

EQUIVARIANT EMBEDDINGS AND COMPACTIFICATIONS OF FREE G -SPACES

NATELLA ANTONYAN

Received 16 November 2001

For a compact Lie group G , we characterize free G -spaces that admit free G -compactifications. For such G -spaces, a universal compact free G -space of given weight and given dimension is constructed. It is shown that if G is finite, the n -dimensional Menger free G -compactum μ^n is universal for all separable, metrizable free G -spaces of dimension less than or equal to n . Some of these results are extended to the case of G -spaces with a single orbit type.

2000 Mathematics Subject Classification: 54H15, 54D35.

1. Introduction. By a G -space, we mean a triple (G, X, α) , where G is a topological group, X is a topological space, and $\alpha : G \times X \rightarrow X$ is a continuous action.

In 1960, Palais proved that every Tychonoff G -space can equivariantly be embedded into a compact Hausdorff G -space provided G is a compact Lie group (see [17, Section 1.5]). This result was extended by de Vries [5] to the case of arbitrary locally compact Hausdorff groups. The local compactness is essential here; it was Megrelishvili who constructed in [14] a continuous action α of a separable, complete metrizable group G on a separable, metrizable space X such that (G, X, α) does not admit an equivariant embedding into a compact G -space. The reader can find other examples of this type in [15].

In this paper, we are mostly interested in *free G -spaces*. Recall that a G -space X is free if, for every $x \in X$, the equality $gx = x$ implies $g = e$, the unity of G . In [2], it is proved that if G is a compact Lie group, then any Tychonoff free G -space can equivariantly be embedded in a locally compact free G -space. In this connection, it is natural to ask the following question.

QUESTION 1.1. Does every free G -space have a G -embedding in a free compact G -space?

One of the purposes of the present paper is to answer this question for G a compact Lie group. Namely, we prove that each finitistic free G -space X has a free G -compactification (Theorem 3.4). In the realm of G -spaces that admit a free G -compactification, we construct a universal, compact, free G -space of given weight and given dimension (Theorem 4.1). This result is extended to the case of the G -spaces with a single orbit type (Theorem 5.2).

2. Preliminaries. Throughout the paper, all topological spaces are assumed to be Tychonoff (i.e., completely regular and Hausdorff). All equivariant or G -maps are assumed to be continuous.

The letter “ G ” will always denote a compact Lie group.

The basic ideas and facts of the theory of G -spaces or topological transformation groups can be found in Bredon [4] and Palais [17].

For the convenience of the reader, however, we recall some more special definitions and facts below.

By e , we will always denote the unity of the group G .

If X is a G -space, for any $x \in X$, we denote the stabilizer (or stationary subgroup) of x by $G_x = \{g \in G \mid gx = x\}$.

If, for all $x \in X$, $G_x = \{e\}$, then we say that the action of G is *free* and X is a *free G -space*.

For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H -saturation of S , that is, $H(S) = \{hs \mid h \in H, s \in S\}$. In particular, $G(x)$ denotes the G -orbit $\{gx \in X \mid g \in G\}$ of x . If $H(S) = S$, then S is said to be an H -invariant set. The G -orbit space is denoted by X/G .

By G/H , we will denote the G -space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

For each subgroup $H \subseteq G$, the H -fixed point set X^H is defined to be the set $\{x \in X \mid H \subseteq G_x\}$.

The family of all subgroups of G which are conjugate to H is denoted by (H) , that is, $(H) = \{gHg^{-1} \mid g \in G\}$. The set (H) is called a G -orbit type (or simply an orbit type). For two orbit types (H_1) and (H_2) , we say that $(H_1) \leq (H_2)$ if and only if $H_1 \subseteq gH_2g^{-1}$ for some $g \in G$. If $(H_1) \leq (H_2)$ and $(H_1) \neq (H_2)$, then we write $(H_1) < (H_2)$. The relation \leq is a partial ordering on the set of all G -orbit types. Since $G_{gx} = gG_xg^{-1}$, for any $x \in X$, $g \in G$, we have $(G_x) = \{G_{gx} \mid g \in G\}$.

We say that a G -space X is of the orbit type (H) , or simply of type (H) , if $(G_x) = (H)$ for every $x \in X$.

In this paper, we will consider only G -spaces that have a single orbit type (H) .

An equivariant map $f : X \rightarrow Y$ of G -spaces is said to be *isovariant* or (*G -isovariant*) if $G_x = G_{f(x)}$ for all $x \in X$.

If X and Y are G -spaces, then $X \times Y$ will always be regarded as a G -space equipped by the diagonal action of G .

A G -compactification of a G -space X is a pair $(b_G, b_G X)$, where $b_G : X \rightarrow b_G X$ is a G -homeomorphic embedding into a compact G -space $b_G X$ such that the image $b_G(X)$ is dense in $b_G X$. Usually, $b_G X$ alone is a sufficient denotation. By $\beta_G X$, we will denote the maximal G -compactification of X .

In the sequel, we will need the following lemma.

LEMMA 2.1 (see [1]). *Let $f : X \rightarrow S$ be an isovariant map of G -spaces. Then, the map $h : X \rightarrow S \times (X/G)$, defined by $h(x) = (f(x), p(x))$ where $p : X \rightarrow X/G$ is the orbit map, is a G -homeomorphic embedding.*

We also recall the well-known and important definition of a slice [17, page 27].

DEFINITION 2.2. A subset S of a G -space X is called an H -slice in X if

- (1) S is H -invariant, that is, $H(S) = S$,
- (2) the saturation $G(S)$ is open in X ,
- (3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$,
- (4) S is closed in $G(S)$.

The saturation $G(S)$ will be said to be an H -tube. If, in addition, $G(S) = X$, then we say that S is a global H -slice in X .

If S is a global H -slice in X , then X is G -homeomorphic to the so-called *twisted product* $G \times_H S$. Recall that $G \times_H S$ is just the H -orbit space of the product $G \times S$ on which H acts by the rule $h(g, s) = (gh^{-1}, hs)$, where $h \in H$ and $(g, s) \in G \times S$. In turn, G acts on $G \times_H S$ by the formula $g'[g, s] = [g'g, s]$, where $g' \in G$, $[g, s] \in G \times_H S$ (see [4, Section 4]).

One of the basic results of the theory of topological transformation groups is the Slice theorem, which asserts the following: if X is a G -space and $x \in X$, then there exists a G_x -slice $S \subset X$ containing the point x (see, e.g., [17, Theorem 1.7.18] or [4, Chapter II, Theorem 5.4]).

An important consequence of the Slice theorem is that if X is a G -space with the orbits all of the same type, then the orbit map $X \rightarrow X/G$ is a locally trivial fibration [4, Chapter II, Theorem 5.8].

In what follows, \cong_G will mean “*is G -homeomorphic*.”

We write $\tilde{X} = X/G$ for the orbit space of X .

The following definition is due to Jaworowski [12] even for G -spaces of finitely many orbit types.

DEFINITION 2.3. We say that a G -space X with a single orbit type (H) is of *finite structure* if the orbit map $p : X \rightarrow \tilde{X}$ has a finite trivializing cover, that is to say, there exists a finite open cover $\{U_1, \dots, U_n\}$ of \tilde{X} such that each $p^{-1}(U_i)$ is G -equivalent to $(G/H) \times U_i$, that is, there exists a G -homeomorphism $f_i : p^{-1}(U_i) \rightarrow (G/H) \times U_i$ such that $\pi(f_i(x)) = p(x)$ for every $x \in p^{-1}(U_i)$.

Here, we remark that the claim “ $p : X \rightarrow \tilde{X}$ has a finite trivializing cover” is equivalent to “ X can be covered by finitely many H -tubes.” Namely, in this form, we will use the definition in what follows.

It is evident from Definition 2.3 that any invariant subspace of a G -space of finite structure is again a G -space of finite structure.

3. G -compactifications of a single orbit type. Recall that the cone $\text{con}(X)$ over a compact metric space X is the quotient set $[0, 1] \times X / \{0\} \times X$ equipped with the quotient topology. This topology is metrizable too (see [10, Chapter VI, Lemma 1.1]). The image of the point $(t, x) \in [0, 1] \times X$ under the canonical projection $p : [0, 1] \times X \rightarrow \text{con}(X)$ will be denoted by tx , and we will simply

write θ (think of zero) instead of $0x$; this is the vertex of the cone. It is convenient to call the number t in tx the norm of tx and denote it by $\|tx\|$.

If X_1, \dots, X_k are compact metric spaces, the join $X_1 * \dots * X_k$ is defined to be the subset of the product $\text{con}(X_1) \times \dots \times \text{con}(X_k)$ consisting of all those points (t_1x_1, \dots, t_kx_k) for which $\sum_{i=1}^k t_i = 1$. Below, we will consider the case when $X_1 = \dots = X_k = G/H$, where H is a closed subgroup of G . In this case, G acts coordinatewise on the k -fold join $G/H * \dots * G/H$ by left translations; so, $G/H * \dots * G/H$ is a G -space, which we will denote shortly by $(G/H)^{*k}$.

In what follows, by a Euclidean G -space, we mean a real Euclidean space E on which G acts by means of orthogonal transformations.

It is convenient to introduce the following notion that is closely related to the notion of the finite structure introduced by Jaworowski (see [Section 2](#)).

DEFINITION 3.1. We say that a G -space X is of Euclidean type if there exists an isovariant map $f : X \rightarrow E$ into a Euclidean G -space E .

In [12], Jaworowski proved that each normal G -space of finite structure is of Euclidean type. Here, we need the following more precise version of Jaworowski's result.

LEMMA 3.2. *Any normal G -space X of a single orbit type (H) and of finite structure admits an isovariant map into a finite-dimensional, compact, metrizable G -space D of type (H) .*

PROOF. It is known that, under the conditions of the lemma, the orbit map $p : X \rightarrow \tilde{X}$ is a locally trivial fibration (see [4, Chapter II, Theorem 5.8]).

Let $\{U_1, U_2, \dots, U_k\}$ be a finite open cover of the orbit space \tilde{X} such that, for every $1 \leq n \leq k$, $p^{-1}(U_n)$ is equivariantly homeomorphic to the product $G/H \times U_n$, where the group G acts on the left on G/H and acts trivially on U_n .

Further, for each $n \geq 1$, the first projection of the product $p^{-1}(U_n) = G/H \times U_n$ gives an isovariant map $\varphi_n : p^{-1}(U_n) \rightarrow G/H$.

Since the orbit space \tilde{X} is normal, there exists a closed shrinking $\{F_1, \dots, F_k\}$ for $\{U_1, U_2, \dots, U_k\}$ in \tilde{X} , that is, $F_n \subset U_n$ for all $1 \leq n \leq k$ and $\bigcup_{n=1}^k F_n = \tilde{X}$ [9, Theorem 1.7.8].

Let $\psi_n : \tilde{X} \rightarrow [0, 1]$ be a continuous function such that $F_n \subset \psi_n^{-1}(1)$ and $\tilde{X} \setminus U_n \subset \psi_n^{-1}(0)$. Now, we define a map $f_n : X \rightarrow \text{con}(G/H)$ by the formula

$$f_n(x) = \begin{cases} \theta & \text{if } x \notin p^{-1}(U_n), \\ \psi_n(p(x))\varphi_n(x) & \text{if } x \in p^{-1}(U_n). \end{cases} \quad (3.1)$$

It is clear that f_n is an equivariant map, and that its restriction to $p^{-1}(F_n)$ coincides with φ_n and is, therefore, isovariant. We consider the diagonal product

$$f = \Delta_{n=1}^k f_n : X \rightarrow (\text{con}(G/H))^k. \quad (3.2)$$

Then, f is an equivariant map. Since $\sum_{n=1}^k \|f_n(x)\| = 1$, for all $x \in X$, we conclude that $f(x)$ belongs, in fact, to the k -fold join $(G/H)^{*k}$.

If $x \in X$ and $x \in p^{-1}(F_n)$, then

$$G_{f(x)} = \bigcap_{k=1}^{\infty} G_{f_k(x)} \subset G_{f_n(x)} = G_x. \quad (3.3)$$

On the other hand, $G_x \subset G_{f(x)}$ since f is equivariant. Consequently, $G_x = G_{f(x)}$, that is, f is an isovariant map.

Now, we define D to be the closure of $f(X)$ in $(G/H)^{*k}$. Then, D is finite-dimensional, compact and metrizable. It remains to see that D has the orbit type (H) . Let $d \in D$ be an arbitrary point. Since each orbit type in $(G/H)^{*k}$ is $\leq (H)$, we see that $(G_d) \leq (H)$. On the other hand, since $f(X)$ is dense in D and $f(X)$ is of type (H) , it follows from the Slice theorem [4, Chapter II, Corollary 5.5] that $(H) \leq (G_d)$. Thus, $(G_d) = (H)$, and hence $f : X \rightarrow D$ is the desired map. \square

THEOREM 3.3. *For a G -space X of a single orbit type (H) , the following are equivalent:*

- (1) X has a G -compactification of type (H) ,
- (2) X has an isovariant map in a finite-dimensional, compact metrizable G -space D of type (H) ,
- (3) X is of Euclidean type,
- (4) X has an isovariant map in a compact G -space of type (H) ,
- (5) X has a G -compactification of type (H) and of the same weight w_X .

PROOF. (1) \Rightarrow (2). Let $b_G X$ be a G -compactification of X of type (H) . By Lemma 3.2, there is an isovariant map $\varphi : b_G X \rightarrow D$ in some finite-dimensional, compact metrizable G -space D of type (H) . The restriction $\varphi|_X$ is the desired map.

(2) \Rightarrow (3). Let $\varphi : X \rightarrow D$ be an isovariant map in a finite-dimensional, compact metrizable G -space D of type (H) . Since there exists an equivariant embedding $i : D \rightarrow E$ in a Euclidean G -space E (see [4, Chapter II, Theorem 10.1]), the composition $f = \varphi i$ maps X isovariantly into E .

(3) \Rightarrow (4). Let $\psi : X \rightarrow E$ be an isovariant map in a Euclidean G -space E . Then, by Lemmas 3.2 and 3.6, there exists an isovariant map $j : \psi(X) \rightarrow D$ in a finite-dimensional, compact metrizable G -space D of type (H) . Then, the composition $j\psi : X \rightarrow D$ is the required map.

(4) \Rightarrow (1). Let $f : X \rightarrow Y$ be an isovariant map in a compact G -space Y of type (H) .

Let $p : X \rightarrow X/G$ be the orbit map. By Lemma 2.1, the diagonal product $i = \varphi \Delta p : X \rightarrow Y \times (X/G)$ is an equivariant embedding.

Let B be any compactification of the orbit space X/G . Then, X can be regarded as an invariant subset of the compact G -space $Y \times B$, where G acts on B trivially. Now, the closure \bar{X} of X in $Y \times B$ is a G -compactification of X . Since

Y is of type (H) , we see that $Y \times B$ is also of type (H) . Hence, $b_G X = \overline{X}$ is a G -compactification of X of type (H) .

(2) \Rightarrow (5) can be proved like the implication (4) \Rightarrow (1) using D instead of Y . In that case, if we choose the compactification B of X/G such that $w(B) = w(X/G)$ [8, Theorem 3.5.2], then the G -compactification $b_G X$ will have the weight $w(b_G X) = w(X/G)$ because $w(D) = \aleph_0$. It remains only to observe that $w(X) = w(X/G)$.

(5) \Rightarrow (1) is evident. \square

Recall that a paracompact space X is said to be *finitistic* if every open cover of X has a refinement ω of a finite order, that is, there is a natural number n such that any point $x \in X$ can belong at most to n elements of ω (see [19]).

Evidently, each compact space, as well as each paracompact finite-dimensional space, is finitistic.

A wide class of G -spaces that admit G -compactifications of a single orbit type is provided by the following theorem.

THEOREM 3.4. *Every finitistic G -space X of type (H) has a G -compactification $b_G X$ of the same type (H) and of the same weight wX .*

For the proof, we need the following result, which was established first by Milnor for finite-dimensional spaces (cited in [17, Theorem 1.8.2]).

LEMMA 3.5. *Let X be a finitistic space and let $\{U_\alpha\}$ be an open covering of X . Then, there exist a natural number n and an open covering $\{V_{i\beta}\}_{\beta \in B_i}$, $i = 0, \dots, n$, of X refining $\{U_\alpha\}$ such that $V_{i\beta} \cap V_{i\beta'} = \emptyset$ whenever $\beta \neq \beta'$ and $0 \leq i \leq n$.*

PROOF. As X is finitistic, there are a natural number n and a refinement $\{W_\mu\}$ of $\{U_\alpha\}$ such that the order of the cover $\{W_\mu\}$ is at most n . Let $\{\varphi_\mu\}$ be a locally finite partition of unity with $\varphi_\mu^{-1}((0, 1]) \subset W_\mu$. For every $0 \leq i \leq n$, let B_i be the set of all subsets β of the indexing set of the cover $\{W_\mu\}$ with cardinality $|\beta| = i + 1$. Given $\beta = (\mu_0, \dots, \mu_i) \in B_i$, we set

$$V_{i\beta} = \{x \in X \mid \varphi_{\mu_j}(x) > 0 \text{ and } \varphi_\mu(x) < \varphi_{\mu_j}(x) \ \forall 0 \leq j \leq i, \mu \notin \beta\}. \quad (3.4)$$

As in a neighborhood of any point x , only a finite number of φ_μ is not identically zero, and it follows that each $V_{i\beta}$ is open.

Let us check that $V_{i\beta} \cap V_{i\beta'} = \emptyset$ if $\beta \neq \beta'$. Indeed, since $|\beta| = i + 1 = |\beta'|$ and $\beta \neq \beta'$, we infer that there are $\mu \in \beta \setminus \beta'$ and $\mu' \in \beta' \setminus \beta$. Now, if $x \in V_{i\beta} \cap V_{i\beta'}$, it then follows that $\varphi_{\mu(x)} < \varphi_{\mu'}(x) < \varphi_\mu(x)$, a contradiction.

Check that $\{V_{i\beta}\}$ is a covering for X . If $x \in X$ and μ_0, \dots, μ_m are all the indices with $\varphi_{\mu_k}(x) > 0$ so arranged that

$$\varphi_{\mu_0}(x) = \varphi_{\mu_1}(x) = \dots = \varphi_{\mu_i}(x) > \varphi_{\mu_{i+1}}(x) \geq \dots \geq \varphi_{\mu_m}(x), \quad (3.5)$$

then, evidently, $x \in V_{m\beta}$, where $\beta = \{\mu_0, \dots, \mu_m\}$. Since

$$x \in \bigcap_{j=0}^m \text{supp } \varphi_{\mu_j} \subset \bigcap_{j=0}^m W_{\mu_j} \quad (3.6)$$

and $\{W_\mu\}$ has order $\leq n$, it follows that $m \leq n$. Consequently, $i \leq n$, and, clearly, $x \in V_{i\{\mu_0, \dots, \mu_i\}}$. Thus, $\{V_{i\beta}\}_{\beta \in B_i}$, $i = 0, \dots, n$, is an open cover of X , and since $V_{i\beta} \subset W_\mu$ for every $\mu \in i$, we see that $\{V_{i\beta}\}$ refines the cover $\{W_\mu\}$, and hence, the original cover $\{U_\alpha\}$. Thus, $\{V_{i\beta}\}$ is the desired cover. \square

LEMMA 3.6. *Every finitistic G -space X having only one orbit type is of finite structure.*

PROOF. Let (H) be the only orbit type of X . Let $\{S_\alpha\}$ be a family of H -slices in X such that $X = \bigcup G(S_\alpha)$. Then, $G(S_\alpha) \cong_G (G/H) \times p(S_\alpha)$ and the sets $p(G(S_\alpha)) = p(S_\alpha)$ constitute an open cover of the orbit space X/G . Now, by [6], X/G is also finitistic, so, by the preceding lemma, we can find a natural number n and an open cover $\{\tilde{U}_{i\beta}\}_{\beta \in B_i}$, $i = 0, \dots, n$ of X/G which refines $\{p(S_\alpha)\}$ and is such that $\tilde{U}_{i\beta} \cap \tilde{U}_{i\beta'} = \emptyset$ if $\beta \neq \beta'$. Then, the set $U_{i\beta} = p^{-1}(\tilde{U}_{i\beta})$ is an H -slice in $\tilde{U}_{i\beta}$, that is, $G(U_{i\beta}) \cong_G (G/H) \times \tilde{U}_{i\beta}$ [17, Proposition 1.7.2], and $\tilde{U}_{i\beta} = p(U_{i\beta})$. It then follows that the union $U_i = \bigcup_{\beta \in B_i} U_{i\beta}$ is an H -slice over $\tilde{U}_i = \bigcup_{\beta \in B_i} \tilde{U}_{i\beta}$ (see [17, Proposition 1.7.3]). Thus, $G(U_i) \cong_G (G/H) \times \tilde{U}_i$, and hence $\{\tilde{U}_i\}_{i=1}^n$ is a finite trivializing cover for X/G . \square

PROOF OF THEOREM 3.4. It follows from Lemmas 3.6 and 3.2 that X is of Euclidean type. Now, the claim follows from Theorem 3.3. \square

PROPOSITION 3.7. *If a G -space X of type (H) admits a G -compactification $b_G X$ of the same type (H) , then its maximal G -compactification $\beta_G X$ is also of the same type (H) .*

PROOF. Indeed, there exists a G -map $f : \beta_G X \rightarrow b_G X$. Hence, $(G_t) \leq (G_{f(t)}) = (H)$ for every $t \in \beta_G X$. On the other hand, since X is dense in $\beta_G X$ and X is of type (H) , it follows from the Slice theorem that $(H) \leq (G_t)$ for every $t \in \beta_G X$ (see [4, Chapter II, Corollary 5.5]). Thus, $(G_t) = (H)$ for every $t \in \beta_G X$. \square

The following is an example of a free \mathbb{Z}_2 -action on the Hilbert cube with a removed point, which does not have a free \mathbb{Z}_2 -compactification.

EXAMPLE 3.8 (see [12]). Let $X = [-1, 1]^\infty \setminus \{0\}$, where $0 = (0, 0, \dots) \in [-1, 1]^\infty$, and $G = \mathbb{Z}_2$, the cyclic group of order two. So, X is the Hilbert cube with a removed point. Consider the free action of \mathbb{Z}_2 on X defined by the standard involution $\{x_i\} \rightarrow \{-x_i\}$. It turns out that the free \mathbb{Z}_2 -space X does not have a free \mathbb{Z}_2 -compactification.

Assume the contrary. Then, by Theorem 3.3, there exists an isovariant map $f : X \rightarrow E$ in a Euclidean \mathbb{Z}_2 -space. Since X is a free \mathbb{Z}_2 -space, $f^{-1}(0) = \emptyset$, where 0 denotes the origin of E . Clearly, the radial retraction $r : E \setminus \{0\} \rightarrow S$ onto the

unit sphere of E is an isovariant map. Hence, the composition $\varphi = rf : X \rightarrow S$ is isovariant too.

Let S^k be a sphere of arbitrary dimension $k > 0$, considered as a G -space with the antipodal action of \mathbb{Z}_2 .

CLAIM 1. Each sphere S^k can \mathbb{Z}_2 -equivariantly be embedded into the \mathbb{Z}_2 -space X .

Indeed, it suffices to show that the \mathbb{Z}_2 -maps from S^k to $[-1, 1]$ separate points of S^k . Let $a, b \in S^k$, $a \neq b$. If $b = -a$, then we first choose a continuous map $f : S^k \rightarrow [-1, 1]$ with $f(a) = 1$ and $f(b) = -1$ and then define $f'(x) = (f(x) - f(-x))/2$, $x \in S^k$. Clearly, f' is a \mathbb{Z}_2 -map with $f'(a) = 1$ and $f'(b) = -1$. If $b \neq -a$, then we first choose a continuous map $f : S^k \rightarrow [-1, 1]$ with $f(a) = f(-b) = 1$ and $f(b) = f(-a) = -1$ and then define $f'(x) = (f(x) - f(-x))/2$, $x \in S^k$. Clearly, f' is a \mathbb{Z}_2 -map with $f'(a) = 1$ and $f'(b) = -1$.

Now, by [Claim 1](#) there exists a G -embedding $i : S^k \hookrightarrow X$. The composition $q = \varphi i : S^k \rightarrow S$ is then an equivariant (i.e., an antipodal) map. But, according to the classical Borsuk-Ulam theorem (see, e.g., [18, Chapter 5, Section 8, Corollary 8]), there is no such a map for $k > \dim S$.

This example also has the following interesting property in spirit of Douwen's paper [20].

COROLLARY 3.9. *Let $f : X \rightarrow X$ be the standard involution on the Hilbert cube with a removed point ([Example 3.8](#)). Then, the Stone-Ćech compactification $\beta f : \beta X \rightarrow \beta X$ has a fixed point.*

PROOF. Indeed, otherwise βX is a free \mathbb{Z}_2 -compactification of X , which contradicts the claim of [Example 3.8](#). \square

4. Universal finite-dimensional compact free G -spaces. In this section, we prove the following theorem.

THEOREM 4.1. *For every infinite cardinal number τ and for every nonnegative integer $n \geq \dim G$, there exists a compact free G -space \mathcal{F}_τ^n with $w(\mathcal{F}_\tau^n) = \tau$, $\dim(\mathcal{F}_\tau^n) = n$ which is universal in the following sense: \mathcal{F}_τ^n contains a G -homeomorphic copy of any free G -space X of Euclidean type with $wX \leq \tau$ and $\dim \beta_G X \leq n$. In particular, \mathcal{F}_τ^n contains a G -homeomorphic copy of each paracompact free G -space X with $wX \leq \tau$ and $\dim X \leq n$.*

We notice that a similar result for the nonfree case was established earlier in [13].

Before proceeding with the proof, we will establish the following lemma.

LEMMA 4.2. *Let X be a paracompact free G -space. Then, the following two properties are fulfilled:*

- (1) $\dim X = \dim(X/G) + \dim G$;
- (2) $\dim \beta_G X = \dim X$.

PROOF. (1) Let $p : X \rightarrow X/G$ be the orbit map. It is well known [4, Chapter II, Theorem 5.8] that p is a locally trivial fibration with fibers homeomorphic to G . Let $\{U_\alpha\}$ be an open trivializing cover of the orbit space X/G , that is, $p^{-1}(U_\alpha) \cong_G G \times U_\alpha$. By compactness of G , the map p is closed, and by a theorem of E. Michael [8, Theorem 5.1.13], the orbit space X/G is paracompact, too. Then, there exists a locally finite closed cover $\{F_\alpha\}$ of X/G such that $F_\alpha \subset U_\alpha$ for each index α . It follows that $p^{-1}(F_\alpha) \cong_G G \times F_\alpha$ and the family $\{p^{-1}(F_\alpha)\}$ constitute a locally finite closed cover of X . Then, according to the Sum theorem [9, Theorem 3.1.10], $\dim X = \max_\alpha \{\dim p^{-1}(F_\alpha)\}$. But $\dim p^{-1}(F_\alpha) = \dim(G \times F_\alpha)$. Being a closed subset of a paracompact space, F_α is itself paracompact. On the other hand, G is a polyhedron. Hence, Morita's theorem [16] is applicable here and, accordingly this logarithmic law holds true: $\dim(G \times F_\alpha) = \dim G + \dim F_\alpha$. Thus, we have $\dim X = \dim G + \max_\alpha \{\dim F_\alpha\}$. Applying once more the sum theorem, we get $\dim(X/G) = \max_\alpha \{\dim F_\alpha\}$. Consequently, $\dim X = \dim(X/G) + \dim G$.

(2) We will use the formula $\beta(X/G) = (\beta_G X)/G$ (see [3]). Consider two cases.

(a) Let $\dim X < \infty$. Then, X has finite structure (Lemma 3.6) and then $\beta_G X$ is a free G -space (Proposition 3.7). Applying twice the equality established in the previous step, we get

$$\begin{aligned} \dim \beta_G X &= \dim(\beta_G X)/G + \dim G \\ &= \dim \beta(X/G) + \dim G \\ &= \dim(X/G) + \dim G \\ &= \dim X. \end{aligned} \tag{4.1}$$

(b) Let $\dim X = \infty$. By Claim 1, we have $\dim X = \dim G + \dim(\beta_G X)/G$, which implies that $\dim(\beta_G X)/G = \infty$. But the orbit map does not rise dimension [6]; in particular,

$$\dim \beta_G X = \dim(\beta_G X)/G = \infty = \dim X. \tag{4.2}$$

□

The following lemma in the nonfree case was proved by Megrelishvili [13] even for noncompact acting groups.

LEMMA 4.3. *Let $f : X \rightarrow Y$ be a G -map of a compact free G -space X into a compact G -space Y . Then, there exist a compact free G -space Z and G -maps $\varphi : X \rightarrow Z$, $\psi : Z \rightarrow Y$ such that $f = \psi\varphi$ and $\dim Z \leq \dim X$, $wZ \leq wY$.*

PROOF. We will first prove the claim in case when Y is a free G -space, too. Consider the induced map $f' : X/G \rightarrow Y/G$. By Mardešić's factorization theorem [9, Theorem 3.3.2], there exist a compact space Z' and continuous maps $\varphi' : X/G \rightarrow Z'$, $\psi' : Z' \rightarrow Y/G$ such that $f' = \psi'\varphi'$ and $\dim Z' \leq \dim(X/G)$, $wZ' \leq w(Y/G)$.

Denote by p the orbit map $Y \rightarrow Y/G$. It is well known [11, Chapter IV, Proposition 4.1] that we have the following (pullback) commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & Y \\ \downarrow \pi & & \downarrow p \\ Z' & \xrightarrow{\psi'} & Y/G \end{array} \quad (4.3)$$

where Z is a compact G -space with $Z/G=Z'$, $\pi : Z \rightarrow Z'$ —the orbit map and ψ —an equivariant map that induces the map ψ' . In fact, Z is the G -invariant subset of $Z' \times Y$ defined as follows: $Z = \{(z', y) \mid \psi'(z') = p(y)\}$, where G acts on $Z' \times Y$ by $g(z', y) = (gz', y)$ for $g \in G$ and $(z', y) \in Z' \times Y$. Thus, Z is a compact free G -space and $\psi : Z \rightarrow Y$ is the restriction of the second projection $Z' \times Y \rightarrow Y$.

Now, we define $\varphi : X \rightarrow Z$ by $\varphi(x) = (\varphi'q(x), f(x))$, where $q : X \rightarrow X/G$ is the orbit map. It is easy to check that $f = \psi\varphi$.

On the other hand, $wZ = wZ' = w(Y/G) = wY$.

Let us check that $\dim Z \leq \dim X$. As Z is a paracompact free G -space, we can apply Lemma 4.2, according to which $\dim Z = \dim Z' + \dim G \leq \dim(X/G) + \dim G = \dim X$.

Now we pass to the general case. By Lemma 3.2, there is an isovariant map $h : X \rightarrow D$ to a compact free G -space D . Consider the product $T = h(X) \times Y$ and the map $r : X \rightarrow T$ defined by $r(x) = (h(x), f(x))$, $x \in X$. Since X is free and h is isovariant, we infer that T is a free G -space. It is clear that r is equivariant and $wT = wY$. Now, we apply the preceding case, according to which there exist a compact G -space Z and G -maps $\varphi : X \rightarrow Z$, $\psi_1 : Z \rightarrow T$ such that $\dim Z \leq \dim X$, $wZ \leq wT$ and $r = \psi_1\varphi$. Observe that $wT = wY$ because $wh(X) = \mathfrak{N}_0$; so, $wZ \leq wY$. Put $\psi = \pi_2\psi_1$, where $\pi_2 : T \rightarrow Y$ is the second projection. Then, $\psi : Z \rightarrow Y$ is a G -map such that $f = \psi\varphi$. It remains to observe that Z is a free G -space; this is immediate from the equivariance of ψ_1 and from the freeness of T . \square

PROOF OF THEOREM 4.1. Let B_τ be a universal Tychonoff G -cube of weight τ (see [3]), that is, B_τ is a G -space homeomorphic to the Tychonoff cube $[0, 1]^\tau$ and contains a G -homeomorphic copy of every G -space of weight $\leq \tau$. Let $\{Y_t\}_{t \in T}$ be the family of all invariant free subsets $Y_t \subset B_\tau$ of Euclidean type such that $\dim \beta_G Y_t \leq n$. This family is nonempty because the group G belongs to it. For each $t \in T$, we denote by i_t the identical embedding of Y_t into B_τ . Consider the discrete sum $Y = \bigoplus_{t \in T} \beta_G Y_t$, which naturally becomes a G -space. By Proposition 3.7, each $\beta_G Y_t$ is a free G -space. Consequently, Y is a paracompact free G -space. As $\dim \beta_G Y_t \leq n$ for all $t \in T$, then, by the Sum theorem [9, Theorem 3.1.10], we have $\dim Y \leq n$. Consequently, by Lemma 4.2, $\dim \beta_G Y = \dim Y \leq n$.

Next, each map $i_t : Y_t \rightarrow B_\tau$ can be extended to a G -map $i'_t : \beta_G Y_t \rightarrow B_\tau$ (see [17, Section 5]); so, a map $i : Y \rightarrow B_\tau$ arises defined by $i(y) = i'_t(y)$ for $y \in \beta_G Y_t$. Applying once more [17, Section 5], we extend i to a G -map $j : \beta_G Y \rightarrow B_\tau$. As Y has finite structure according to Proposition 3.7, $\beta_G Y$ is a compact free G -space. By virtue of Lemma 4.3, there exist a compact free G -space \mathcal{F}_τ^n and G -maps $\varphi : \beta_G Y \rightarrow \mathcal{F}_\tau^n$, $\psi : \mathcal{F}_\tau^n \rightarrow B_\tau$ such that $i = \psi\varphi$ and $\dim \mathcal{F}_\tau^n \leq n$, $w\mathcal{F}_\tau^n \leq wB_\tau = \tau$. We claim that \mathcal{F}_τ^n is the desired G -space.

Indeed, let X be an arbitrary free G -space such that $\dim X \leq n$ and $wX \leq \tau$. Since X is equivariantly embeddable in B_τ , there exists a $t \in T$ such that Y_t is G -homeomorphic to X . As the restriction of i on Y_t is a homeomorphism, the restriction $\varphi|_{Y_t}$ is also a homeomorphism. Besides, $\varphi|_{Y_t}$ is equivariant. Thus, X is equivariantly embeddable in \mathcal{F}_τ^n .

If X is paracompact, then, by Lemma 4.2, $\dim \beta_G X = \dim X \leq n$, and hence X can be embedded equivariantly in \mathcal{F}_τ^n .

To complete the proof, it remains to see that $\dim \mathcal{F}_\tau^n = n$ and $w\mathcal{F}_\tau^n = \tau$. As \mathcal{F}_τ^n contains an equivariant homeomorphic copy of the n -dimensional, compact free G -space $G \times I^k$ with $k = n - \dim G$, we infer that $\dim \mathcal{F}_\tau^n = n$. On the other hand, the discrete sum Z of τ many copies of G is a metrizable free G -space of weight $wZ = \tau$, and hence \mathcal{F}_τ^n contains an equivariant homeomorphic copy of Z . This yields that $w\mathcal{F}_\tau^n = \tau$. \square

From Theorem 4.1, the following corollary follows immediately.

COROLLARY 4.4. *Any paracompact free G -space X has a free G -compactification $b_G X$ of weight $w(b_G X) \leq wX$ and of dimension $\dim b_G X \leq \dim X$.*

COROLLARY 4.5. *Let G be a finite group. Then, for any integer $n \geq 0$, there is a free action of G on the Menger compactum μ^n such that every separable, metrizable, free G -space X with $\dim X \leq n$ admits an equivariant embedding into μ^n .*

PROOF. By the preceding corollary, X has a compact, metrizable, free G -compactification $b_G X$ of $\dim b_G X \leq \dim X$. It remains to apply Dranishnikov's result [7, Corollary and Theorem 3] to the effect that there is a unique free action of G on the Menger compactum μ^n such that μ^n contains an equivariant homeomorphic copy of each compact, metrizable, free G -space of dimension less than or equal to n . \square

5. The case of G -spaces of a single orbit type. In this section, we generalize Theorem 4.1 to the case of G -spaces of Euclidean type that may not be free, but have a single orbit type.

Let H be a closed subgroup of G and X be a G -space of type (H) . Let $N(H)$ be the normalizer of H in G and $W(H) = N(H)/H$, the Weyl group. Below, for any $n \in N(H)$, we denote by \tilde{n} the lateral class nH . The group $W(H)$ acts freely on X^H , the H -fixed point set of X . At the same time, $W(H)$ acts on G/H by the

formula

$$\tilde{n} * gH = gn^{-1}H, \quad \tilde{n} \in W(H), \quad gH \in G/H. \quad (5.1)$$

The twisted product $(G/H) \times_{W(H)} X^H$ is just the $W(H)$ -orbit space of the product $G/H \times X^H$ endowed with the diagonal action of $W(H)$. It is well known (see [4, Chapter II, Corollary 5.11]) that X is G -homeomorphic to the G -space $(G/H) \times_{W(H)} X^H$, equipped with the action of G given by the formula

$$g' * [gH, x] = [g'gH, x], \quad g' \in G, \quad [gH, x] \in (G/H) \times_{W(H)} X^H. \quad (5.2)$$

LEMMA 5.1. *If H is a closed subgroup of G and Y is a free $W(H)$ -space, then the twisted product $T = (G/H) \times_{W(H)} Y$ has only one orbit type (H) . Besides, $wT = wY$ and $\dim T = \dim Y + \dim(G/N(H))$.*

PROOF. Indeed, let $[gH, x]$ be a point of $(G/H) \times_{W(H)} Y$ fixed under an element $g' \in G$. Then, $[g'gH, x] = [gH, x]$ or, equivalently, $(g'gH, x) = (\tilde{n} * gH, \tilde{n}x)$, for some $n \in N(H)$. Then, $g'gH = gn^{-1}H$ and $x = \tilde{n}x$. Since $W(H)$ acts freely on Y , the equality $x = \tilde{n}x$ implies that $n \in H$. The equality $g'gH = gn^{-1}H$ yields that $g' = gn^{-1}hg^{-1}$ for some $h \in H$, and hence, $g' \in gHg^{-1}$. Consequently, the stabilizer of $[gH, x]$ is just the group gHg^{-1} , and hence, the G -space $(G/H) \times_{W(H)} Y$ has only one orbit type (H) .

Since $w(G/H) \leq \mathfrak{N}_0$, we see that

$$w((G/H) \times_{W(H)} Y) \leq wY. \quad (5.3)$$

On the other hand, Y is a subset of T , so $wY \leq wT$.

For the second equality, by Lemma 4.2 and by the above quoted Morita's theorem [16], we have

$$\begin{aligned} \dim T &= \dim((G/H) \times_{W(H)} Y) \\ &= \dim((G/H) \times Y) - \dim W(H) \\ &= \dim(G/H) + \dim Y - (\dim N(H) - \dim H) \\ &= \dim Y + \dim G - \dim H - \dim N(H) + \dim H \\ &= \dim Y + \dim G - \dim N(H) \\ &= \dim Y + \dim(G/N(H)). \end{aligned} \quad (5.4)$$

□

THEOREM 5.2. *For every closed subgroup $H \subset G$, every infinite cardinal number τ and for every nonnegative integer $n \geq \dim G$, there exists a compact G -space $\mathcal{F}_\tau^n(H)$ of type (H) with $w(\mathcal{F}_\tau^n(H)) = \tau$, $\dim(\mathcal{F}_\tau^n(H)) = n$ which is universal in the following sense: $\mathcal{F}_\tau^n(H)$ contains a G -homeomorphic copy of any G -space X of Euclidean type and of the single orbit type (H) such that $wX \leq \tau$ and $\dim \beta_G X \leq n$. In particular, $\mathcal{F}_\tau^n(H)$ contains a G -homeomorphic copy of each paracompact G -space X of type (H) with $wX \leq \tau$ and $\dim X \leq n$.*

PROOF. Let $k = n - \dim(N(H)/H)$. Then, we have

$$\begin{aligned}
 k &= n - \dim(G/N(H)) \\
 &= n - \dim G + \dim N(H) \\
 &\geq \dim N(H) - \dim H \\
 &= \dim(N(H)/H) \\
 &= \dim W(H).
 \end{aligned} \tag{5.5}$$

Hence, by [Theorem 4.1](#), there exists a universal compact free $W(H)$ -space \mathcal{F}_τ^k of dimension k and weight τ .

Set $\mathcal{F}_\tau^n(H) = (G/H) \times_{W(H)} \mathcal{F}_\tau^k$. By [Lemma 5.1](#), $\mathcal{F}_\tau^n(H)$ is a compact G -space of the single orbit type (H) . We claim that it is the required one.

Indeed, by [Lemma 5.1](#),

$$\begin{aligned}
 w((G/H) \times_{W(H)} \mathcal{F}_\tau^k) &= w\mathcal{F}_\tau^k = \tau, \\
 \dim \mathcal{F}_\tau^n(H) &= \dim((G/H) \times_{W(H)} \mathcal{F}_\tau^k) \\
 &= \dim \mathcal{F}_\tau^k + \dim(G/N(H)) \\
 &= k + \dim(G/N(H)) = n.
 \end{aligned} \tag{5.6}$$

Now, if X is a G -space with the single orbit type (H) such that $wX \leq \tau$ and $\dim X \leq n$, then, since $X = (G/H) \times_{W(H)} X^H$, it follows from [Lemma 5.1](#) that $w(X^H) \leq \tau$ and $\dim X^H \leq k$.

By [Theorem 4.1](#), there is a $W(H)$ -equivariant embedding $f : X^H \hookrightarrow \mathcal{F}_\tau^k$. Then, the map $F : (G/H) \times_{W(H)} X^H \rightarrow (G/H) \times_{W(H)} \mathcal{F}_\tau^k$, generated by f , is a G -equivariant embedding. We recall that F is defined as follows: $F([gh, x]) = [gh, f(x)]$ for every $[gh, x] \in (G/H) \times_{W(H)} X^H$ (see [[17](#), Theorem 1.7.10]).

It remains only to recall that

$$X = (G/H) \times_{W(H)} X^H, \quad \mathcal{F}_\tau^n(H) = (G/H) \times_{W(H)} \mathcal{F}_\tau^k. \tag{5.7}$$

This completes the proof. \square

From [Theorem 5.2](#), the following corollary follows immediately.

COROLLARY 5.3. *Any paracompact G -space X of a single orbit type (H) has a G -compactification $b_G X$ of the same orbit type (H) such that $w(b_G X) \leq wX$ and $\dim b_G X \leq \dim X$.*

REFERENCES

- [1] S. A. Antonyan, *Retraction properties of an orbit space*, Mat. Sb. (N.S.) **137(179)** (1988), no. 3, 300-318 (Russian), translation in Math. USSR-Sb. **65** (1990), no. 2, 305-321.
- [2] ———, *Based free compact Lie group actions on the Hilbert cube*, Mat. Zametki **65** (1999), no. 2, 163-174 (Russian), translation in Math. Notes **65** (1999), no. 1-2, 135-143.

- [3] S. A. Antonyan and Yu. M. Smirnov, *Universal objects and bicomact extensions for topological groups of transformations*, Dokl. Akad. Nauk SSSR **257** (1981), no. 3, 521–526 (Russian), translation in Soviet Math. Dokl. **23** (1981), no. 2, 279–284.
- [4] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York, 1972.
- [5] J. de Vries, *On the existence of G -compactifications*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 3, 275–280.
- [6] S. Deo and H. S. Tripathi, *Compact Lie group actions on finitistic spaces*, Topology **21** (1982), no. 4, 393–399.
- [7] A. N. Dranishnikov, *Free actions of 0-dimensional compact groups*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 1, 212–228 (Russian), translation in Math. USSR-Izv. **32** (1989), no. 1, 217–232.
- [8] R. Engelking, *General Topology*, Monografie Matematyczne, vol. 60, PWN—Polish Scientific Publishers, Warsaw, 1977.
- [9] ———, *Dimension Theory*, North-Holland Mathematical Library, vol. 19, North-Holland Publishing, Amsterdam, 1978.
- [10] S.-T. Hu, *Theory of Retracts*, Wayne State University Press, Detroit, 1965.
- [11] D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966.
- [12] J. Jaworowski, *An equivariant extension theorem and G -retracts with a finite structure*, Manuscripta Math. **35** (1981), no. 3, 323–329.
- [13] M. G. Megrelishvili, *A factorization theorem and universal compact spaces for G -spaces*, Russian Math. Surveys **38** (1983), no. 6, 125–126.
- [14] ———, *A Tikhonov G -space that does not have compact G -extension and G -linearization*, Russian Math. Surveys **43** (1988), no. 2, 177–178.
- [15] M. G. Megrelishvili and T. Scarr, *Constructing Tychonoff G -spaces which are not G -Tychonoff*, Topology Appl. **86** (1998), no. 1, 69–81.
- [16] K. Morita, *On the dimension of product spaces*, Amer. J. Math. **75** (1953), 205–223.
- [17] R. S. Palais, *The classification of G -spaces*, no. 36, Mem. Amer. Math. Soc., 1960.
- [18] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
- [19] R. G. Swan, *A new method in fixed point theory*, Comment. Math. Helv. **34** (1960), 1–16.
- [20] E. K. van Douwen, *βX and fixed-point free maps*, Topology Appl. **51** (1993), no. 2, 191–195.

Natella Antonyan: Departamento de Matemáticas, Facultad de Ciencias, UNAM, Ciudad Universitaria, 04510, México D.F., Mexico; Departamento de Matemáticas-DIA, Instituto Tecnológico de Monterrey, Calle del Puente 222, Ejidos de Huipulco, Tlalpan, 14380 México D.F., Mexico

E-mail address: antonyan@servidor.unam.mx; nantonya@itesm.mx



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

