EXPRESSION FOR A GENERAL ELEMENT OF AN $SO(n)$ MATRIX

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We derive the expression for a general element of an $SO(n)$ matrix. All elements are obtained from a single element of the matrix. This has applications in recently developed methods for computing Lyapunov exponents.

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1. Introduction. Matrix representations of the $SO(n)$ group have played an important role in mathematical physics [5, 6]. They continue to be used in many fields to this day [4, 7, 8]. They also play a crucial role in new methods for computing Lyapunov exponents [2, 3].

In this paper, we obtain the expression for a general element of an $SO(n)$ matrix $Q^{(n)}$ for $n \geq 3$. This offers significant advantages in generalizing the recent Lyapunov spectrum calculation methods [2, 3] to higher dimensions. We demonstrate that expressions for all elements can be obtained from the expression of a single matrix element by suitable operations. As an example of the application of these results, we derive the elements of an $SO(3)$ matrix in Section 3. The standard expressions are obtained as expected.

2. General element of an $SO(n)$ matrix. In this section, we derive the expression for a general element of an $SO(n)$ matrix denoted by $Q^{(n)}$ (for $n \geq 3$). In all the expressions below, it is implicitly assumed that $n \geq 3$.

We start by deriving the expression for the element $Q_{1n}^{(n)}$. Then we prove that all other elements of $Q^{(n)}$ can be obtained from this single element and give explicit expressions for these elements. This method is based on the representation of the group $SO(n)$ as a product of $n(n-1)/2$ $n \times n$ matrices, which are simple rotations in the $(i-j)$ th coordinates [1].

PROPOSITION 2.1. *An $SO(n)$ matrix $Q^{(n)}$ can be represented as the following product of simple rotations (see [1]):*

$$Q^{(n)} = O^{(1,2)} O^{(1,3)} \dots O^{(1,n)} \dots O^{(n-1,n)}, \quad (2.1)$$

where $O^{(i,j)}$ is given as

$$O_{kl}^{(i,j)} = \begin{cases} 1, & \text{if } k = 1 \neq i, j; \\ \cos \theta_r, & \text{if } k = l = i \text{ or } j; \\ \sin \theta_r, & \text{if } k = i, l = j; \\ -\sin \theta_r, & \text{if } k = j, l = i; \\ 0, & \text{otherwise,} \end{cases} \quad (2.2)$$

where $r = (i-1)(2n-i)/2 + j - i$.

Let

$$\begin{aligned} T^{(1)} &= O^{(1,2)} O^{(1,3)} \dots O^{(1,n)}, \\ T^{(2)} &= O^{(2,3)} O^{(2,4)} \dots O^{(2,n)}, \\ &\vdots \\ T^{(k)} &= O^{(k,k+1)} O^{(k,k+2)} \dots O^{(k,n)}, \\ &\vdots \\ T^{(n-1)} &= O^{(n-1,n)}. \end{aligned} \quad (2.3)$$

We see that the matrix $T^{(1)}$ depends only on the first $(n-1)$ θ_i 's, namely, $\theta_1, \theta_2, \dots, \theta_{n-1}$, and the matrix $T^{(2)}$ depends only on the next $(n-2)$ θ_i 's, namely, $\theta_n, \theta_{n+1}, \dots, \theta_{2n-3}$, and so on. Finally, the matrix $T^{(n-1)}$ depends only on one θ_i , namely, $\theta_{n(n-1)/2}$. Thus, a general matrix $T^{(k)}$ is parameterized by the following θ_i 's, namely, $\theta_{m(n,k)}, \theta_{m(n,k)+1}, \dots, \theta_{p(n,k)}$, where $m(n,k)$ and $p(n,k)$ are given by

$$m(n,k) = \frac{(k-1)(2n-k)+2}{2}, \quad (2.4)$$

$$p(n,k) = \frac{k(2n-k-1)}{2}. \quad (2.5)$$

Therefore,

$$Q^{(n)} = T^{(1)} T^{(2)} \dots T^{(n-1)}. \quad (2.6)$$

The matrix $T^{(k)}$ ($k = 1, 2, \dots, n-1$) is given by

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & R^{(k)} & \\ 0 & & & \end{bmatrix}, \quad (2.7)$$

where $R^{(k)}$ is an $(n-k+1) \times (n-k+1)$ matrix parameterized by $\theta_{m(n,k)+1}, \theta_{m(n,k)+2}, \dots, \theta_{p(n,k)}$, where $m(n,k)$ and $p(n,k)$ are given by (2.4) and (2.5),

respectively. The elements of $R^{(k)}$ are given as follows:

$$R_{11}^{(k)} = \prod_{r=m(n,k)}^{p(n,k)} \cos \theta_r, \quad (2.8)$$

$$R_{12}^{(k)} = \sin \theta_{m(n,k)}, \quad (2.9)$$

and for $j = 3, 4, \dots, n - (k - 1)$,

$$R_{1j}^{(k)} = \left(\prod_{r=0}^{j-3} \cos \theta_{m(n,k)+r} \right) \sin \theta_{m(n,k)+j-2}. \quad (2.10)$$

The second row ($j = 1, 2, \dots, n - (k - 1)$) is given by

$$R_{2j}^{(k)} = \frac{\partial}{\partial \theta_{m(n,k)}} R_{1j}^{(k)}. \quad (2.11)$$

The rest of the rows ($i = 3, 4, \dots, n - (k - 1)$ and $j = 1, 2, \dots, n - (k - 1)$) are given by

$$R_{ij}^{(k)} = \frac{\partial}{\partial \theta_{m(n,k)+i-2}} \mathcal{F}_{ij}^{(k)}, \quad (2.12)$$

where $\mathcal{F}_{ij}^{(k)}$ = Coefficient of $\prod_{r=0}^{i-3} \cos \theta_{m(n,k)+r}$ in $R_{1j}^{(k)}$.

Putting everything together, from (2.6) we have the following lemma.

LEMMA 2.2. *Let $Q^{(n)}$ be an $SO(n)$ matrix ($n \geq 3$). Then the element $Q_{1n}^{(n)}$ is given by the expression*

$$Q_{1n}^{(n)} = \sum_{j_{n-2}=2}^3 \sum_{j_{n-3}=2}^4 \cdots \sum_{j_2=2}^{n-1} \sum_{j_1=2}^n R_{1,j_1}^{(1)} R_{j_1-1,j_2}^{(2)} R_{j_2-1,j_3}^{(3)} \cdots R_{j_{n-2}-1,2}, \quad (2.13)$$

where $j_{n-1} = 2$.

Next, we prove that all other elements of $Q^{(n)}$ can be obtained from the single element $Q_{1n}^{(n)}$ (derived above). To show this, we need some preliminary results contained in Lemmas 2.3 and 2.4 proved below.

LEMMA 2.3. *Consider a general $SO(n)$ matrix $Q^{(n)}$ ($n \geq 3$). The expressions for $Q_{in}^{(n)}$'s, $i = 1, 2, \dots, n - 1$, do not involve the term $\cos \theta_{p(n,1)} (= \cos \theta_{n-1})$ in them.*

PROOF. We can write the matrix $Q^{(n)}$ as

$$Q^{(n)} = R^{(1)} \Gamma \quad (\text{since } T^{(1)} = R^{(1)}), \quad (2.14)$$

where Γ is of the form

$$\Gamma = T^{(2)}T^{(3)} \dots T^{(n-1)} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A^{(n-1)} & \\ 0 & & & \end{bmatrix}. \quad (2.15)$$

Here $A^{(n-1)}$ is a general $\text{SO}(n-1)$ matrix parameterized by $\theta_n, \theta_{n+1}, \dots, \theta_{n(n-1)/2}$. Thus, $Q_{in}^{(n)}$ ($i = 1, 2, \dots, n-1$) is given by

$$Q_{in}^{(n)} = \sum_{k=2}^n R_{ik}^{(1)} A_{k-1, n-1}^{(n-1)}. \quad (2.16)$$

From this equation, we see that $R_{i1}^{(1)}$'s ($i = 1, 2, \dots, n-1$) are absent in the expressions for $Q_{in}^{(n)}$ ($i = 1, 2, \dots, n-1$). Also, by (2.9), (2.10), (2.11), and (2.12), which give the expressions for $R_{ij}^{(k)}$'s, we see that the term $\cos \theta_{n-1}$ is absent in all the $R_{ik}^{(1)}$'s, where $i = 1, 2, \dots, n-1$ and $k = 2, 3, \dots, n$. Finally, $A^{(n-1)}$ is parameterized by $\theta_n, \theta_{n+1}, \dots, \theta_{n(n-1)/2}$ and hence does not contain the term $\cos \theta_{n-1}$. Therefore, $Q_{in}^{(n)}$ ($i = 1, 2, \dots, n-1$) does not involve the term $\cos \theta_{n-1}$. This proves the lemma. $\square$

LEMMA 2.4. For $n \geq 3$, $Q_{nn}^{(n)} = \prod_{k=1}^{n-1} \cos \theta_{p(n,k)}$, where

$$p(n, k) = \frac{k(2n-k-1)}{2}. \quad (2.17)$$

This lemma is easily proved by mathematical induction and hence we omit the proof.

We are now in a position to prove that we can obtain all rows of $Q^{(n)}$ given only the first row.

LEMMA 2.5. Let $Q^{(n)}$ be an $\text{SO}(n)$ matrix ($n \geq 3$). Let $Q_{1i}^{(n)}$, $i = 1, 2, \dots, n$, be its first row. Then the second row is given by the following equation:

$$Q_{2l}^{(n)} = \frac{\partial Q_{1l}^{(n)}}{\partial \theta_1}, \quad l = 1, 2, \dots, n. \quad (2.18)$$

The other rows are given by the following expression:

$$Q_{il}^{(n)} = \frac{\partial \mathcal{B}_{il}^{(n)}}{\partial \theta_{i-1}}, \quad i = 3, 4, \dots, n; \quad l = 1, 2, \dots, n, \quad (2.19)$$

where

$$\mathcal{B}_{il}^{(n)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } Q_{1l}^{(n)}. \quad (2.20)$$

PROOF. A general $SO(n)$ matrix $Q^{(n)}$ is given by

$$Q^{(n)} = T^{(1)}\Gamma, \quad (2.21)$$

where $T^{(1)}$ and Γ are given by (2.3) and (2.15), respectively. The matrix $T^{(1)}$ is parameterized by the following $(n-1)$ θ 's, namely, $\theta_1, \theta_2, \dots, \theta_{n-1}$ while Γ is given by (2.15), where $A^{(n-1)}$ is an $SO(n-1)$ matrix, parameterized by $(n-1)(n-2)/2$ θ 's, namely, $\theta_n, \theta_{n+1}, \dots, \theta_{n(n-1)/2}$. Thus, $Q_{i1}^{(n)}$, $i = 1, 2, \dots, n$, is given by

$$Q_{i1}^{(n)} = R_{i1}^{(1)}. \quad (2.22)$$

Using this equation and (2.11), we obtain

$$Q_{21}^{(n)} = \frac{\partial Q_{11}^{(n)}}{\partial \theta_1}. \quad (2.23)$$

Also, from (2.12), we have

$$R_{i1}^{(1)} = \frac{\partial \mathcal{J}_{i1}^{(1)}}{\partial \theta_{i-1}}, \quad i = 3, 4, \dots, n, \quad (2.24)$$

where (see (2.22) and (2.20))

$$\mathcal{J}_{i1}^{(1)} = \mathcal{B}_{i1}^{(n)}. \quad (2.25)$$

Thus,

$$\frac{\partial \mathcal{B}_{i1}^{(1)}}{\partial \theta_{i-1}} = \frac{\partial \mathcal{J}_{i1}^{(1)}}{\partial \theta_{i-1}} = R_{i1}^{(1)} = Q_{i1}^{(n)}, \quad i = 3, 4, \dots, n. \quad (2.26)$$

Now, for $l = 2, 3, \dots, n$, we have

$$Q_{il}^{(n)} = \sum_{k=2}^n R_{ik}^{(1)} A_{k-1, l-1}^{(n-1)}. \quad (2.27)$$

Putting $i = 1$, we get

$$Q_{1l}^{(n)} = \sum_{k=2}^n R_{1k}^{(1)} A_{k-1, l-1}^{(n-1)}. \quad (2.28)$$

Since $A_{k-1, l-1}^{(n-1)}$'s do not involve the first $(n-1)$ θ 's, namely, $\theta_1, \theta_2, \dots, \theta_{n-1}$, we obtain (for $k = 2, 3, \dots, n$)

$$\frac{\partial}{\partial \theta_1} (R_{1k}^{(1)} A_{k-1, l-1}^{(n-1)}) = R_{2k}^{(1)} A_{k-1, l-1}^{(n-1)}. \quad (2.29)$$

Summing over k ($k = 2, 3, \dots, n$) and using (2.28) and (2.23), we get

$$\frac{\partial}{\partial \theta_1} Q_{1l}^{(n)} = Q_{2l}^{(n)}, \quad l = 1, 2, \dots, n. \quad (2.30)$$

Thus, the second row of $Q^{(n)}$, namely, $Q_{2l}^{(n)}$ ($l = 1, 2, \dots, n$) obeys the hypothesis (2.18). We will now prove the hypothesis for the rest of its rows.

Let

$$\mathcal{J}_{il}^{(1)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } R_{1l}^{(1)}, \quad i = 3, 4, \dots, n, \quad (2.31)$$

$$\mathcal{C}_{ik} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } R_{1k}^{(1)} A_{k-1, l-1}^{(n-1)}, \quad i = 3, 4, \dots, n; \quad k = 2, 3, \dots, n. \quad (2.32)$$

Therefore, (see (2.28) and (2.20))

$$\sum_{k=2}^n \mathcal{C}_{ik} = \mathcal{B}_{il}^{(n)}. \quad (2.33)$$

Since $A_{k-1, l-1}^{(n-1)}$'s do not involve $\theta_1, \theta_2, \dots, \theta_{n-1}$, we have from (2.32)

$$\mathcal{C}_{ik} = A_{k-1, l-1}^{(n-1)} \mathcal{J}_{ik}^{(1)}, \quad (2.34)$$

where $\mathcal{J}_{ik}^{(1)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } R_{1k}^{(1)}$.

Thus, (see (2.12))

$$\frac{\partial \mathcal{C}_{ik}}{\partial \theta_{i-1}} = A_{k-1, l-1}^{(n-1)} \frac{\partial \mathcal{J}_{ik}^{(1)}}{\partial \theta_{i-1}} = A_{k-1, l-1}^{(n-1)} R_{ik}^{(1)}. \quad (2.35)$$

Summing both sides over k ($k = 2, 3, \dots, n$), we obtain

$$\sum_{k=2}^n \frac{\partial \mathcal{C}_{ik}}{\partial \theta_{i-1}} = \sum_{k=2}^n R_{ik}^{(1)} A_{k-1, l-1}^{(n-1)} = Q_{il}^{(n)}. \quad (2.36)$$

But, from (2.33),

$$\sum_{k=2}^n \frac{\partial \mathcal{C}_{ik}}{\partial \theta_{i-1}} = \frac{\partial (\sum_{k=2}^n \mathcal{C}_{ik})}{\partial \theta_{i-1}} = \frac{\partial \mathcal{B}_{il}^{(n)}}{\partial \theta_{i-1}}. \quad (2.37)$$

Thus,

$$\frac{\partial \mathcal{B}_{il}^{(n)}}{\partial \theta_{i-1}} = Q_{il}^{(n)}, \quad (2.38)$$

where $\mathcal{B}_{il}^{(n)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } Q_{1l}^{(n)}$ for $l = 2, 3, \dots, n$.

Combining the above equation with (2.26), we obtain the following:

$$Q_{il}^{(n)} = \frac{\partial \mathcal{B}_{il}^{(n)}}{\partial \theta_{i-1}}, \quad i = 3, 4, \dots, n; \quad l = 1, 2, \dots, n, \quad (2.39)$$

where $\mathcal{B}_{il}^{(n)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } Q_{1l}^{(n)}$. Thus, (2.30) and (2.39) prove the lemma. $\square$

We next prove a result analogous to [Lemma 2.5](#), but for columns instead of rows. Combining [Lemmas 2.5](#) and [2.6](#) will give us the desired result of obtaining all elements of $Q^{(n)}$ from a single element.

LEMMA 2.6. *For $n \geq 3$, given the n th column of $Q^{(n)}$, the $(n-1)$ th column is given by the following expression:*

$$Q_{i,n-1}^{(n)} = \frac{\partial Q_{in}^{(n)}}{\partial \theta_{p(n,n-1)}}, \quad i = 1, 2, \dots, n. \quad (2.40)$$

The other columns are given by

$$Q_{il}^{(n)} = \frac{\partial \mathcal{D}_{il}^{(n)}}{\partial \theta_{p(n,l)}}, \quad i = 1, 2, \dots, n; \quad l = 1, 2, \dots, n-2, \quad (2.41)$$

where $\mathcal{D}_{il}^{(n)} =$ Coefficient of $\prod_{m=l+1}^{n-1} \cos \theta_{p(n,m)}$ in $Q_{in}^{(n)}$.

The proof of this lemma is by induction and is straightforward (though laborious). So we omit the proof.

[Lemma 2.6](#) implies that given the last column of $Q^{(n)}$, we can derive the other columns. In particular, given $Q_{1n}^{(n)}$ ([Lemma 2.2](#)), we can obtain the first row. Once the first row is known, using [Lemma 2.5](#), all other rows can be derived. Therefore, we see that from one element of $Q^{(n)}$, namely, $Q_{1n}^{(n)}$ we can generate the whole $SO(n)$ matrix by performing suitable operations. Thus we have proved the following theorem.

THEOREM 2.7. *Consider an $n \times n$ $SO(n)$ matrix $Q^{(n)}$ ($n \geq 3$). The expression for $Q_{1n}^{(n)}$ is given by*

$$Q_{1n}^{(n)} = \sum_{j_{n-2}=2}^3 \sum_{j_{n-3}=2}^4 \cdots \sum_{j_2=2}^{n-1} \sum_{j_1=2}^n R_{1,j_1}^{(1)} R_{j_1-1,j_2}^{(2)} R_{j_2-1,j_3}^{(3)} \cdots R_{j_{n-2}-1,2}, \quad (2.42)$$

where $j_{n-1} = 2$ and the matrices $R^{(k)}$ are given by [\(2.9\)](#), [\(2.10\)](#), [\(2.11\)](#), and [\(2.12\)](#). All other elements of $Q^{(n)}$ can be derived from this single element. Elements of the first row are given by

$$Q_{1,n-1}^{(n)} = \frac{\partial Q_{1n}^{(n)}}{\partial \theta_{p(n,n-1)}}, \quad (2.43)$$

$$Q_{1l}^{(n)} = \frac{\partial (\mathcal{D}_{1l}^{(n)})}{\partial \theta_{p(n,l)}}, \quad l = 1, 2, \dots, n-2, \quad (2.44)$$

where $\mathcal{D}_{1l}^{(n)} =$ Coefficient of $\prod_{m=l+1}^{n-1} \cos \theta_{p(n,m)}$ in $Q_{1n}^{(n)}$. Elements of the second row are given by

$$Q_{2l}^{(n)} = \frac{\partial Q_{1l}^{(n)}}{\partial \theta_1}, \quad l = 1, 2, \dots, n. \quad (2.45)$$

Elements of the remaining rows are given by

$$Q_{il}^{(n)} = \frac{\partial \mathcal{B}_{il}^{(n)}}{\partial \theta_{i-1}}, \quad i = 3, 4, \dots, n; \quad l = 1, 2, \dots, n, \quad (2.46)$$

where $\mathcal{B}_{il}^{(n)} = \text{Coefficient of } \prod_{r=1}^{i-2} \cos \theta_r \text{ in } Q_{1l}^{(n)}$.

3. Example: SO(3). We will now derive the SO(3) matrix using [Theorem 2.7](#). We will first get the expression for $Q_{13}^{(3)}$ (see [\(2.42\)](#)):

$$Q_{13}^{(3)} = R_{12}^{(1)} R_{12}^{(2)} + R_{13}^{(1)} R_{22}^{(2)}. \quad (3.1)$$

From [\(2.9\)](#) and [\(2.10\)](#), we have

$$R_{12}^{(1)} = \sin \theta_1, \quad R_{13}^{(1)} = \cos \theta_1 \sin \theta_2. \quad (3.2)$$

From [\(2.9\)](#) and [\(2.11\)](#), we get

$$R_{12}^{(2)} = \sin \theta_3, \quad R_{22}^{(2)} = \cos \theta_3. \quad (3.3)$$

Therefore, we obtain

$$Q_{13}^{(3)} = \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3. \quad (3.4)$$

From [\(2.43\)](#), $Q_{12}^{(3)}$ is given as

$$Q_{12}^{(3)} = \frac{\partial Q_{13}^{(3)}}{\partial \theta_3} = \sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3, \quad (3.5)$$

and from [\(2.44\)](#), $Q_{11}^{(3)}$ is given as

$$Q_{11}^{(3)} = \frac{\partial \mathcal{D}_{11}^{(3)}}{\partial \theta_2}, \quad (3.6)$$

where $\mathcal{D}_{11}^{(3)} = \text{Coefficient of } \prod_{m=2}^2 \cos \theta_{p(3,m)} \text{ in } Q_{13}^{(3)}$. Thus,

$$Q_{11}^{(3)} = \cos \theta_1 \cos \theta_2. \quad (3.7)$$

The second row of $Q^{(3)}$ is given by [\(2.45\)](#):

$$Q_{2l}^{(3)} = \frac{\partial Q_{1l}^{(3)}}{\partial \theta_1}, \quad l = 1, 2, 3. \quad (3.8)$$

Therefore,

$$\begin{aligned} Q_{21}^{(3)} &= -\sin \theta_1 \cos \theta_2, \\ Q_{22}^{(3)} &= \cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3, \\ Q_{23}^{(3)} &= \cos \theta_1 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3. \end{aligned} \quad (3.9)$$

The last row is given by (2.46):

$$Q_{3l}^{(3)} = \frac{\partial \mathcal{B}_{3l}^{(3)}}{\partial \theta_2}, \quad l = 1, 2, 3, \quad (3.10)$$

where $\mathcal{B}_{3l}^{(3)}$ = Coefficient of $\prod_{r=1}^1 \cos \theta_r$ in $Q_{1l}^{(3)}$. Therefore, we have

$$\begin{aligned} Q_{31}^{(3)} &= -\sin \theta_2, \\ Q_{32}^{(3)} &= -\cos \theta_2 \sin \theta_3, \\ Q_{33}^{(3)} &= \cos \theta_2 \cos \theta_3. \end{aligned} \quad (3.11)$$

The $Q^{(3)}$ matrix that we have derived agrees with the standard representation as expected.

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