{k_1}$ on $A[n]$ (resp., $K(n)$) with center q on A, and

$$k_1 + k_2 \le \operatorname{tr}\deg_{A/q}^{k_{w_{k_2}}} = \operatorname{tr}\deg_{k_{w_{k_2}|K}}^{k_{w_{k_2}}} + \operatorname{tr}\deg_{A/q}^{k_{w_{k_2}|K}} \le \operatorname{tr}\deg_K^{K(n)} + t \le n + t, \tag{2.10}$$

where $w_{k_2|K}$ is the restriction of w_{k_2} to K. Consequently, $s \le n + t$.

Conversely, take $k \le t$. Then there exists a valuation v' on K with center q on A such that $k \le \operatorname{tr}\deg_{A/q}^{k_{v'}}$. According to Proposition 2.1, there exists a valuation w_0 on $A[n]$ extending v' with $\operatorname{tr}\deg_{k_{v'}}^{k_{w_0}} = n$, and

$$n + k \le \operatorname{tr}\deg_{k_{v'}}^{k_{w_0}} + \operatorname{tr}\deg_{A/q}^{k_{v'}} = \operatorname{tr}\deg_{A/q}^{k_{w_0}} \le s. \tag{2.11}$$

(b) The proof follows immediately from (a) and the preceding results. □

Corollary 2.12. *If the transcendence degree of L on K is infinite, then there is no upper bound on $\operatorname{tr}\deg_{A/q}^{k_w} + ht(m(w)/qA_w)$, with w running through all valuations on L with center q on A.*

Proof. The proof is immediate from the preceding proposition. □

3. The symbol $\delta((0), Q)$ in $A[n]$. Throughout this section, A will be an integral domain, $(0) \ne q$ a prime ideal of A, K the quotient field of A, and n will be a nonnegative integer.

Lemma 3.1. *Let Q be a superior of q in $A[X]$ such that there exists $a \in A$ with $X - a \in Q$. Then, for each valuation v of K with center q on A, there exists* *a valuation w of $K(X)$ with center Q on $A[X]$, extending v, and such that*

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[X]/Q)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + 1. \tag{3.1}$$

Proof. Let δ be a strictly positive element of the value group G_v of v. We define the valuation w on $K(X)$ as follows: if

$$f(X) = b_0 + \cdots + b_n(X-a)^n, \tag{3.2}$$

then

$$w(f(X)) = \inf\{v(a_i) + i\delta \mid i \in \{0,\dots,n\}\}. \tag{3.3}$$

It is well known (see [1]) that $\operatorname{tr}\deg^{k_w}_{k_v} = 1$. We show that w is a valuation on $A[X]$ with center Q on $A[X]$. For $f(X) = b_0 + \cdots + b_n(X-a)^n \in A[X]$, we have

$$w(f(X)) = \inf\{v(b_i) + i\delta \mid i \in \{0,\dots,n\}\} \geq 0. \tag{3.4}$$

If $f(X) \in m(w) \cap A[X]$, then $b_0 \in m(v) \cap A = q$ and $f(X) \in Q$.

Conversely, let $g(X) = a_0 + \cdots + a_m(X-a)^m \in Q \subset A[X]$. For each $i \in \{1,\dots,m\}$, $v(a_i) + i\delta > 0$ and $a_0 \in m(v) \cap A = q$, hence

$$w(g(X)) = \inf\{v(a_i) + i\delta \mid i \in \{0,\dots,m\}\} > 0, \tag{3.5}$$

that is, $g(X) \in m(w) \cap A[X]$. Thus, we have

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[X]/Q)} = \operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A/q)} = \operatorname{tr}\deg^{k_w}_{k_v} + \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + 1. \tag{3.6}$$

□

Lemma 3.2. *Let Q be a superior of q in $A[X]$ and let v be a valuation on K with center q on A. Then there exists a valuation w on $K(X)$ with center Q on $A[X]$ extending v such that* $\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[X]/Q)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + 1$.

Proof. (a) Assume that A is integrally closed in the algebraic closure K' of K. We have two different cases:

(1) q is a maximal ideal of A, $Q' = Q/q[X]$ is generated by $g(X) = X^n + \overline{a}_{n-1}X^{n-1} + \cdots + \overline{a}_0 \in (A/q)[X]$. Let $a_i \in A$ be a representant of $\overline{a}_i \in (A/q)$. Then

$$f(X) = a_0 + \cdots + a_{n-1}(X-a)^{n-1} + X^n = \prod_{i=1}^{m}(X-r_i)^{\alpha_i} \in K[X]. \tag{3.7}$$

Since r_i is integral over A, so $r_i \in A$, and

$$g(X) = \prod_{i=1}^{m}(X-\overline{r_i})^{\alpha_i} \in Q'. \tag{3.8}$$

Then there exists $j \in \{1,\dots,m\}$ such that $X - r_j \in Q$. We conclude by Lemma 3.2.

(2) Now, let q be any prime ideal in A. Let $S = A - q$; we have $(S^{-1}Q)$ which is a superior to qA_q in $A_q[X]$ and A_q is integrally closed in K'. The valuation v has center qA_q on A_q, so there exists a valuation w of $K[X]$ with center $S^{-1}Q$ on $A_q[X]$ extending v, with

$$\begin{aligned} \operatorname{tr}\deg^{k_w}_{\mathrm{Fr}(A_q[X]/S^{-1}Q)} &= \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)} + 1, \\ \operatorname{tr}\deg^{k_w}_{\mathrm{Fr}(A[X]/Q)} &= \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)} + 1. \end{aligned} \tag{3.9}$$

(b) Let A' be the integral closure of A in the algebraic closure K' of K. Let v' be a valuation on K' extending v. The integral closure of A in K' is the intersection of all the valuation rings on K' that contain A, as v is a valuation on A, so v' is a valuation on A'. Let q' be the center of v' on A', $q' \cap A = q$, and $q'[X] \cap A[X] = q[X]$. The closure $A'[X]$ is integral over $A[X]$, so there exists a prime ideal Q' of $A'[X]$ such that Q' is a superior of q', and Q' lies over Q. According to (a), there exists a valuation w' of $K'(X)$ with center Q' on $A'[X]$ extending v' with $\operatorname{tr}\deg^{k_{w'}}_{\mathrm{Fr}(A'[X]/Q')} = \operatorname{tr}\deg^{k_{v'}}_{\mathrm{Fr}(A'/q')} + 1$. Let w be the restriction of w' to $K(X)$; w is a valuation on $A[X]$,

$$\begin{aligned} m(w) \cap A[X] &= m(w') \cap K(X) \cap A[X] \\ &= m(w') \cap A'[X] \cap A[X] = Q' \cap A[X] = Q, \end{aligned} \tag{3.10}$$

and w prolongs v. Also

$$\operatorname{tr}\deg^{k_{v'}}_{\mathrm{Fr}(A'/q')} + \operatorname{tr}\deg^{\mathrm{Fr}(A'/q')}_{\mathrm{Fr}(A/q)} = \operatorname{tr}\deg^{k_{v'}}_{\mathrm{Fr}(A/q)} = \operatorname{tr}\deg^{k_{v'}}_{k_v} + \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)}. \tag{3.11}$$

It follows from Proposition 1.3 that $\operatorname{tr}\deg^{k_{v'}}_{k_v} \le \operatorname{tr}\deg^{K'}_{K} = 0$, hence

$$\begin{aligned} \operatorname{tr}\deg^{k_{v'}}_{\mathrm{Fr}(A'/q')} &= \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)}, \\ \operatorname{tr}\deg^{k_{w'}}_{\mathrm{Fr}(A'[X]/Q')} &= \operatorname{tr}\deg^{k_w}_{\mathrm{Fr}(A[X]/Q)}, \\ \operatorname{tr}\deg^{k_w}_{\mathrm{Fr}(A[X]/Q)} &= \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)} + 1. \end{aligned} \tag{3.12}$$

□

Remark 3.3. Let Q be a prime ideal of $A[X]$ lying over q. Then, for each valuation v of K with center q on A, there exists a valuation w of $K(X)$ extending v and with center Q on $A[X]$ such that

$$\operatorname{tr}\deg^{k_w}_{\mathrm{Fr}(A[X]/Q)} = \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)} + \operatorname{ht}(Q/q[X]). \tag{3.13}$$

Indeed, Lemma 3.2 implies the case where Q is a superior of q. If $Q = q[X]$, let w be the canonical extension of v to $K(X)$. It is well known (see Section 1) that $\operatorname{tr}\deg^{k_w}_{\mathrm{Fr}((A/q)[X])} = \operatorname{tr}\deg^{k_v}_{\mathrm{Fr}(A/q)}$.

Theorem 3.4. *Let Q be a prime ideal of $A[n]$ lying over q. Then, for each valuation v of K with center q on A, there exists a valuation w of $K(n)$ extending v and with center Q on $A[n]$ such that*

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[n]/Q)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + \operatorname{ht}(Q/q[n]). \tag{3.14}$$

Proof. One proceeds by induction on n. The case $n = 1$ follows from Remark 3.3. Assume that the statement is true for $n-1$. Let $Q_1 = Q \cap A[X_1]$. Then there exists a valuation w_1 of $K(X_1)$ extending v, with center Q_1 on $A[X_1]$, and

$$\operatorname{tr}\deg^{k_{w_1}}_{\operatorname{Fr}(A[X_1]/Q_1)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + \operatorname{ht}(Q_1/q[X_1]), \tag{3.15}$$

and there exists a valuation w of $K(X_1)(X_2,\dots,X_n) = K(n)$ extending w_1, with center Q on $A[n]$, and

$$\begin{aligned}\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[n]/Q)} &= \operatorname{tr}\deg^{k_{w_1}}_{\operatorname{Fr}(A[X_1]/Q_1)} + \operatorname{ht}(Q/Q_1[X_2,\dots,X_n]) \\ &= \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + \operatorname{ht}(Q_1/q[X_1]) + \operatorname{ht}(Q/Q_1[X_2,\dots,X_n]).\end{aligned} \tag{3.16}$$

We conclude by remarking that

$$\operatorname{ht}(Q_1/q[X_1]) + \operatorname{ht}(Q/Q_1[X_2,\dots,X_n]) = \operatorname{ht}(Q/q[X_1,\dots,X_n]). \tag{3.17}$$

□

Notation 3.5. Take $q_1 \subset q_2$ in $\operatorname{Spec}(A)$. We will denote by $\delta(q_1,q_2)$ the maximal value d for which there exists a valuation v on $\operatorname{Fr}(A/q_1)$ with center q_2/q_1 on A/q_1 such that $\operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q_2)} = d$.

Jaffard has shown in [2] that $\delta((0),q_2)$ is the greatest number n such that there exists a chain of valuations $v_0 < \cdots < v_n$ on A with center q_2 on A.

Corollary 3.6. *Let Q be a prime ideal of $A[n]$ lying over q. Then*

$$\delta((0),Q) = \delta((0),q) + \operatorname{ht}(Q/q[n]). \tag{3.18}$$

Proof. In the case where $Q = q[n]$, the result is well known (see [2]).

Suppose that $Q \neq q[n]$. For each valuation v of K with center q on A, there exists a valuation w of $K(n)$ with center Q on $A[n]$, extending v, with

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[n]/Q)} = \operatorname{tr}\deg^{k_v}_{\operatorname{Fr}(A/q)} + \operatorname{ht}(Q/q[n]) \le \delta((0),Q), \tag{3.19}$$

and consequently

$$\delta((0),Q) \ge \delta((0),q) + \operatorname{ht}(Q/q[n]). \tag{3.20}$$

Conversely, let w' be a valuation on $K(n)$ with center Q on $A[n]$ and let v' be its restriction to K. The valuation v' has center q on A and

$$\operatorname{tr}\deg^{k_{w'}}_{\operatorname{Fr}(A[n]/Q)} + \operatorname{tr}\deg^{\operatorname{Fr}(A[n]/Q)}_{\operatorname{Fr}(A/q)} = \operatorname{tr}\deg^{k_{w'}}_{k_{v'}} + \operatorname{tr}\deg^{k_{v'}}_{\operatorname{Fr}(A/q)} \le n + \delta((0),q), \tag{3.21}$$

and therefore

$$\begin{aligned} \operatorname{tr}\deg^{k_{w'}}_{\operatorname{Fr}(A[n]/Q)} &\le \delta((0),q) + ht(Q/q[n]), \\ \delta((0),Q) &\le \delta((0),q) + \operatorname{ht}(Q/q[n]). \end{aligned} \tag{3.22}$$

□

Proposition 3.7. *Let q_1 be a prime ideal of A and $Q_1 \subset Q_2$ two prime ideals of $A[n]$ lying over q_1. Then*

$$\delta(Q_1,Q_2) = \operatorname{ht}(Q_2/q_1[n]) - \operatorname{ht}(Q_1/q_1[n]) - 1. \tag{3.23}$$

Proof. Let $T = A - q_1$. Then $T^{-1}(A[n]/Q_1)$ is an $\operatorname{Fr}(A/q_1)$-algebra of finite type, and therefore a Noetherian domain according to [3]. Then

$$\begin{aligned} \delta(Q_1,Q_2) &= \delta((0),Q_2/Q_1) = \delta((0),T^{-1}(Q_2/Q_1)) \\ &= \operatorname{ht}(T^{-1}(Q_2/Q_1)) - 1 = \operatorname{ht}(Q_2/Q_1) - 1 \\ &= \operatorname{ht}(Q_2/q_1[n]) - ht(Q_1/q_1[n]) - 1. \end{aligned} \tag{3.24}$$

□

We finish this section studying the case of trivial valuations and we assume that $q = (0)$.

Proposition 3.8. *Let Q be a prime ideal of $A[n]$ lying over (0). Then there exists a valuation w of $K(n)$ with center Q on $A[n]$ such that*

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[n]/Q)} = \begin{cases} \operatorname{ht}(Q) - 1 & \textit{if } Q \ne 0, \\ 0 & \textit{if } Q = 0. \end{cases} \tag{3.25}$$

Proof. If $Q = (0)$, then it suffices to take for w the trivial valuation on $K(n)$. We suppose that $Q \ne (0)$. If $n = 1$, then for each $w \in A(Q)$, according to the preceding result, $\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[X]/Q)} \le \delta((0),Q) = 0$, and therefore

$$\operatorname{tr}\deg^{k_w}_{\operatorname{Fr}(A[X]/Q)} = \operatorname{ht}(Q) - 1 = 0. \tag{3.26}$$

Take $n > 1$, assume that the property holds for $n-1$, and let $Q_1 = Q \cap A[X_1]$. Then there exists a valuation w_1 of $K(X_1)$ with center Q_1 on $A[X_1]$ and $\operatorname{tr}\deg^{k_{w_1}}_{\operatorname{Fr}(A[X_1]/Q_1)} = 0$. If w_1 is the trivial valuation, then there exists a valuation w of $K(X_1)(X_2,\dots,X_n) = K(n)$ with center Q on $A[X_1][X_2,\dots,X_n] = A[n]$ and

$\operatorname{tr deg}^{k_w}_{\operatorname{Fr}(A[n]/Q)} = \operatorname{ht}(Q) - 1$. If w_1 is not trivial, then $Q_1 \neq (0)$ and it follows from Theorem 3.4 that there exists a valuation w on $K(n)$ extending w_1 and with center Q on $A[n]$, and

$$\begin{aligned}\operatorname{tr deg}^{k_w}_{\operatorname{Fr}(A[n]/Q)} &= \operatorname{tr deg}^{k_{w_1}}_{\operatorname{Fr}(A[X_1]/Q_1)} + \operatorname{ht}(Q/Q_1[X_2,\dots,X_n]) \\ &= \operatorname{ht}(Q) - \operatorname{ht}(Q_1) \\ &= \operatorname{ht}(Q) - 1.\end{aligned} \tag{3.27}$$

□

4. Valuations on $A[n]$ centered on the same ideal and extending the same valuation. Let v be a valuation on K with center q on A and Q a prime ideal of $A[n]$ lying over q. We will use the following notation:

(a) $A(Q) = \{w \mid w$ is a valuation on $A[n]$ with center $Q\}$;
(b) $A(v) = \{w \mid w$ is a valuation on $A[n]$ extending $v\}$;
(c) $A(v,Q) = \{w \mid w$ is a valuation on $A[n]$ with center Q extending $v\}$.

Lemma 4.1. *Let w be a valuation on $K(n)$. Then w is maximal in $A(v,Q)$ if and only if w is maximal in $\{w'' \mid w''$ is a valuation on $A_v[n]$ with center $Q_1 = m(w) \cap A_v[n]\}$ and Q_1 is maximal in $\{Q' \mid Q' \in \operatorname{Spec}(A_v[n])$ with $Q' \cap A[n] = Q\}$.*

Proof. Suppose that w is maximal in $A(v,Q)$. Assume that there exists $Q_2 \in \operatorname{Spec}(A_v[n])$ such that $Q_1 \subset Q_2$ and $Q_2 \cap A[n] = Q$. By Proposition 1.4, there exists a valuation w_2 on $A_v[n]$ with center Q_2, and $w < w_2$ such that $m(w_2) \cap A[n] = Q$ and $A_v \subseteq A_{w_2} \cap K \subseteq A_w \cap K = A_v$. Thus, w_2 extends v with center Q on $A[n]$, which is a contradiction.

Assume that $w < w'$, with w' a valuation on $A_v[n]$ with center Q_1. Then $m(w') \cap A[n] = Q$ and w' extends v, that is, w' is a valuation on $A[n]$ with center Q and extending v, which is impossible.

Conversely, if $w < w'$ with w' a valuation on $A[n]$ extending v and with center Q, then $m(w) \cap A_v[n] = Q_1 \subset m(w') \cap A_v[n]$, but $(m(w') \cap A_v[n]) \cap A[n] = m(w') \cap A[n] = Q$, which is again a contradiction. □

Remark 4.2. Take $w \in A(v,Q)$. Let $Q_1 = m(w) \cap A_v[n]$ and assume that Q_1 is maximal in $\{Q' \mid Q' \in \operatorname{Spec}(A_v[n]),\ Q' \cap A[n] = Q\}$. The ideal $Q_1/m(v)[n]$ is maximal in $\{Q' \mid Q' \in \operatorname{Spec}(k_v[n]),\ Q' \cap (A/q)[n] = Q/q[n]\}$. According to [5], we have

$$\operatorname{ht}(Q_1/m(v)[n]) - \operatorname{ht}(Q/q[n]) = \inf\left(\operatorname{tr deg}^{k_v}_{A/q}, \operatorname{tr deg}^{A[n]/Q}_{A/q}\right). \tag{4.1}$$

Theorem 4.3. *For $w \in A(v,Q)$, the following assertions are equivalent:*

(a) *w is maximal in $A(v,Q)$;*
(b) $\inf(\operatorname{tr deg}^{k_w}_{k_v}, \operatorname{tr deg}^{k_w}_{A[n]/Q}) = 0$.

Proof. First, suppose that w is maximal in $A(v,Q)$.

Case 1. The transcendence degree of $k(v)$ on $k(q)$ is finite. Let $Q_1 = m(w) \cap A_v[n]$. Then

$$\begin{aligned} &\operatorname{tr}\deg^{k_w}_{A[n]/Q} + n - \operatorname{ht}(Q/q[n]) \\ &\quad = \operatorname{tr}\deg^{k_w}_{A/q} = \operatorname{tr}\deg^{k_w}_{A_v[n]/Q_1} + \operatorname{tr}\deg^{A_v[n]/Q_1}_{A[n]/Q} + n - \operatorname{ht}(Q/q[n]). \end{aligned} \tag{4.2}$$

According to Lemma 4.1, $\operatorname{tr}\deg^{k_w}_{A_v[n]/Q_1} = 0$, and

$$\operatorname{tr}\deg^{A_v[n]/Q_1}_{A[n]/Q} + n - \operatorname{ht}(Q/q[n]) = n - \operatorname{ht}(Q_1/m(v)[n]) + \operatorname{tr}\deg^{k_v}_{A/q}, \tag{4.3}$$

and therefore

$$\begin{aligned} \operatorname{tr}\deg^{A_v[n]/Q_1}_{A[n]/Q} &= \operatorname{ht}(Q/q[n]) - \operatorname{ht}(Q_1/m(v)[n]) + \operatorname{tr}\deg^{k_v}_{A/q}, \\ \operatorname{tr}\deg^{k_w}_{A[n]/Q} &= \operatorname{tr}\deg^{A_v[n]/Q_1}_{A[n]/Q} \\ &= \operatorname{ht}(Q/q[n]) - \operatorname{ht}(Q_1/m(v)[n]) + \operatorname{tr}\deg^{k_v}_{A/q}. \end{aligned} \tag{4.4}$$

Remark 4.2 implies that

$$\operatorname{tr}\deg^{k_w}_{A[n]/Q} = \operatorname{tr}\deg^{k_v}_{A/q} - \inf\left(\operatorname{tr}\deg^{k_v}_{A/q}, \operatorname{tr}\deg^{A[n]/Q}_{A/q}\right). \tag{4.5}$$

If $\operatorname{tr}\deg^{k_w}_{A[n]/Q} \neq 0$, then $\operatorname{tr}\deg^{k_w}_{A[n]/Q} = \operatorname{tr}\deg^{k_v}_{A/q} + \operatorname{ht}(Q/q[n]) - n$, and therefore

$$\operatorname{tr}\deg^{k_w}_{A/q} = \operatorname{tr}\deg^{k_w}_{k_v} + \operatorname{tr}\deg^{k_v}_{A/q} = \operatorname{tr}\deg^{k_v}_{A/q}, \tag{4.6}$$

that is, $\operatorname{tr}\deg^{k_w}_{k_v} = 0$.

Case 2. The transcendence degree of $k(v)$ on $k(q)$ is infinite. We will show that $\operatorname{tr}\deg^{k_w}_{k_v} = 0$. According to Lemma 4.1 and Remark 4.2,

$$\operatorname{ht}(Q_1/m(v)[n]) - \operatorname{ht}(Q/q[n]) = \operatorname{tr}\deg^{A[n]/Q}_{A/q} = n - \operatorname{ht}(Q/q[n]), \tag{4.7}$$

and therefore $\operatorname{ht}(Q_1/m(v)[n]) = n$. Thus,

$$\operatorname{tr}\deg^{k_w}_{A_v[n]/Q_1} + \operatorname{tr}\deg^{A_v[n]/Q_1}_{A_v/m(v)} = \operatorname{tr}\deg^{k_w}_{k_v}, \tag{4.8}$$

so

$$\operatorname{tr}\deg^{k_w}_{k_v} = \operatorname{tr}\deg^{A_v[n]/Q_1}_{A_v/m(v)} = n - ht(Q_1/m(v)[n]) = 0. \tag{4.9}$$

Conversely, assume that $\inf(\operatorname{tr}\deg^{k_w}_{k_v}, \operatorname{tr}\deg^{k_w}_{A[n]/Q}) = 0$. It follows from Corollary 2.9 that w is maximal in $A(v,Q)$. □

Proposition 4.4. *If $w_0 \in A(v,Q)$ is not maximal in $A(v,Q)$, then there exists w_1 in $A(v,Q)$ such that $w_0 < w_1$ and $\operatorname{tr}\deg^{k_{w_1}}_{A[n]/Q} = \operatorname{tr}\deg^{k_{w_0}}_{A[n]/Q} - 1$.*

Proof. Let $Q_0 = m(w_0) \cap A_v[n]$. According to Lemma 4.1, $\operatorname{tr deg}_{A_v[n]/Q_0}^{k_{w_0}} \neq 0$ or Q_0 is not maximal in $\{Q' \mid Q' \in \operatorname{Spec}(A_v[n]) \text{ with } Q' \cap A[n] = Q\}$.

If $\operatorname{tr deg}_{A_v[n]/Q_0}^{k_{w_0}} \neq 0$, then it follows from Lemma 2.3 that there exists a valuation w_1 on $A_v[n]$ with center Q_0 such that $w_0 < w_1$ and $\operatorname{tr deg}_{A_v[n]/Q_0}^{k_{w_1}} = \operatorname{tr deg}_{A_v[n]/Q_0}^{k_{w_0}} - 1$, that is, $\operatorname{tr deg}_{A[n]/Q}^{k_{w_1}} = \operatorname{tr deg}_{A[n]/Q}^{k_{w_0}} - 1$. The valuation w_1 extends v since $A_v \subseteq A_{w_1} \cap K \subseteq A_{w_0} \cap K = A_v$.

If $\operatorname{tr deg}_{A_v[n]/Q_0}^{k_{w_0}} = 0$, let Q_1 be a prime ideal of $A_v[n]$ lying over Q with $Q_0 \subset Q_1$ and $\operatorname{ht}(Q_1/Q_0) = 1$. According to Proposition 1.4 and Theorem 2.4, there exists a valuation w_1 on $A_v[n]$ with center Q_1 such that $w_0 < w_1$ and $\operatorname{tr deg}_{A_v[n]/Q_1}^{k_{w_1}} = 0$; w_1 is a valuation on $A[n]$ with center Q. Then

$$\begin{aligned} \operatorname{tr deg}_{A[n]/Q}^{k_{w_0}} &= \operatorname{tr deg}_{A[n]/Q}^{A_v[n]/Q_0} = \operatorname{ht}(Q/q[n]) - \operatorname{ht}(Q_0/m(v)[n]) + \operatorname{tr deg}_{A/q}^{k_v}, \\ \operatorname{tr deg}_{A[n]/Q}^{k_{w_1}} &= \operatorname{ht}(Q/q[n]) - \operatorname{ht}(Q_1/m(v)[n]) + \operatorname{tr deg}_{A/q}^{k_v}, \end{aligned} \tag{4.10}$$

and therefore

$$\begin{aligned} \operatorname{tr deg}_{A[n]/Q}^{k_{w_0}} - \operatorname{tr deg}_{A[n]/Q}^{k_{w_1}} &= \operatorname{ht}(Q_1/m(v)[n]) - \operatorname{ht}(Q_0/m(v)[n]) \\ &= \operatorname{ht}(Q_1/Q_0) = 1. \end{aligned} \tag{4.11}$$

□

Theorem 4.5. *For $w \in A(v,Q)$, let $d(w,v,Q) = \operatorname{Sup}\{k \mid k \in \mathbb{N}, \exists k+1$ valuations $w = w_0 < \cdots < w_k$ in $A(v,Q)\}$. Then*

$$d(w,v,Q) = \inf\left(\operatorname{tr deg}_{k_v}^{k_w}, \operatorname{tr deg}_{A[n]/Q}^{k_w}\right). \tag{4.12}$$

Proof. Let $k = d(w,v,Q)$. According to Proposition 4.4, there exist $w_0 < \cdots < w_k$ in $A(v,Q)$, with $\operatorname{tr deg}_{A[n]/Q}^{k_{w_i}} = \operatorname{tr deg}_{A[n]/Q}^{k_{w_0}} - i$ for each $i \in \{0,\dots,k\}$ and w_k maximal in $A(v,Q)$, that is,

$$\operatorname{tr deg}_{A[n]/Q}^{k_{w_k}} = \operatorname{tr deg}_{A/q}^{k_v} - \inf\left(\operatorname{tr deg}_{A/q}^{k_v}, \operatorname{tr deg}_{A/q}^{A[n]/Q}\right) = \operatorname{tr deg}_{A[n]/Q}^{k_w} - k, \tag{4.13}$$

and therefore

$$\begin{aligned} k &= \operatorname{tr deg}_{A[n]/Q}^{k_w} - \operatorname{tr deg}_{A/q}^{k_v} + \inf\left(\operatorname{tr deg}_{A/q}^{k_v}, \operatorname{tr deg}_{A/q}^{A[n]/Q}\right) \\ &= \inf\left(\operatorname{tr deg}_{k_v}^{k_w}, \operatorname{tr deg}_{A[n]/Q}^{k_w}\right). \end{aligned} \tag{4.14}$$

□

Proposition 4.6. *For $w \in A(v,Q)$,*

$$\begin{aligned} s &= \sup\{k \mid \exists k+1 \text{ valuations } w_k < \cdots < w_0 = w \text{ on } A(v,Q)\} \\ &= h = \inf(\operatorname{ht}(m(w)/QA_w)), \operatorname{ht}(m(w)/m(v)A_w). \end{aligned} \tag{4.15}$$

Proof. Let $k \in \mathbb{N}$ and let $w_k < \cdots < w_0 = w$ be valuations on $A(v,Q)$. As $A(v,Q) \subseteq A(v) \cap A(Q)$, $k \le \inf(\mathrm{ht}(m(w)/QA_w), \mathrm{ht}(m(w)/m(v)A_w))$, and consequently $s \le h$.

The converse inequality is trivial if $s = \infty$. So let $s < \infty$ and assume $s < h$. Take $w_k < \cdots < w_0 = w$ in $A(v,Q)$. Then

$$\begin{aligned} \mathrm{ht}\,(m(w)/QA_w) &= \mathrm{ht}\,(m(w)/m(w_k)) + \mathrm{ht}\,(m(w_k)/QA_{w_k}) \\ &= k + \mathrm{ht}\,(m(w_k)/QA_{w_k}), \end{aligned} \tag{4.16}$$

and therefore $\mathrm{ht}(m(w_k)/QA_{w_k}) \neq 0$, and there exists $w'_{k+1} \in A(Q)$ with $w'_{k+1} < w_k$. As

$$\begin{aligned} \mathrm{ht}\,(m(w)/m(v)A_w) &= \mathrm{ht}\,(m(w)/m(w_k)) + \mathrm{ht}\,(m(w_k)/m(v)A_{w_k}) \\ &= k + \mathrm{ht}\,(m(w_k)/m(v)A_{w_k}), \end{aligned} \tag{4.17}$$

we have that $\mathrm{ht}(m(w_k)/m(v)A_{w_k}) \neq 0$, and therefore there exists $w''_{k+1} \in A(v)$ with $w''_{k+1} < w_k$. Thus, $A_{w_k} \subset A_{w'_{k+1}}$, $A_{w_k} \subset A_{w''_{k+1}}$, $A_{w'_{k+1}} = (A_{w_k})_{P'}$, and $A_{w''_{k+1}} = (A_{w_k})_{P''}$ with P' and P'' two prime ideals of A_{w_k}. As P' and P'' are comparable, $A_{w'_{k+1}} \subseteq A_{w''_{k+1}}$ or $A_{w''_{k+1}} \subseteq A_{w'_{k+1}}$. If $w''_{k+1} \le w'_{k+1}$, $m(w''_{k+1}) \subseteq m(w'_{k+1})$ and $m(v) = m(w''_{k+1}) \cap A_v \subseteq m(w'_{k+1}) \cap A_v$, that is, $m(w'_{k+1}) \cap A_v = m(v)$ and w'_{k+1} extends v, it will be a contradiction since $w'_{k+1} \in A(v,Q)$.

If $w'_{k+1} \le w''_{k+1}$, then $m(w'_{k+1}) \subseteq m(w''_{k+1}) \subset m(w)$, and then $m(w''_{k+1}) \cap A[n] = Q$ and $w''_{k+1} \in A(v,Q)$, which is again a contradiction.

We conclude that $s = h$ and finish the proof. □

Corollary 4.7. *For $w \in A(v,Q)$, the following assertions are equivalent:*

(a) *w is minimal in $A(v,Q)$;*

(b) $\inf(\mathrm{ht}(m(w)/QA_w), \mathrm{ht}(m(w)/m(v)A_w)) = 0$*;*

(c) *w is minimal in $A(Q)$ or in $A(v)$.*

Notation 4.8. For each valuation w on $A(v,Q)$, let $l(w)$ be the maximal length of a chain of valuations on $A(v,Q)$ passing through w. The maximum value of $l(w)$, where w runs through the set $A(v,Q)$, will be denoted by $d(A(v,Q))$.

Theorem 4.9. *For each valuation w on $A(v,Q)$,*

(a) $l(w) = \inf(\mathrm{ht}(m(w)/QA_w), \mathrm{ht}(m(w)/m(v)A_w)) + \inf(\mathrm{tr\,deg}_{k_v}^{k_w}, \mathrm{tr\,deg}_{A[n]/Q}^{k_w})$*;*

(b) $d(A(v,Q)) = \mathrm{tr\,deg}_{k(q)}^{k_v} + \mathrm{ht}(Q/q[n])$.

Proof. (a) The proof follows immediately from Theorem 4.5 and Proposition 4.6.

(b) Let $w' \in A(v,Q)$. Then

$$\mathrm{tr\,deg}_{A[n]/Q}^{k_{w'}} + n - \mathrm{ht}\,(Q/q[n]) = \mathrm{tr\,deg}_{k(q)}^{k_{w'}} = \mathrm{tr\,deg}_{k(v)}^{k_{w'}} + \mathrm{tr\,deg}_{k(q)}^{k_v}, \tag{4.18}$$

thus

$$\begin{aligned}\operatorname{tr}\deg^{k_{w'}}_{A[n]/Q} &= \operatorname{tr}\deg^{k_v}_{k(q)} + \operatorname{ht}\left(Q/q[n]\right) + \left(\operatorname{tr}\deg^{k_{w'}}_{k(v)} - n\right) \\ &\leq \operatorname{tr}\deg^{k_v}_{k(q)} + \operatorname{ht}\left(Q/q[n]\right).\end{aligned} \tag{4.19}$$

Therefore, $d(A(v,Q)) \leq \operatorname{tr}\deg^{k_v}_{k(q)} + \operatorname{ht}(Q/q[n])$.

Theorem 3.4 implies the existence of $w \in A(v,Q)$ satisfying $\operatorname{tr}\deg^{k_w}_{A[n]/Q} = \operatorname{tr}\deg^{k_v}_{k(q)} + ht(Q/q[n])$, and the converse inequality follows. □

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