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LOCAL SPECTRAL THEORY FOR 2×2 OPERATOR MATRICES

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We discuss the spectral properties of the operator $M_C \in \mathcal{L}(X\oplus Y)$ defined by $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, $C \in \mathcal{L}(Y,X)$, and X, Y are complex Banach spaces. We prove that $(S_{A^*} \cap S_B) \cup \sigma(M_C) = \sigma(A) \cup \sigma(B)$ for all $C \in \mathcal{L}(Y,X)$. This allows us to give a partial positive answer to Question 3 of Du and Jin (1994) and generalizations of some results of Houimdi and Zguitti (2000). Some applications to the similarity problem are also given.

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1. Introduction. Let X and Y be complex Banach spaces and let $\mathcal{L}(X)$, $\mathcal{L}(Y)$, and $\mathcal{L}(Y,X)$ be the algebras of all continuous linear operators on X, Y, and from Y to X, respectively. For $T \in \mathcal{L}(X)$, we denote by $\sigma(T)$ its spectrum, $\sigma_p(T)$ its point spectrum, T^* its adjoint operator, and R_T its resolvent map. Let $x \in X$ and $\lambda_0 \in \mathbb{C}$, we say that λ_0 is in the *local resolvent* of T at x, denoted by $\rho_T(x)$, if the equation

$$(T-\lambda)f(\lambda) = x \tag{1.1}$$

admits an analytic solution in a neighborhood of λ_0. The set $\sigma_T(x) = \mathbb{C}\setminus\rho_T(x)$ is called the *local spectrum* of T at x.

If for every $x \in X$ any two solutions of (1.1) agree on their common domain, T is said to have the *single-valued extension property* (SVEP). It is obvious that T has the SVEP if and only if the zero function is the only analytic function which satisfies $(T-\lambda)f(\lambda) = 0$.

For $T \in \mathcal{L}(X)$ and F a closed set of $\mathbb{C}$, denote by the set $X_T(F) := \{x \in X,\ \sigma_T(x) \subset F\}$ the *analytic spectral space.* The analytic residuum S_T is the set of $\lambda_0 \in \mathbb{C}$ for which there exist a neighborhood G_{λ_0} and $f : G_{\lambda_0} \to X$ a nonzero analytic function such that $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in G_{\lambda_0}$. We say that the operator T has the *Dunford condition C* (DCC) if $X_T(F)$ is closed whenever F is closed. It is clear that T has the SVEP if and only if $S_T = \emptyset$. The *surjective spectrum* of T is given by $\sigma_{su}(T) := \{\lambda \in \mathbb{C} / \ \mathbb{T}-\lambda \text{ is not surjective}\}$. It is known that $\sigma_{su}(T) = \cup_{x\in X}\sigma_T(x)$ and that $\sigma(T) = S_T \cup \sigma_{su}(T)$ (see [6, 7, 8]). In particular, if T has the SVEP, we obtain that $\sigma(T) = \sigma_{su}(T)$. A complete study of basic notions of local spectral theory can be found in [1, 4].

The operator $M_C \in \mathcal{L}(X \oplus Y)$ is defined by $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, $C \in \mathcal{L}(Y,X)$, and X, Y are complex Banach spaces. In [2], a number of natural questions concerning the relation between the spectrums of these operators have been considered. Our interest in this paper is to develop some new conditions under which we have the equality $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. Houimdi and Zguitti [3] give a positive answer to this question when the operator B has the SVEP.

Our main result gives a partial answer to [2, Question 3], and generalizes results from [3]. At the end, we give a generalization of [3, Proposition 3.2] related to the DCC for the operator M_C and some applications to the similarity of orbits.

We collect in the following proposition some useful spectral properties of the operator M_C from [2, 3], that can be also obtained easily.

Proposition 1.1. *If $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y,X)$, then*

(1) $\sigma_p(A) \subset \sigma_p(M_C) \subset \sigma_p(A) \cup \sigma_p(B)$,
(2) $\sigma_{ap}(A) \subset \sigma_{ap}(M_C) \subset \sigma_{ap}(A) \cup \sigma_{ap}(B)$,
(3) $S_A \subset S_{M_C} \subset S_A \cup S_B$,
(4) $\sigma_{\mathrm{su}}(B) \subset \sigma_{\mathrm{su}}(M_C) \subset \sigma_{\mathrm{su}}(A) \cup \sigma_{\mathrm{su}}(B)$,
(5) $\sigma_{ap}(A) \cup \sigma_{\mathrm{su}}(B) \subset \sigma(M_C)$,
(6) $\sigma(A) \cup \sigma(B) = S_{A^*} \cup S_B \cup \sigma(M_C)$.

2. Spectral theory of the operator M_C. We study in the sequel spectral theory of M_C, we refine the inclusions given in Proposition 1.1, and provide some properties of local spectrum.

Using Leiterer's theorem (see [6, Theorem 3.2.1, page 212] and [5]), we derive the following proposition.

Proposition 2.1. *If A, B, and C are given, then*

$$S_B \subset \sigma_{\mathrm{su}}(A) \cup S_{M_C}. \tag{2.1}$$

Proof. Let $\lambda_0 \in S_B \setminus \sigma_{\mathrm{su}}(A)$. There exist a neighborhood V_{λ_0} and a nonzero analytic function $g : V_{\lambda_0} \to Y$ satisfying

$$(B-\mu)g(\mu) = 0, \qquad V_{\lambda_0} \cap \sigma_{\mathrm{su}}(A) = \varnothing. \tag{2.2}$$

By Leiterer's theorem, there exists an analytic function $f : V_{\lambda_0} \to X$ satisfying

$$(A-\mu)f(\mu) = -Cg(\mu) \quad \forall \mu \in V_{\lambda_0}. \tag{2.3}$$

The nonzero analytic function $f \oplus g : V_{\lambda_0} \to X \oplus Y$ defined by $(f \oplus g)(\mu) = f(\mu) \oplus g(\mu)$ satisfies

$$(M_C - \mu)\big(f(\mu) \oplus g(\mu)\big) = 0 \quad \forall \mu \in V_{\lambda_0}, \tag{2.4}$$

hence $\lambda_0 \in S_{M_C}$. □

The following corollary is immediate from Proposition 2.1 and Proposition 1.1(4).

Corollary 2.2. *For given operators A, B, and C,*

$$\sigma(B) \subset \sigma_{\mathrm{su}}(A) \cup \sigma(M_C). \tag{2.5}$$

To establish our main theorem, we first claim the following proposition related to the local spectrum of the operator M_C, which generalizes [3, Proposition 2.1].

Proposition 2.3. *If A, B, and C are given,*

$$S_B \cup \sigma_A(x) = S_B \cup \sigma_{M_C}(x \oplus 0) \quad \forall x \in X. \tag{2.6}$$

Proof. If $\lambda_0 \notin (S_B \cup \sigma_{M_C}(x \oplus 0))$, then there exist a neighborhood V_{λ_0} and a nonzero analytic function $h : V_{\lambda_0} \to X \oplus Y$ satisfying

$$(M_C - \mu)h(\mu) = x \oplus 0 \quad \forall \mu \in V_{\lambda_0}. \tag{2.7}$$

Let $h = h_1 \oplus h_2$ with $h_1 : V_{\lambda_0} \to X$ and $h_2 : V_{\lambda_0} \to Y$ analytic functions. We obtain

$$(A - \mu)h_1(\mu) + Ch_2(\mu) = x, \qquad (B - \mu)h_2(\mu) = 0. \tag{2.8}$$

As $\lambda_0 \notin S_B$, we conclude that $h_2 = 0$ on V_{λ_0}. Hence

$$(A - \mu)h_1(\mu) = x \quad \forall \mu \in V_{\lambda_0}, \tag{2.9}$$

thus $\lambda_0 \notin \sigma_A(x)$. The reverse inclusion is clear. □

The following result generalizes [3, Corollary 2.2].

Corollary 2.4. *For given operators A, B, and C,*

$$S_B \cup \sigma_{\mathrm{su}}(M_C) = \sigma_{\mathrm{su}}(A) \cup \sigma(B). \tag{2.10}$$

Proof. By Proposition 2.1, we have $\sigma_{\mathrm{su}}(M_C) \subset \sigma_{\mathrm{su}}(A) \cup \sigma_{\mathrm{su}}(B)$. Hence

$$S_B \cup \sigma_{\mathrm{su}}(M_C) \subset \sigma_{\mathrm{su}}(A) \cup \sigma(B). \tag{2.11}$$

For the converse, we obtain by Proposition 2.3 that

$$S_B \cup \sigma_A(x) = S_B \cup \sigma_{M_C}(x \oplus 0) \quad \forall x \in X, \tag{2.12}$$

hence

$$S_B \cup \sigma_{\mathrm{su}}(A) \subset S_B \cup \sigma_{\mathrm{su}}(M_C). \tag{2.13}$$

Since $\sigma_{su}(B) \subset \sigma_{su}(M_C)$, we check that

$$\sigma_{su}(A) \cup \sigma(B) \subset S_B \cup \sigma_{su}(M_C). \tag{2.14}$$

□

We are now in a position to derive a generalization of [3, Theorem 2.1].

Theorem 2.5. *For given operators A, B, and C,*

$$(S_{A^*} \cap S_B) \cup \sigma(M_C) = \sigma(A) \cup \sigma(B). \tag{2.15}$$

Proof. In the first step, we prove that

$$S_B \cup \sigma(M_C) = \sigma(A) \cup \sigma(B). \tag{2.16}$$

By Corollary 2.4 and Proposition 1.1(3), we obtain that

$$\sigma(A) \cup \sigma(B) \subset S_B \cup \sigma(M_C). \tag{2.17}$$

The converse is clear since the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ is always true. In the second step, remark that

$$M_C{}^* := \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix}. \tag{2.18}$$

Thus, in the same way, we obtain that

$$S_{A^*} \cup \sigma(M_C) = \sigma(A) \cup \sigma(B). \tag{2.19}$$

Hence the theorem is proved. □

We give in what follows a new condition under which we have the desired equality.

Corollary 2.6. *For given operators A, B, and C, if $S_{A^*} \cap S_B = \emptyset$, then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.*

3. Applications. In this section, we give some necessary conditions such that M_C has the SVEP and provide some results on similarity orbits and the range of generalized derivation using Theorem 2.5. The following result gives a partial characterization of the property of the single extension property of the operator M_C and discusses the converse of [3, Proposition 3.1].

Proposition 3.1. *For given operators A, B, and C such that the surjective spectrum of the operator A has an empty interior,*

$$S_{M_C} = \emptyset \Longleftrightarrow S_A = S_B = \emptyset. \tag{3.1}$$

In particular M_C has SVEP if and only if A and B have the SVEP.

Proof. If A and B have the SVEP, so is the case of M_C. (See Proposition 2.1(3).) On the other hand, the relations

$$S_B \subset \sigma_{su}(A) \cup S_{M_C}, \qquad S_A \subset S_{M_C} \subset S_A \cup S_B \tag{3.2}$$

imply that $S_A = S_B = \emptyset$. The proof is complete. □

The next proposition provides a generalization of [3, Proposition 3.2].

Proposition 3.2. *If A and B are given and F is a closed set such that $S_B \subset F$, then the following assertion holds true: if there exists $C \in \mathcal{L}(Y,X)$ such that the analytic subspace $(X \oplus Y)_{M_C}(F)$ is closed, then $X_A(F)$ is also closed.*

Proof. Let (x_n) be a sequence of elements of $X_A(F)$ which converges to $x \in X$. By Proposition 2.3, we have

$$S_B \cup \sigma_A(x_n) = S_B \cup \sigma_{M_C}(x_n \oplus 0) \quad \forall x \in X. \tag{3.3}$$

We deduce that $\sigma_{M_C}(x_n \oplus 0) \subset F \cup S_B$. Thus, $\sigma_{M_C}(x_n \oplus 0) \subset F$. Since $(X \oplus Y)_{M_C}(F)$ is closed, we have $\sigma_{M_C}(x \oplus 0) \subset F$. Then

$$\sigma_A(x) \subset S_B \cup \sigma_A(x) = S_B \cup \sigma_{M_C}(x \oplus 0). \tag{3.4}$$

We derive that $\sigma_A(x) \subset F \cup S_B \subset F$, then $x \in X_A(F)$, hence $X_A(F)$ is closed. □

Our second application is related to the similarity problem. Let A, B, and M_C be as above and consider $\delta_{A,B}$ the *derivation operator* defined by $\delta_{A,B}(X) = AX - XB$ for $X \in \mathcal{L}(X)$. Let $\operatorname{Im}(\delta_{A,B})$ be its range and denote by H_1, H_2, and H_3 the following classes of operators:

$$\begin{aligned} H_1 &:= \{C \in \mathcal{L}(Y,X) \text{ such that } C \in \operatorname{Im}(\delta_{A,B})\}, \\ H_2 &:= \{C \in \mathcal{L}(Y,X) \text{ such that } M_C \text{ is similar to } M_O\}, \\ H_3 &:= \{C \in \mathcal{L}(Y,X) \text{ such that } \sigma(M_C) = \sigma(M_O)\}. \end{aligned} \tag{3.5}$$

It is obvious that

$$H_1 \subset H_2 \subset H_3. \tag{3.6}$$

The above inclusions received a lot of interest (see [9, 10]). In general, any of these inclusions can be strict. We construct some classes of operators such that $H_1 \neq H_2$ and $H_2 \neq H_3$.

Remark first by Corollary 2.6 that if $S_{A^*} \cap S_B = \emptyset$, then $H_3 = \mathcal{L}(Y,X)$. Suppose in the sequel that $X = Y$ and let S be the unilateral shift. We consider, respectively, the operators $A := \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ and $M_C := \begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$.

By [9, Theorem 6], if M_C and M_O are similar, then C is a commutator, that is, $C \neq \lambda I + K$ for any $\lambda \in \mathbb{C}$ and K a compact operator. Hence, choosing A so

that $S_{A^*} \cap S_A = \emptyset$ and considering $C = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and K a compact operator, we get $M_C \in H_3 \setminus H_2$. Moreover, for $C = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$, we conclude by [9] that $M_C \in H_2 \setminus H_1$. Consequently, the inclusions in (3.6) are all strict for $M_C = \begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ with $A = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$, and S is the unilateral shift.

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