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ASYMPTOTICS OF INTEGRODIFFERENTIAL MODELS WITH INTEGRABLE KERNELS II

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Nonlinear singularly perturbed Volterra integrodifferential equations with weakly singular kernels are investigated using singular perturbation methods, the Mellin transform technique, and the theory of fractional integration.

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1. Introduction. The following singularly perturbed nonlinear Volterra integrodifferential equation is considered:

$$\varepsilon y'(t) = g(t) + \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{k(t,s)}{(t-s)^\beta} h(s, y(s))\, ds, \quad 0 < t \le T,\ y(0) = y_0. \tag{1.1}$$

Here, ε is a positive parameter satisfying $0 < \varepsilon \ll 1$ and $0 < \beta < 1$. The functions g and h and the Volterra kernel k are continuously differentiable. Moreover, $k(t,t)$ and $\partial_2 h(t, \psi(t))$ are assumed to be nonzero for all $0 \le t \le T$, for a continuously differentiable function ψ. These functions may as well depend regularly on ε but it is assumed here that they are independent. As for the case when $h(t,y) = y$ investigated in [6], (1.1) exhibits an initial layer or a layer of rapid transition at $t = 0$. It will be shown that the order of magnitude of the initial layer thickness is $O(\varepsilon^{1/(2-\beta)})$, $\varepsilon \to 0$. In this region, referred as the inner layer, the solution $y(t;\varepsilon)$ of (1.1) changes rapidly while in the rest of the domain, called the outer region, the solution varies slowly. To guarantee the existence of decaying solutions in the initial layer, appropriate layer stability assumption will be imposed.

It is known from the standard theory of Volterra integral equations that this problem has a continuous solution $y(t;\varepsilon)$, $0 \le t \le T$, for all $\varepsilon > 0$. However, if one is interested in the solution for small values of ε, perturbation methods have to be applied. The structure of the integrodifferential equation suggests a regular approximation of the form

$$y_{\text{app}}(t;\varepsilon) \approx \sum_{n=0}^{\infty} \varepsilon^n y_n(t), \quad \varepsilon \longrightarrow 0. \tag{1.2}$$

Assuming that the regular approximation has indeed the structure given in (1.2), the leading order term $y_0(t)$ satisfies

$$0 = g(t) + \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{k(t,s)}{(t-s)^\beta} h(s, y_0(s))\, ds, \quad 0 \le t \le T. \tag{1.3}$$

This is a Volterra integral equation of the first kind. For this equation to have a continuous solution, $g(t)$ cannot be merely continuous; the forcing function must be smoother than the desired solution. Even if (1.3) has a solution $y_0(t)$ in $C[0,T]$, it may not approximate $y(t;\varepsilon)$ uniformly for all $0 \le t \le T$ as $\varepsilon \to 0$, especially if $y_0(0) \ne \lim_{\varepsilon\to 0} y(0,\varepsilon)$. This situation confirms that problem (1.1) is singularly perturbed, and therefore, a second term in (1.2) is needed to correct the nonuniformity. Thus, one introduces a new scaled variable with a different magnitude in order to obtain a uniformly valid approximation. The idea is that if the initial layer region is described in terms of the new time scale, the layer region is stretched in such a way that its neighborhood gets magnified unproportionally to the rest of the domain under consideration, and therefore, no rapid variation in the solution should be exhibited. This new variable is, therefore, called the inner variable or the stretched variable. The two widely used techniques that employ the inner and outer expansions are the method of "matched asymptotic expansion" and the "additive decomposition" method. In the investigation presented here, the additive decomposition method is applied. Assuming that the initial layer stability condition holds, it is shown here that $y(t;\varepsilon)$ converges uniformly to $y_0(t)$ as $\varepsilon \to 0$, for all $0 \le t \le T$.

The additive decomposition method was first applied to study (1.1) in [1]. A variety of interesting examples have been solved in [1]. However, the analysis fails to reveal the general structure of the formal approximation as the case $\beta = 0$ is considered simultaneously with the case $0 < \beta < 1$. Also, the validity of the given formal approximation in [1] is not demonstrated. The linear singularly perturbed Volterra integral equations with weakly singular kernels have been studied in [4]. It is shown in [4] that the initial layer correction term can be written in terms of the Mittag-Leffler function. The Mittag-Leffler function decays algebraically at infinity. The results in [4] reveal that singularly perturbed Volterra integral equations with weakly singular kernels exhibit narrower initial layer regions compared with similar equations with continuous kernels and integrodifferential equations with weakly singular kernels. The linear version of (1.1) has been investigated in [6] using the theory of fractional integration. It is demonstrated in [6] that the linear scalar singularly perturbed Volterra integrodifferential equation has a wider initial layer width, of order $O(\varepsilon^{1/(2-\beta)})$, $\varepsilon \to 0$, and that the formal approximate solution is an asymptotic solution up to the order of magnitude of $O(\varepsilon^{1/(2-\beta)})$, $\varepsilon \to 0$. The rescaling of the initial layer width and magnitude (of the solution) in [2, 3, 4, 5, 6] reveals differences in the order of magnitudes of the initial layer thickness and magnitudes of the solution in the layer region for the cases $\beta = 0$ and $\beta \ne 0$. This, in fact, suggests

different starting asymptotic approximations. This paper aims at extending the results in [6] for the nonlinear problem. The main, new, and interesting result is the proof of asymptotic correctness.

The method of additive decomposition will be used to construct an approximate solution of (1.1). This will be proceeded by the construction of approximations valid in the inner and outer regions. The inner layer approximation is assumed to be negligible in the outer region, and therefore, a decay in the initial layer is crucial. In the application of the additive decomposition method, exponential decay in the initial layer simplifies the analysis as transcendentally small terms can be omitted from the asymptotic expansions. However, in the case considered here, the inner layer solution decays algebraically and this challenges the technique. Also, there are abnormalities when balancing terms of similar orders of ε. This shows the difficulties in analyzing the cases $\beta = 0$ and $0 < \beta < 1$ simultaneously. The abnormality occurs especially when one tries to derive the first- (and higher-) order terms in the formal approximate solution. Therefore, this paper will restrict attention to the leading order solution.

In Section 2, a review of some known results which are applied later in the analysis is presented. In Section 3, the application of the additive decomposition technique to integrodifferential equations of type (1.1) is described and the leading order formal solution is derived. In Section 4, assumptions imposed on the data are stated and the formal approximate solution is proved to have the required properties. It is also shown in Section 4 that if $y_0(t;\varepsilon)$ satisfies (1.1) approximately with a residual $\rho(t;\varepsilon)$, then $\rho(t;\varepsilon) = O(\varepsilon)$, $\varepsilon \to 0$. Finally, in Section 5, the theorem on asymptotic correctness is presented which says: under given conditions, if $y_0(t;\varepsilon)$ is a formal approximate solution of (1.1) and $y(t;\varepsilon)$ is the exact solution, then $|y(t;\varepsilon) - y_0(t;\varepsilon)| = O(\varepsilon^{1/(2-\beta)})$, $\varepsilon \to 0$, uniformly for all $0 \le t \le T$ and all sufficiently small values of ε.

2. Mathematical preliminaries. The following results will be applied in the presentation, and therefore, for convenience, are stated below.

The first result is that of applying the Mellin transformation to the asymptotic evaluation of integrals. One will find a detailed discussion in [7, 17]. The Parseval formula for Mellin transforms will be applied in Section 4, and therefore, is particularly stated.

The Mellin transform of a locally integrable function $y(t)$ on $(0,\infty)$ is defined by

$$M[y;z] = \int_0^\infty t^{z-1} y(t)\,dt \tag{2.1}$$

when the integral converges.

Theorem 2.1. *Suppose that $M[f;1-z]$ and $M[h;z]$ are defined and holomorphic, each in some vertical strip, whose boundary is determined by the analytical structure of the corresponding function as $t \to 0$ and as $t \to \infty$. Suppose further that these strips overlap. Then the integral*

$$I(t) = \int_0^\infty f(s)h(ts)ds \tag{2.2}$$

can be described in terms of the Mellin-Barnes integral:

$$I(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f(s);1-z]M[h(ts);z]dz, \tag{2.3}$$

where $\operatorname{Re}(z) = c$ *lies in the overlapping strip.*

The second result involves the application of fractional calculus to obtain solutions of integrodifferential equations with weakly singular kernels.

Theorem 2.2. *The Volterra integrodifferential equation*

$$\phi'(t) = \kappa_0 + \frac{\kappa_1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\phi(s)ds, \quad 0 < t,\ \phi(0) = \phi_0, \tag{2.4}$$

where $1 > \beta > 0$, κ_0 and κ_1 are constants, has the solution

$$\phi(t) = \phi_0 E_{2-\beta}(\kappa_1 t^{2-\beta}) + \kappa_0 \int_0^t E_{2-\beta}(\kappa_1 s^{2-\beta})ds. \tag{2.5}$$

Here, E_γ is the Mittag-Leffler function of order γ, defined by

$$E_\gamma(\lambda t^\gamma) = \sum_{n=0}^\infty \frac{\lambda^n t^{n\gamma}}{\Gamma(n\gamma+1)}, \quad t > 0,\ \gamma > 0,\ \lambda \in \mathbb{C}. \tag{2.6}$$

The third and last result is the nonlinear generalization of the Gronwall's inequality. This has been proved in [8].

Theorem 2.3. *Let $\phi, \psi : [0,\zeta) \to [0,\infty)$, $\varphi : [0,\zeta)\times[0,\infty) \to [0,\infty)$ be continuous such that*

$$0 \le \varphi(t,u) - \varphi(t,v) \le M(t,u)(u-v), \quad t \in [0,\zeta),\ 0 \le v \le u, \tag{2.7}$$

where M is nonnegative and continuous on $[0,\zeta)\times[0,\infty)$. Then for every nonnegative continuous solution of the inequality

$$y(t) \le \phi(t) + \psi(t)\int_0^t \varphi(s,y(s))ds, \quad t \in [0,\zeta), \tag{2.8}$$

the following estimate holds:

$$y(t) \le \phi(t) + \psi(t) \int_0^t \varphi(s,\phi(s)) \exp\left(\int_s^t M(\sigma,\phi(\sigma))\psi(\sigma)d\sigma\right)ds \tag{2.9}$$

for all $t \in [0,\zeta)$.

3. Heuristic analysis and formal solution. To start, one seeks an approximate solution $y_{\text{app}}(t;\varepsilon)$ in the form

$$y_{\text{app}}(t;\varepsilon) = u(t;\varepsilon) + v\left(\frac{t}{\varepsilon^\alpha};\varepsilon\right), \quad \tau = \frac{t}{\varepsilon^\alpha},\ \alpha > 0, \tag{3.1}$$

and requires that

$$\lim_{\tau\to\infty} v(\tau;\varepsilon) = 0. \tag{3.2}$$

The inner layer function $v(t/\varepsilon^\alpha;\varepsilon)$, which is the second term needed in (1.2), corrects the nonuniformity in the initial layer. It is assumed that $u(t;\varepsilon)$ and $v(\tau;\varepsilon)$ have asymptotic expansions of the form

$$u(t;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n(t), \qquad v(\tau;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^{n\alpha} v_n(\tau), \tag{3.3}$$

as $\varepsilon \to 0$, so that

$$y_{\text{app}}(t;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n(t) + \sum_{n=0}^{\infty} \varepsilon^{n\alpha} v_n\left(\frac{t}{\varepsilon^\alpha}\right), \quad \varepsilon \longrightarrow 0. \tag{3.4}$$

Moreover, one requires that for all $n \ge 0$,

$$\lim_{\tau\to\infty} v_n(\tau) = 0. \tag{3.5}$$

The substitution of (3.4) in (1.1), the expression of all terms in terms of the inner variable τ, and the examination of the dominant balance in the relation yield

$$\operatorname{Ord}(\varepsilon^{1-\alpha}) = \operatorname{Ord}(\varepsilon^{\alpha(1-\beta)}), \quad \varepsilon \longrightarrow 0. \tag{3.6}$$

Hence, one chooses

$$\alpha = \frac{1}{2-\beta}. \tag{3.7}$$

This implies that the singularly perturbed equation (1.1) possesses an initial layer width of order $(\varepsilon^{1/(2-\beta)})$, $\varepsilon \to 0$, meaning that the solution $y(t;\varepsilon)$ of (1.1) is slowly varying for $O(\varepsilon^{1/(2-\beta)}) \le t \le T$ as $\varepsilon \to 0$, but changes rapidly on a small interval $0 \le t \le O(\varepsilon^{1/(2-\beta)})$. Therefore, the initial layer region for problem (1.1) is thicker compared with similar equations with continuous kernels (see [2, 3, 5]) and integral equations with weakly singular kernels (see [4]).

3.1. Derivation of the formal approximate solution. From this point, the attention is restricted to

$$y_0(t;\varepsilon) = u_0(t) + v_0\left(\frac{t}{\varepsilon^\alpha}\right) \tag{3.8}$$

since $y_{\text{app}}(t;\varepsilon) \sim y_0(t;\varepsilon)$, $\varepsilon \to 0$. During the derivation of the formal solution, the following will be assumed that

$$v_0(\tau) \sim c_0\tau^{-\beta_0}, \quad \tau \longrightarrow \infty, \tag{3.9}$$

where c_0 and β_0 are constants, $\beta_0 > 1$.

Now, let

$$y(t;\varepsilon) := y_0(t;\varepsilon) = u_0(t) + v_0\left(\frac{t}{\varepsilon^\alpha}\right). \tag{3.10}$$

Suppose that $y_0(t;\varepsilon)$ satisfies (1.1) approximately, with a residual $\rho(t;\varepsilon)$, then

$$\begin{aligned}&\varepsilon u_0'(t) + \varepsilon^{1-\alpha}v_0'\left(\frac{t}{\varepsilon^\alpha}\right)\\ &\quad = g(t) + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^\beta}h\left(s, u_0(s) + v_0\left(\frac{s}{\varepsilon^\alpha}\right)\right)ds - \rho(t;\varepsilon),\end{aligned} \tag{3.11}$$

or equivalently,

$$\begin{aligned}\rho(t;\varepsilon) = {}& -\varepsilon u_0'(t) - \varepsilon^{1-\alpha}v_0'\left(\frac{t}{\varepsilon^\alpha}\right) + g(t) + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^\beta}h(s,u_0(s))ds\\ &+ \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^\beta}\left\{h\left(s, u_0(s) + v_0\left(\frac{s}{\varepsilon^\alpha}\right)\right) - h(s,u_0(s))\right\}ds.\end{aligned} \tag{3.12}$$

To obtain the leading order outer equation, one considers (3.12) in the limit, as $\varepsilon \to 0$ and $t > 0$ fixed. However, the last integral in (3.12) can be written as

$$I_\varepsilon(t) = \frac{t^{1-\beta}}{\Gamma(1-\beta)}\int_0^1 \frac{k(t,ts)}{(1-s)^\beta}\left\{h\left(ts, u_0(ts) + v_0\left(\frac{ts}{\varepsilon^\alpha}\right)\right) - h(ts,u_0(ts))\right\}ds. \tag{3.13}$$

For a fixed $t > 0$, one can rewrite $I_\varepsilon(t)$ as

$$I_\varepsilon(\theta;t) = \varepsilon^{\alpha(1-\beta)}\theta^{1-\beta}\int_0^\infty \phi(s)\psi(s\theta;t)ds, \tag{3.14}$$

where $\theta = t/\varepsilon^\alpha$,

$$\begin{aligned}\phi(s) &= \begin{cases}\dfrac{1}{\Gamma(1-\beta)}(1-s)^{-\beta}, & 0 \le s < 1,\\ 0, & 1 \le s < \infty,\end{cases}\\ \psi(s\theta;t) &= k(t,ts)\{h(ts, u_0(ts) + v_0(\theta s)) - h(ts,u_0(ts))\}.\end{aligned} \tag{3.15}$$

By suppressing the dependence of ψ on t, the following lemma establishes that when $t > 0$ is fixed, $I_\varepsilon(t) = O(\varepsilon^\alpha)$, $\varepsilon \to 0$.

Lemma 3.1. *Consider*

$$I_\beta(\theta) = \int_0^\infty \phi(s)\psi(s\theta)ds. \tag{3.16}$$

Now, suppose that $M[\phi;1-z]$ and $M[\psi;z]$ are defined and holomorphic, each in some vertical strip determined by the asymptotic behaviors of ϕ and ψ. Then it follows that, as $\theta \to \infty$,

$$I_\beta(\theta) = O(\theta^{-1}). \tag{3.17}$$

Proof. The form of the integral in (3.16) suggests the application of Mellin transform technique in the determination of an asymptotic expansion of $I_\beta(\theta)$, $\theta \to \infty$. This is a well-known technique which involves asymptotic behaviors of functions in the integrand. One should note that Taylor theorem implies that $v_0(\theta)$ and $\psi(\theta)$ are O-equivalent,

$$\psi(\tau) = \operatorname{Ord}\left(v_0(\tau)\right), \quad \tau \longrightarrow 0,\ \tau \longrightarrow \infty. \tag{3.18}$$

It is known from (3.8) that $v_0(\tau) = O(1)$, $\tau \to 0$, and it has been assumed in (3.9) that $v_0(\tau) = O(\tau^{-\beta_0})$, $\tau \to \infty$. Since $I_\beta(\theta)$ is absolutely convergent, the analytical strips of $M[\phi;1-z]$ and $M[\psi;z]$ overlap. Let $\operatorname{Re}\{z\} = c$ lay in the overlapping strip, then the Parseval formula (2.3) implies that

$$I_\beta(\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \theta^{-z} M[\phi;1-z]M[\psi;z]dz, \tag{3.19}$$

where the following identity has been used:

$$M[\psi(s\theta);z] = \theta^{-z}M[\psi;z]. \tag{3.20}$$

One observes from (2.1) that

$$M[\phi;z] = \frac{\Gamma(z)}{\Gamma(z+1-\beta)}, \tag{3.21}$$

and therefore,

$$I_\beta(\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \theta^{-z} M[\psi;z]\frac{\Gamma(1-z)}{\Gamma(2-\beta-z)}dz. \tag{3.22}$$

The asymptotic evaluation of (3.22), as $\theta \to \infty$, involves the asymptotic behavior of $\psi(\tau)$, $\tau \to \infty$, which in turn involves the asymptotic behavior of $v_0(\tau)$, $\tau \to \infty$, the result assumed in (3.9). The asymptotic relation in (3.18) implies that

$$\psi(\theta) \sim c_0\theta^{-\beta_0}, \quad \theta \longrightarrow \infty,\ \beta_0 > 1. \tag{3.23}$$

It can be shown that $M[\psi;z]$ has a simple pole at $z=\beta_0$ and the singular part of the Laurent expansion of $M[\psi;z]$ about this point is given by

$$\frac{-c_0}{z-\beta_0}. \tag{3.24}$$

To compute the asymptotic behavior of I_β, the vertical path is displaced to the right. In doing so, the pole implied by $\Gamma(1-z)$ at $z=1$ is encountered before that of $M[\psi;z]$ since $\beta_0>1$, and hence, it provides the leading order contribution. Computing the relevant residues, one finds that

$$I_\beta(\theta)\sim\frac{1}{\Gamma(1-\beta)}M[\psi;1]\theta^{-1},\quad \theta\longrightarrow\infty, \tag{3.25}$$

where

$$M[\psi;1]=\int_0^\infty \psi(s)ds. \tag{3.26}$$

□

Substituting this result into $I_\varepsilon(\theta;t)$ yields, for a fixed $t>0$,

$$I_\varepsilon(t)=O(\varepsilon^\alpha),\quad \varepsilon\longrightarrow 0. \tag{3.27}$$

Then, if $\rho(t;\varepsilon)=o(1)$ as $\varepsilon\to 0$, the leading order outer equation is obtained from (3.12) by fixing $t>0$ and letting ε tend to zero. This gives

$$0=g(t)+\frac{1}{\Gamma(1-\beta)}\int_0^t\frac{k(t,s)}{(t-s)^\beta}h(s,u_0(s))ds,\quad t\geq 0. \tag{3.28}$$

The equation governing the leading order inner layer solution follows by substituting (3.28) in (3.12) and expressing all terms in terms of τ:

$$\begin{aligned}
&\rho(\varepsilon^\alpha\tau;\varepsilon)+\varepsilon u_0'(\varepsilon^\alpha\tau)+\varepsilon^{1-\alpha}v_0'(\tau)\\
&\quad=\frac{\varepsilon^{\alpha(1-\beta)}}{\Gamma(1-\beta)}\int_0^\tau\frac{k(\varepsilon^\alpha\tau,\varepsilon^\alpha\sigma)}{(\tau-\sigma)^\beta}\{h(0,u_0(0)+v_0(\sigma))-h(0,u_0(0))\}d\sigma\\
&\qquad+\frac{\varepsilon^{\alpha(1-\beta)}}{\Gamma(1-\beta)}\int_0^\tau\frac{k(\varepsilon^\alpha\tau,\varepsilon^\alpha\sigma)}{(\tau-\sigma)^\beta}\Big(\{h(\varepsilon^\alpha\sigma,u_0(\varepsilon^\alpha\sigma)+v_0(\sigma))-h(\varepsilon^\alpha\sigma,u_0(\varepsilon^\alpha\sigma))\}\\
&\qquad\qquad-\{h(0,u_0(0)+v_0(\sigma))-h(0,u_0(0))\}\Big)d\sigma.
\end{aligned} \tag{3.29}$$

This equation is equivalent to

$$\begin{aligned}
\rho(\varepsilon^\alpha\tau;\varepsilon)&=-\varepsilon u_0'(\varepsilon^\alpha\tau)-\varepsilon^{1-\alpha}v_0'(\tau)\\
&\quad+\frac{\varepsilon^{\alpha(1-\beta)}}{\Gamma(1-\beta)}\int_0^\tau\frac{k(0,0)}{(\tau-\sigma)^\beta}\{h(0,u_0(0)+v_0(\sigma))-h(0,u_0(0))\}d\sigma\\
&\quad+O(\varepsilon),\quad \varepsilon\longrightarrow 0.
\end{aligned} \tag{3.30}$$

Multiplying throughout by $\varepsilon^{\alpha-1}$, fixing $\tau > 0$, and letting ε tend to zero, one has by assuming $\varepsilon^{\alpha-1}\rho_0(\varepsilon^{\gamma}\tau;\varepsilon) = o(1)$, as $\varepsilon \to 0$,

$$v_0'(\tau) = \frac{k(0,0)}{\Gamma(1-\beta)}\int_0^{\tau}(\tau-\sigma)^{-\beta}\{h(0,u_0(0)+v_0(\sigma)) - h(0,u_0(0))\}d\sigma, \qquad \tau > 0,\ v_0(0) = y_0 - u_0(0). \tag{3.31}$$

If $u_0(t)$ satisfies (3.28) and $v_0(\tau)$ obeys (3.31), it follows from (3.12) that

$$\begin{aligned}\rho(t;\varepsilon) = -\varepsilon u_0'(t) - \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(0,0)}{(t-s)^{\beta}}\left\{h\left(0,u_0(0)+v_0\left(\frac{s}{\varepsilon^{\alpha}}\right)\right) - h(0,u_0(0))\right\}ds \\ + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^{\beta}}\left\{h\left(s,u_0(s)+v_0\left(\frac{s}{\varepsilon^{\alpha}}\right)\right) - h(s,u_0(s))\right\}ds.\end{aligned} \tag{3.32}$$

4. Properties of the formal solution. From this section on, it will be assumed that

(H_g) the function $g(t) \in C^2[0,T]$ and that $g(0) = 0$;

(H_h) the nonlinear term $h(t,y)$ is at least twice continuously differentiable with respect to both t and y;

(H_k) $k(t,s)$ is a C^2-function on $0 \le s \le t \le T$ and that there exists a positive constant η such that

$$\begin{gathered} k(t,t)\partial_2 h(t,u_0(t)) \le -\eta, \quad 0 \le t \le T, \\ k(0,0)\partial_2 h(0,\upsilon) \le -\eta \end{gathered} \tag{4.1}$$

for all υ between $u_0(0)$ and y_0.

Note that (H_k) is the initial layer stability condition. It forces initial layer solutions to decay and also restricts generally the size of the initial layer jump, see Proposition 4.1. This assumption may look restrictive, but it is satisfied by most physical models since it is the criterion for general stability. Examples of such models include problems in reactor dynamics (see [12, 13, 14]) and reaction diffusion models, see, for example, [16].

It is shown in this section that there are unique solutions $u_0(t)$ and $v_0(\tau)$ that solve (3.28) and (3.31), respectively, and that they have the important properties assumed in their derivation.

Equation (3.28) is a Volterra integral equation of the first kind for $u_0(t)$. If one puts

$$h(t,u_0(t)) = p(t), \tag{4.2}$$

the resulting equation

$$0 = g(t) + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^{\beta}}p(s)ds, \quad t \ge 0, \tag{4.3}$$

is a linear Volterra equation of the first kind. A unique solution $p(t) \in C^1[0,T]$ exists under given conditions (H_g) and $k(t,t) \neq 0$, see [10]. Then the existence and uniqueness of a continuous solution $u_0(t)$ of (4.2) follows from the implicit function theorem and the fact that $\partial_2 h(t,u_0(t)) \neq 0$ (H_k). The cases $k(t,s) = k$ a constant and $k(t,s) = k(t-s)$ would give the function $p(t)$ exactly. Numerical approximation of $u_0(t)$ from (4.2) should then be easy.

Proposition 4.1. *Suppose that (H_g), (H_h), and (H_k) hold. Then (3.31) has a* C^∞*-solution* v_0 *satisfying*

$$v_0(\tau) \sim \frac{y_0 - u_0(0)}{\Gamma(\beta-1)}\tau^{\beta-2}, \quad \tau \longrightarrow \infty. \tag{4.4}$$

Proof. Consider (3.31); the standard theory of Volterra integrodifferential equations (see, e.g., [13, 15]) ensures the existence of $v_0(\tau) \in C^\infty[0,\infty)$. Moreover, for $v_0(0)$ sufficiently small, $v_0(\tau) \to 0$, $\tau \to \infty$.

Using the identity

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(\xi)d\xi \tag{4.5}$$

with

$$\varphi(\xi) = h(0, u_0(0) + \xi v_0(\tau)), \tag{4.6}$$

one writes

$$h(0,u_0(0)+v_0(\tau)) - h(0,u_0(0)) = \int_0^1 \partial_2 h(0,u_0(0)+\xi v_0(\tau))d\xi v_0(\tau). \tag{4.7}$$

Thus, (3.31) becomes

$$\begin{gathered} v_0'(\tau) = \frac{1}{\Gamma(1-\beta)}\int_0^\tau (\tau-\sigma)^{-\beta}\int_0^1 k(0,0)\partial_2 h(0,u_0(0)+\xi v_0(\sigma))d\xi v_0(\sigma)d\sigma, \\ v_0(0) = y_0 - u_0(0). \end{gathered} \tag{4.8}$$

Assumption (H_k) implies that $v_0(\tau)$ is nonincreasing if $v_0(0) > 0$ and nondecreasing if $v_0(0) < 0$ and that $u_0(0) + \xi v_0(\tau)$ lies between $u_0(0)$ and $u_0(0) + v_0(0)$. Integrating both sides of (4.8), one obtains

$$\begin{aligned} &v_0(\tau) - v_0(0) \\ &\quad = \frac{1}{\Gamma(2-\beta)}\int_0^\tau (\tau-\sigma)^{1-\beta}\int_0^1 k(0,0)\partial_2 h(0,u_0(0)+\xi v_0(\sigma))d\xi v_0(\sigma)d\sigma. \end{aligned} \tag{4.9}$$

It follows that, when $v_0(0) > 0$,

$$v_0(\tau) \le v_0(0) - \frac{\eta}{\Gamma(2-\beta)} \int_0^\tau (\tau-\sigma)^{1-\beta} v_0(\sigma)\,d\sigma \tag{4.10}$$

and when $v_0(0) < 0$,

$$v_0(\tau) \ge v_0(0) - \frac{\eta}{\Gamma(2-\beta)} \int_0^\tau (\tau-\sigma)^{1-\beta} v_0(\sigma)\,d\sigma. \tag{4.11}$$

Thus,

$$|v_0(\tau)| \le |v_0(0)| - \frac{\eta}{\Gamma(2-\beta)} \int_0^\tau (\tau-\sigma)^{1-\beta} |v_0(\sigma)|\,d\sigma, \quad \tau > 0. \tag{4.12}$$

In particular, $|v_0(\tau)| \le |v_0(0)|$, $\tau \ge 0$. It follows from comparison theorems and the application of the Laplace transform method on

$$\phi(\tau) = |v_0(0)| - \frac{\eta}{\Gamma(2-\beta)} \int_0^\tau (\tau-\sigma)^{1-\beta} \phi(\sigma)\,d\sigma, \quad \tau > 0, \tag{4.13}$$

that

$$|v_0(\tau)| \le |y_0 - u_0(0)|\,|E_{2-\beta}(-\eta\tau^{2-\beta})|, \quad \tau \ge 0. \tag{4.14}$$

Here, $E_{2-\beta}$ is the Mittag-Leffler function defined in (2.6). The asymptotic expansions of the Mittag-Leffler function established in [4, 6] imply that

$$v_0(\tau) \sim (y_0 - u_0(0)) \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\tau^{(\beta-2)i}}{\Gamma(1-(2-\beta)i)}, \quad \tau \longrightarrow \infty. \tag{4.15}$$

This completes the proof. □

Proposition 4.2. *Suppose that $\rho(t;\varepsilon)$ satisfies (3.32), then there exists a positive constant c_1 which does not depend on ε and ε_0, such that*

$$|\rho(t;\varepsilon)| \le c_1\varepsilon, \quad \varepsilon \longrightarrow 0, \tag{4.16}$$

for all $0 \le t \le T$ and all $0 < \varepsilon \le \varepsilon_0$.

Proof. Equation (3.32) can also be written as

$$\begin{aligned} \rho(t;\varepsilon) = {} & -\varepsilon u_0'(t) \\ & + \frac{t^{1-\beta}}{\Gamma(1-\beta)} \int_0^1 (1-s)^{-\beta} \Big[\psi\Big(t, ts, v_0\Big(\frac{ts}{\varepsilon^\alpha}\Big)\Big) \\ & \qquad - \psi\Big(0, 0, v_0\Big(\frac{ts}{\varepsilon^\alpha}\Big)\Big)\Big] ds, \quad t \ge 0, \end{aligned} \tag{4.17}$$

where

$$\psi\left(t,ts,v_0\left(\frac{ts}{\varepsilon^\alpha}\right)\right)=k(t,ts)\left[h\left(ts,u_0(ts)+v_0\left(\frac{ts}{\varepsilon^\alpha}\right)\right)-h(ts,u_0(ts))\right]. \tag{4.18}$$

Consider the integral on the right-hand side of

$$I(t;\varepsilon)=\int_0^1(1-s)^{-\beta}\left[\psi\left(t,ts,v_0\left(\frac{ts}{\varepsilon^\alpha}\right)\right)-\psi\left(0,0,v_0\left(\frac{ts}{\varepsilon^\alpha}\right)\right)\right]ds. \tag{4.19}$$

At $t=0$, $I(0;\varepsilon)=0$. For any fixed $t>0$ and ε sufficiently small, Lemma 3.1 establishes that $\psi(t,ts,v_0(s/\varepsilon^\alpha))$ and $\psi(0,0,v_0(s/\varepsilon^\alpha))$ assume the same asymptotic expansion. Therefore, for all $0\le t\le T$ and all sufficiently small values of ε, $I(t;\varepsilon)=0$. It then follows that

$$|\rho(t;\varepsilon)|\le c_1\varepsilon \tag{4.20}$$

uniformly for all $0\le t\le T$ and all $0<\varepsilon\le\varepsilon_0$, where c_1 is a constant which depends on $u_0{}'(t)$ and T. □

5. Proof of asymptotic correctness. In this section, it is proved that the formal approximation defined in (3.8) is, indeed, an asymptotic approximation to the solution $y(t;\varepsilon)$ of (1.1). The method is to adopt the theory of [9] on developing a rigorous theory of singular perturbation.

Define the remainder

$$\chi(t;\varepsilon)=y(t;\varepsilon)-y_0(t;\varepsilon),\quad 0\le t\le T. \tag{5.1}$$

Substituting this in (1.1) and using (3.11) gives

$$\begin{aligned}\varepsilon\chi'(t;\varepsilon)&\\ &=\rho(t;\varepsilon)+\frac{1}{\Gamma(1-\beta)}\int_0^t\frac{k(t,s)}{(t-s)^\beta}\{h(s,y_0(s;\varepsilon)+\chi(s;\varepsilon))-h(s,y_0(s;\varepsilon))\}ds,\\ &\qquad 0<t\le T,\ \chi(0;\varepsilon)=0.\end{aligned} \tag{5.2}$$

Applying Taylor's theorem, one has an equivalent perturbed linear equation

$$\begin{aligned}\varepsilon\chi'(t;\varepsilon)&=G(\chi)(t;\varepsilon)+\frac{1}{\Gamma(1-\beta)}\int_0^t\frac{k(t,s)}{(t-s)^\beta}\partial_2h(s,y_0(s;\varepsilon))\chi(s;\varepsilon)ds,\\ &\qquad 0<t\le T,\ \chi(0;\varepsilon)=0,\end{aligned} \tag{5.3}$$

where $G(\chi)(t;\varepsilon)$ is a sum of a small term and a term independent of χ. Moreover,

$$G(\chi)(t;\varepsilon) = \rho(t;\varepsilon) + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{k(t,s)}{(t-s)^\beta}\varphi(s,\chi,y_0;\varepsilon)\,ds, \tag{5.4}$$

where

$$\varphi(t,\chi,y_0;\varepsilon) = \chi(t;\varepsilon)^2\int_0^1 (1-\mu)\partial_{22}h(t,y_0(t;\varepsilon)+\mu\chi(t;\varepsilon))\,d\mu. \tag{5.5}$$

Under given conditions on the data, (5.3) has a continuous solution $\chi(t;\varepsilon)$ on $[0,T]$, $T>0$, for all $\varepsilon>0$. The interest here is in the boundedness of $\chi(t;\varepsilon)$ for sufficiently small values of ε.

Let $\gamma(t,s;\varepsilon)$ denote the differential resolvent kernel for the kernel in (5.3). Then $\gamma(t,s;\varepsilon)$ satisfies $\gamma(t,t;\varepsilon)=1$, $\gamma(t,s;\varepsilon)=0$, $s>t$, and

$$\varepsilon\partial_1\gamma(t,s;\varepsilon) = \frac{1}{\Gamma(1-\beta)}\int_s^t \frac{k(t,\sigma)}{(t-\sigma)^\beta}\partial_2 h(\sigma,y_0(\sigma;\varepsilon))\gamma(\sigma,s;\varepsilon)\,d\sigma, \tag{5.6a}$$

$$\varepsilon\partial_2\gamma(t,s;\varepsilon) = -\frac{1}{\Gamma(1-\beta)}\int_s^t \gamma(t,\sigma;\varepsilon)\frac{k(\sigma,s)}{(\sigma-s)^\beta}\partial_2 h(s,y_0(s;\varepsilon))\,d\sigma. \tag{5.6b}$$

Again, under given conditions, the existence of a continuous solution $\gamma(t,s;\varepsilon)$ of (5.6) is guaranteed, see [11, Chapter 10]. To this point, one may use the variation-of-constants formula to write (5.3) as

$$\chi(t;\varepsilon) = \frac{1}{\varepsilon}\int_0^t \gamma(t,s;\varepsilon)G(\chi)(s;\varepsilon)\,ds, \quad 0\le t\le T. \tag{5.7}$$

To verify (5.7), left multiply (5.3) by $\gamma(t,s;\varepsilon)$ and integrate with respect to s,

$$\begin{aligned}\int_0^t &\gamma(t,s;\varepsilon)\chi'(s;\varepsilon)\,ds\\ &= \frac{1}{\varepsilon}\int_0^t \gamma(t,s;\varepsilon)G(\chi)(s;\varepsilon)\,ds\\ &\quad+\frac{1}{\varepsilon\Gamma(1-\beta)}\int_0^t \gamma(t,s;\varepsilon)\int_0^s \frac{k(s,\sigma)}{(s-\sigma)^\beta}\partial_2 h(\sigma,y_0(\sigma;\varepsilon))\chi(\sigma;\varepsilon)\,d\sigma\,ds.\end{aligned} \tag{5.8}$$

Applying integration by parts on the left-hand side and changing the order of integration in the second term on the right-hand side, one obtains

$$\begin{aligned}\chi'(t;\varepsilon) &= \frac{1}{\varepsilon}\int_0^t \gamma(t,s;\varepsilon)G(\chi)(s;\varepsilon)\,ds\\ &\quad+\int_0^t\Big\{\partial_2\gamma(t,s;\varepsilon)\\ &\qquad+\frac{1}{\varepsilon\Gamma(1-\beta)}\int_s^t \gamma(t,\sigma;\varepsilon)\frac{k(\sigma,s)}{(\sigma-s)^\beta}\partial_2 h(s,y_0(\sigma;\varepsilon))\,d\sigma\Big\}\chi(s;\varepsilon)\,ds.\end{aligned} \tag{5.9}$$

But according to (5.6b), the second integral on the right-hand side is zero, so (5.7) is verified.

The following lemma proves that the differential resolvent $\gamma(t,s;\varepsilon)$ is uniformly bounded for all $0 \le s \le t \le T$ and all sufficiently small ε.

Lemma 5.1. *Suppose that (H_g), (H_h), and (H_k) hold. Let $\gamma(t,s;\varepsilon)$ satisfy (5.6). Then there exist positive constants c_2 and ε_0 such that*

$$\int_0^t |\gamma(t,s;\varepsilon)|\,ds \le c_2\varepsilon^{\alpha}, \quad 0 \le t \le T,\ 0 < \varepsilon \le \varepsilon_0. \tag{5.10}$$

Proof. Consider (5.6a) and integrate both sides with respect to t to obtain

$$\gamma(t,s;\varepsilon) = 1 + \frac{1}{\varepsilon\Gamma(2-\beta)}\int_s^t a(t,\sigma;\varepsilon)(t-\sigma)^{1-\beta}\gamma(\sigma,s;\varepsilon)\,d\sigma, \tag{5.11}$$

where

$$a(t,s;\varepsilon) = (1-\beta)\partial_2 h\big(s,y_0(s;\varepsilon)\big)\int_0^1 \frac{k(s+(t-s)\sigma,s)}{\sigma^{\beta}}\,d\sigma. \tag{5.12}$$

If t is replaced by $t+s$ in (5.11) and change of variables is performed, one has

$$\gamma(t+s,s;\varepsilon) = 1 + \frac{1}{\varepsilon\Gamma(2-\beta)}\int_0^t a(t+s,\sigma+s;\varepsilon)(t-\sigma)^{1-\beta}\gamma(\sigma+s,s;\varepsilon)\,d\sigma. \tag{5.13}$$

One will note that

$$a(t,t;\varepsilon) = (1-\beta)\partial_2 h\big(t,y_0(t;\varepsilon)\big)\int_0^1 \frac{k(t,t)}{\sigma^{\beta}}\,d\sigma, \quad 0 \le t \le T. \tag{5.14}$$

It follows from (H_k) that there is a positive number $0 < \delta < \eta$ such that

$$a(t,t;\varepsilon) \le -\delta < 0 \tag{5.15}$$

for all $0 \le t \le T$ and all $0 < \varepsilon \le \varepsilon_0$. It can be shown that $a(t,s;\varepsilon)$ does not change the sign on $0 \le t \le T$. Thus, there exists a positive constant δ_0 such that

$$a(t,s;\varepsilon) \le -\delta_0, \quad 0 \le s \le t \le T,\ 0 < \varepsilon \le \varepsilon_0. \tag{5.16}$$

Therefore, the differential resolvent γ satisfies

$$\begin{gathered} 0 \le \gamma(t+s,s;\varepsilon) \le 1, \quad 0 \le s \le t \le T,\ 0 < \varepsilon \le \varepsilon_0, \\ \gamma(t+s,s;\varepsilon) \le 1 - \frac{\delta_0}{\varepsilon\Gamma(2-\beta)}\int_0^t (t-\sigma)^{1-\beta}\gamma(\sigma+s,s;\varepsilon)\,d\sigma. \end{gathered} \tag{5.17}$$

In this form, s is simply a parameter and one may write the above inequality as

$$\tilde{y}(t;\varepsilon) \le 1 - \frac{\delta_0}{\varepsilon\Gamma(2-\beta)} \int_0^t (t-\sigma)^{1-\beta} \tilde{y}(\sigma;\varepsilon)\, d\sigma, \tag{5.18}$$

and the proof of Lemma 5.1 may be directly applied to (5.18).

Thus, to prove the lemma, it suffices to show that there exist a positive constant c_3 such that

$$\int_0^t \tilde{y}(s;\varepsilon)\, ds \le c_3 \varepsilon^{\alpha}, \quad 0 \le t \le T,\ 0 < \varepsilon \le \varepsilon_0. \tag{5.19}$$

It has been proved in [6] that

$$\tilde{y}(t;\varepsilon) \le 2\alpha e^{-\epsilon_1 (\varrho/\varepsilon)^{\alpha} t} \cos(\epsilon_2 t), \tag{5.20a}$$

$$\int_0^t \tilde{y}(s;\varepsilon)\, ds \le \frac{4\alpha\varepsilon^{\alpha}}{\epsilon_1 \varrho}, \tag{5.20b}$$

for all $0 \le t \le T$ and all $0 < \varepsilon \le \varepsilon_0$, where ϵ_1, ϵ_2, and ϱ are positive constants. This completes the proof of Lemma 5.1. □

The main result in this paper is presented in the following theorem.

Theorem 5.2. *Suppose that (H_g), (H_h), and (H_k) are satisfied and that Lemma 5.1 holds. Then (1.1) has a continuous solution $y(t;\varepsilon)$ with the property that, if $y_0(t;\varepsilon)$ is the formal approximate solution, then there are positive constants $c^{\star}$ and ε_0, where $c^{\star}$ does not depend on ε such that*

$$\left| y(t;\varepsilon) - y_0(t;\varepsilon) \right| \le c^{\star} \varepsilon^{\alpha} \tag{5.21}$$

for all $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$.

Proof. Consider (5.7) which is equivalent to

$$\begin{aligned} \chi(t;\varepsilon) = {} & \frac{1}{\varepsilon} \int_0^t y(t,s;\varepsilon)\rho(s;\varepsilon)\, ds \\ & + \frac{1}{\varepsilon\Gamma(1-\beta)} \int_0^t \varphi(\chi, y_0, \sigma;\varepsilon) \int_{\sigma}^t \frac{y(t,s;\varepsilon)k(s,\sigma)}{(s-\sigma)^{\beta}}\, ds\, d\sigma, \end{aligned} \tag{5.22}$$

where upon applying (5.20a),

$$\left| \int_{\sigma}^t \frac{y(t,s;\varepsilon)k(s,\sigma)}{(s-\sigma)^{\beta}}\, ds \right| \le \epsilon_3 (t-\sigma)^{1-\beta} \int_0^1 \xi^{-\beta} e^{-\epsilon\xi/\varepsilon^{\alpha}}\, d\xi, \quad 0 \le \sigma \le T. \tag{5.23}$$

Here, ϵ and ϵ_3 are positive constants, where ϵ_3 depends on α, $k(t,s)$, and T. A change of variables implies that

$$\begin{aligned} |\chi(t;\varepsilon)| &\le \frac{1}{\varepsilon}\int_0^t |\gamma(t,s;\varepsilon)|\,|\rho(s;\varepsilon)|\,ds \\ &\quad + \frac{\varepsilon^{-\alpha}\epsilon_4}{\Gamma(1-\beta)}\int_0^t |\varphi(\chi,\gamma_0,\sigma;\varepsilon)| \int_0^\infty \xi^{-\beta}e^{-\epsilon\xi}d\xi\,d\sigma, \end{aligned} \tag{5.24}$$

where ϵ_4 is a positive constant. Applying Proposition 4.2, (5.20b), and the definition of a Gamma function, one obtains

$$|\chi(t;\varepsilon)| \le c_2\varepsilon^{\alpha} + c_3\varepsilon^{-\alpha}\int_0^t |\varphi(\chi,\gamma_0,\sigma;\varepsilon)|\,d\sigma, \quad 0 \le t \le T, \tag{5.25}$$

where c_2 and c_3 are positive constants depending on T. Then, Theorem 2.3 yields the required result that

$$|\chi(t;\varepsilon)| \le c_2\varepsilon^{\alpha} + \frac{c_2\varepsilon^{\alpha}}{2}e^{2c_2c_3T} \tag{5.26}$$

uniformly on $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$. □

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