ON HOPF GALOIS HIRATA EXTENSIONS

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Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois and Hirata separable extension of $B^H$. Then $B$ is characterized in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H)\#H$. A sufficient condition is also given for $B$ to be an $H^*$-Galois Azumaya extension of $B^H$.

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1. Introduction. Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois extension of $B^H$. In [3], the class of $H^*$-Galois Azumaya extensions was investigated and in [8], it was shown that $B$ is a Hirata separable extension of $B^H$ if and only if the commutator subring $V_B(B^H)$ of $B^H$ in $B$ is a left $H$-Galois extension of $C$, where $C$ is the center of $B$ (see [8, Lemma 2.1, Theorem 2.6]). The purpose of the present paper is to characterize a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ and the smash product $V_B(B^H)\#H$. Let $B$ be a right $H^*$-Galois extension of $B^H$ such that $B^H = B^H^*$. Then the following statements are equivalent:

1. $B$ is a Hirata separable extension of $B^H$,
2. $V_B(B^H)$ is an Azumaya $C$-algebra and $V_B(V_B(B^H)) = B^H$,
3. $V_B(B^H)$ is a right $H^*$-Galois extension of $C$ and a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule,
4. $V_B(B^H)$ is a right $H^*$-Galois extension of $C$ and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

Moreover, an equivalent condition is given for a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ to be an $H^*$-Galois Azumaya extension which was studied in [3, 7]. Also, let $B$ be a right $H^*$-Galois and Hirata separable extension of $B^H$ and $A$ a subalgebra of $B^H$ over $C$ such that $B^H$ is a projective Hirata separable extension of $A$ containing $A$ as a direct summand as an $A$-bimodule. Then $V_{B^H}(A)$ is a separable subalgebra of $B^H$ over $C$, and there exists an $H$-submodule algebra $D$ in $B$ which is separable over $C$ such that $D^H = V_{B^H}(A)$ and $D \cong V_{B^H}(A) \otimes_Z F$ as Azumaya $Z$-algebras, where $Z$ is the center of $D$ and $F$ is an Azumaya $Z$-algebra in $D$. 
2. Basic definitions and notations. Throughout, $H$ denotes a finite-dimensional Hopf algebra over a field $k$ with comultiplication $\Delta$ and counit $\varepsilon$, $H^*$ the dual Hopf algebra of $H$, $B$ a left $H$-module algebra, $C$ the center of $B$, $B^H = \{ b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H \}$ which is called the $H$-invariants of $B$, and $B \# H$ the smash product of $B$ with $H$, where $B \# H = B \otimes_k H$ such that for all $b \# h$ and $b' \# h'$ in $B \# H$, $(b \# h)(b' \# h') = \sum b(h_1b')\# h_2h'$, where $\Delta(h) = \sum h_1 \otimes h_2$. The ring $B$ is called a right $H^*$-Galois extension of $B^H$ if $B$ is a right $H^*$-comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes B^H B \rightarrow B \otimes B^H$ is a bijection, where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring $A$ of $B$ with the same identity 1, we denote the commutator subring of $A$ in $B$ by $V_B(A)$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i\}$ in $B$, $i = 1, 2, \ldots, m$, for some integer $m$ such that $\sum a_i b_i = 1$ and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$, where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. A ring $B$ is called a Hirata separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule. A right $H^*$-Galois extension $B$ is called an $H^*$-Galois Azumaya extension if $B$ is separable over $B^H$ which is an Azumaya algebra over $C^H$. A right $H^*$-Galois extension $B$ of $B^H$ is called an $H^*$-Galois Hirata extension if $B$ is also a Hirata separable extension of $B^H$. Throughout, an $H^*$-Galois extension means a right $H^*$-Galois extension unless it is stated otherwise.

3. The $H^*$-Galois Hirata extensions. In this section, we will characterize an $H^*$-Galois Hirata extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H) \# H$. A relationship between an $H^*$-Galois Hirata extension and an $H^*$-Galois Azumaya extension is also given. We begin with some properties of an $H^*$-Galois Hirata extension $B$ of $B^H$. Throughout, we assume $B^H = B^{H^*}$.

**Lemma 3.1.** If $A_1$ and $A_2$ are $H^*$-Galois extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.

**Proof.** By [3, Theorem 5.1], there exist $\{x_i, y_i \in A_1 \mid i = 1, 2, \ldots, n\}$ for some integer $n$ such that, for all $h \in H$, $\sum x_i(h y_i) = T(h)1_{A_1}$, where $T \in \mathcal{I}_{H^*}$, the set of right integrals in $H^*$. Let $t \in \mathcal{I}_H$, the set of left integrals in $H$, such that $T(t) = 1$, then $\{x_i, f_i = t(y_i-1) \mid i = 1, 2, \ldots, n\}$ is a dual basis of the finitely generated and projective right module $A_1$ over $A_1^H$. Since $A_1 \subset A_2$ such that $A_1^H = A_2^H$, $\{x_i, f_i \mid i = 1, 2, \ldots, n\}$ is also a dual basis of the finitely generated and projective right module $A_2$ over $A_2^H$. This implies that $A_1 = A_2$. \hfill \Box

**Lemma 3.2.** If $B$ is an $H^*$-Galois Hirata extension of $B^H$, then $B^H$ is a direct summand of $B$ as a $B^H$-bimodule.

**Proof.** We use the argument as given in [2]. Since $B$ is an $H^*$-Galois and a Hirata separable extension of $B^H$, $V_B(B^H)$ is a left $H$-Galois extension of $C$ (see [8, Lemma 2.1, Theorem 2.6]). Hence, $V_B(B^H)$ is a finitely generated and
projective module over $C$ (see [3, Theorem 2.2]). Let $\Omega = \text{Hom}_C(V_B(B^H), V_B(B^H))$. Since $C$ is commutative, $V_B(B^H)$ is a progenerator of $C$. Thus, $B$ is a right $\Omega$-module such that $B \cong V_B(B^H) \otimes_C \text{Hom}_\Omega(V_B(B^H), B) \cong V_B(B^H) \otimes_C B^H$ as $C$-algebras, where $f(1) \in B^H$ for each $f \in \text{Hom}_\Omega(V_B(B^H), B)$ by the proof of [2, Lemma 2.8]. But $V_B(V_B(B^H)) = B^H$ (see [2, Lemma 2.5]), so $B \cong V_B(B^H) \otimes_C B^H$. This implies that $V_B(B^H)$ is an $H^*$-Galois extension of $C$ (see [2, Lemma 2.8]); and so $C$ is a direct summand of $V_B(B^H)$ as a $C$-bimodule (see [2, Corollaries 1.9 and 1.10]). Therefore, $B^H$ is a direct summand of $B$ as a $B^H$-bimodule.

By the proof of Lemma 3.2, $V_B(B^H)$ is an $H^*$-Galois extension of $C$.

**Corollary 3.3.** If $B$ is an $H^*$-Galois Hirata extension of $B^H$, then $V_B(B^H)$ is an $H^*$-Galois extension of $C$.

**Corollary 3.4.** If $B$ is an $H^*$-Galois Hirata extension of $B^H$, then $V_B(B^H) = B^H \cdot V_B(B^H)$ and the centers of $B$, $B^H$, and $V_B(B^H)$ are the same $C$.

**Proof.** By Corollary 3.3, $V_B(B^H)$ is an $H^*$-Galois extension of $C$, so $B^H \cdot V_B(B^H)$ is also an $H^*$-Galois extension of $B^H (= (B^H \cdot V_B(B^H))^H)$ with the same Galois system as $V_B(B^H)$ (see [3, Theorem 5.1]). Noting that $B^H \cdot V_B(B^H) \subseteq B$, we conclude that $B = B^H \cdot V_B(B^H)$ by Lemma 3.1. Moreover, $V_B(V_B(B^H)) = B^H$ (see [8, Lemma 2.5]), so the centers of $B^H$, $V_B(B^H)$, and $B$ are the same $C$.

**Theorem 3.5.** Let $B$ be an $H^*$-Galois extension of $B^H$. The following statements are equivalent:

1. $B$ is a Hirata separable extension of $B^H$,
2. $V_B(B^H)$ is an $H^*$-Galois extension of $C$ and a direct summand of $V_B(B^H)^H$ as a $V_B(B^H)$-bimodule,
3. $V_B(B^H)$ is an Azumaya $C$-algebra and $V_B(V_B(B^H)) = B^H$,
4. $V_B(B^H)$ is an $H^*$-Galois extension of $C$ and $V_B(B^H)^H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

**Proof.** (1)$\Rightarrow$(3). Since $B$ is an $H^*$-Galois and a Hirata separable extension of $B^H$, by Lemma 3.2, $B^H$ is a direct summand of $B$ as a $B^H$-bimodule. Thus, $V_B(V_B(B^H)) = B^H$ and $V_B(B^H)$ is a separable $C$-algebra (see [4, Propositions 1.3 and 1.4]). But the center of $V_B(B^H)$ is $C$ by Corollary 3.4, so $V_B(B^H)$ is an Azumaya $C$-algebra.

(3)$\Rightarrow$(1). Since $V_B(B^H)$ is an Azumaya $C$-algebra and $B$ is a bimodule over $V_B(B^H)$, $B \cong V_B(B^H) \otimes_C V_B(V_B(B^H)) = V_B(B^H) \otimes_C B^H$ as a bimodule over $V_B(B^H)$ (see [1, Corollary 3.6, page 54]). Noting that $B \cong V_B(B^H) \otimes_C B^H$ is also an isomorphism as $C$-algebras and that $V_B(B^H)$ is an Azumaya $C$-algebra, we conclude that $V_B(B^H) \otimes_C B^H$ is a Hirata separable extension of $B^H$; and so $B$ is a Hirata separable extension of $B^H$.

(3)$\Rightarrow$(2). By the proof of (3)$\Rightarrow$(1), $B \cong V_B(B^H) \otimes_C B^H$ such that $V_B(B^H)$ is a finitely generated and projective module over $C$, so $V_B(B^H)$ is an $H^*$-Galois extension of $C$ (see [2, Lemma 2.8]). Moreover, since $V_B(B^H)$ is an Azumaya
C-algebra, \( V_B(B^H) \) is a direct summand of \( V_B(B^H) \otimes_C (V_B(B^H))^* \) as a \( V_B(B^H) \)-bimodule, where \( (V_B(B^H))^* \) is the opposite algebra of \( V_B(B^H) \). But \( V_B(B^H) \otimes_C (V_B(B^H))^* \equiv \text{Hom}_C(V_B(B^H), V_B(B^H)) \equiv V_B(B^H)^H \) (see [3, Theorem 2.2]), so \( V_B(B^H) \) is a direct summand of \( V_B(B^H)^H \) as a \( V_B(B^H) \)-bimodule.

(2) \( \Rightarrow \) (3). Since \( V_B(B^H) \) is an \( H^* \)-Galois extension of \( C, B^H \cdot V_B(B^H) \) is an \( H^* \)-Galois extension of \( (B^H \cdot V_B(B^H))^H \). But \( (B^H \cdot V_B(B^H))^H = B^H \), so \( B^H = V_B(B^H) \) and \( B \) are \( H^* \)-Galois extensions of \( B^H \) such that \( B^H \cdot V_B(B^H) \subset B \). Hence, \( B^H \cdot V_B(B^H) = B \) by Lemma 3.1. Thus, the centers of \( B \) and \( V_B(B^H) \) are the same. Moreover, \( V_B(B^H) \) is a direct summand of \( V_B(B^H)^H \) as a \( V_B(B^H) \)-bimodule by hypothesis, so it is a separable \( C \)-algebra (see [3, Theorem 2.3]). Thus, \( V_B(B^H) \) is an Azumaya \( C \)-algebra. But then \( B \equiv V_B(B^H) \otimes_C V_B(V_B(B^H)) \). On the other hand, by hypothesis, \( V_B(B^H) \) is an \( H^* \)-Galois extension of \( C \), so \( B \equiv V_B(B^H) \otimes_C B \) (see [2, Lemma 2.8]). Therefore, \( V_B(V_B(B^H)) = B^H \).

(3) \( \Leftrightarrow \) (4). Since \( V_B(B^H) \) is an \( H^* \)-Galois extension of \( C \), it is a finitely generated and projective module over \( C \) and \( \text{Hom}_C(V_B(B^H), V_B(B^H)) \equiv V_B(B^H)^H \) (see [3, Theorem 2.2]). But then \( V_B(B^H) \) is a Hirata separable extension of \( C \) if and only if \( V_B(B^H)^H \) is a direct summand of a finite direct sum of \( V_B(B^H) \) as a bimodule over \( V_B(B^H) \) (see [5, Corollary 3]). Thus, \( V_B(B^H) \) is an Azumaya \( C \)-algebra if and only if \( V_B(B^H) \) is an \( H^* \)-Galois extension of \( C \) and \( V_B(B^H)^H \) is a direct summand of a finite direct sum of \( V_B(B^H) \) as a bimodule over \( V_B(B^H) \).

By Theorem 3.5, we can obtain a relationship between the class of \( H^* \)-Galois Hirata extensions and the class of \( H^* \)-Galois Azumaya extensions which were studied in [3, 7].

**Corollary 3.6.** Let \( B \) be an \( H^* \)-Galois Azumaya extension of \( B^H \). Then \( B \) is an \( H^* \)-Galois Hirata extension of \( B^H \) if and only if \( C = C^H \).

**Proof.** (\( \Rightarrow \)) Since \( B \) is an \( H^* \)-Galois Hirata extension of \( B^H \), \( V_B(B^H) \) is an Azumaya algebra over \( C \) and a left \( H \)-Galois extension of \( C \) (see [8, Theorem 2.6]). Hence, \( V_B(V_B(B^H)) = B^H \) (see [8, Lemma 2.5]). Thus, \( C \subset B^H \); and so \( C = C^H \).

(\( \Leftarrow \)) Since \( B \) is an \( H^* \)-Galois Azumaya extension of \( B^H \), \( V_B(B^H) \) is separable over \( C^H \) (see [3, Lemma 4.1]). Since \( B \) is an \( H^* \)-Galois Azumaya extension of \( B^H \) again, \( V_B(B^H) \) is an \( H^* \)-Galois extension of \( (V_B(B^H))^H \) (see [3, Lemma 4.1]), so both \( B^H \cdot V_B(B^H) \) and \( B \) are \( H^* \)-Galois extensions of \( B^H \) such that \( B^H \cdot V_B(B^H) \subset B \). Hence, \( B^H \cdot V_B(B^H) = B \) by Lemma 3.1. This implies that the center of \( V_B(B^H) \) is \( C \). But by hypothesis, \( C = C^H \), so \( V_B(B^H) \) is an Azumaya \( C \)-algebra. Hence, \( V_B(B^H) \) is a Hirata separable extension of \( C \). But \( B = B^H \cdot V_B(B^H) \equiv B^H \otimes_C V_B(B^H) \) as Azumaya \( C \)-algebras, so \( B \) is a Hirata separable extension of \( B^H \). Thus, \( B \) is an \( H^* \)-Galois Hirata extension of \( B^H \).

**Corollary 3.7.** Let \( B \) be an \( H^* \)-Galois Azumaya extension of \( B^H \). Then \( B \) is an \( H^* \)-Galois Azumaya extension of \( B^H \) if and only if \( B \) is an Azumaya \( C^H \)-algebra.
Proof. ($\Rightarrow$) Since $B$ is an $H^*$-Galois Azumaya extension of $B^H$, $B^H$ is an Azumaya $C^H$-algebra and $B$ is separable over $B^H$ (see [3, Theorem 3.4]). Hence, $B$ is separable over $C^H$ by the transitivity of separable extensions. But $B$ is an $H^*$-Galois Azumaya extension of $B^H$ and an $H^*$-Galois Hirata extension of $B^H$ by hypothesis, so $C = C^H$ by Corollary 3.6. This implies that $B$ is an Azumaya $C^H$-algebra.

($\Leftarrow$) By hypothesis, $B$ is an Azumaya $C^H$-algebra. Hence, $C = C^H$. But $B$ is an $H^*$-Galois Hirata extension of $B^H$ again, $B$ is a Hirata separable extension of $B^H$ and a finitely generated and projective module over $B^H$. Thus, $V^H_B(V_B(B^H)) = B^H$ (see [8, Lemma 2.5]); and so $B^H (= V^H_B(V_B(B^H)))$ is an Azumaya subalgebra of $B$ over $C^H$ by the commutator theorem for Azumaya algebras (see [1, Theorem 4.3, page 57]). This proves that $B$ is an $H^*$-Galois Azumaya extension of $B^H$. \( \square \)

4. Invariant subalgebras. For an $H^*$-Galois Hirata extension $B$ as given in Theorem 3.5, let $A$ be a subalgebra of $B^H$ over $C$ such that $B^H$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule. In this section, we show that $V^H_B(A)$ is the $H$-invariant subalgebra of a separable subalgebra $D$ in $B$ over $C$, that is, $D^H = V^H_B(A)$. We denote by $\mathcal{H}$ the set $\{A \mid A$ is a subalgebra of $B^H$ over $C$ such that $B^H$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule}.

**Lemma 4.1.** Let $B$ be an $H^*$-Galois Hirata extension of $B^H$. For any $A \in \mathcal{H}$, $V_B(A)$ is an $H$-submodule algebra of $B$ and separable over $C$, and $(V_B(A))^H = V_B^H(A)$ which is a separable $C$-algebra.

**Proof.** Since $A \in \mathcal{H}$, $B^H$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule. But $B$ is an $H^*$-Galois Hirata extension of $B^H$, so $B$ is a projective Hirata separable extension of $B^H$. Hence, by the transitivity property of projective Hirata separable extensions, $B$ is a projective Hirata separable extension of $A$. Also $B^H$ is a direct summand of $B$ as a $B^H$-bimodule by Lemma 3.2, so $A$ is a direct summand of $B$ as an $A$-bimodule. Thus, $V_B(A)$ is a separable algebra over $C$ (see [6, Theorem 1]). Moreover, it is clear that $(V_B(A))^H = V_B^H(A)$, so $V_B^H(A)$ is a separable $C$-algebra (see Corollary 3.4 and [6, Theorem 1]). \( \square \)

Next we want to show which separable subalgebra of $B^H$ over $C$ is an $H$-invariant subring of an $H$-submodule algebra in $B$. Let $\mathcal{I} = \{E \subset B \mid E$ is a separable $C$-subalgebra of $B^H$ and satisfies the double centralizer property in $B^H$ such that $V_B^H(E) \in \mathcal{H}\}$. Next we show that for any $E \in \mathcal{I}$, $E$ is the $H$-invariant subring of an $H$-submodule algebra $D$ in $B$ which is separable over $C$.

**Theorem 4.2.** Let $E$ be in $\mathcal{I}$. Then there exists an $H$-submodule algebra $D$ in $B$ which is separable over $C$ such that $D^H = E$. 

**Proof.** Since $E$ is in $\mathcal{F}$, $V_{BH}(E)$ is in $\mathcal{F}$ such that $V_{BH}(V_{BH}(E)) = E$. Now by Lemma 4.1, $V_{B}(V_{BH}(E))$ is an $H$-submodule algebra of $B$ and separable over $C$ such that $(V_{B}(V_{BH}(E)))^{H} = V_{BH}(V_{BH}(E))$. But $V_{BH}(V_{BH}(E)) = E$, so
\[
(V_{B}(V_{BH}(E)))^{H} = E. \tag{4.1}
\]
Let $D = V_{B}(V_{BH}(E))$. Then $D$ satisfies the theorem.

By Theorem 4.2, we obtain an expression for the separable $H$-submodule algebra $D$ for a given $E$ in $\mathcal{F}$.

**Corollary 4.3.** By keeping the notations as given in Theorem 4.2, let $Z$ be the center of $E$. Then $D \cong E \otimes_{Z} V_{D}(E)$ as Azumaya $Z$-algebras.

**Proof.** Since $E$ satisfies the double centralizer property in $B^{H}$, $V_{BH}(V_{BH}(E)) = E$. Hence, the centers of $E$ and $V_{B}(E)$ are the same $Z$. Similarly as given in the proof of Lemma 4.1, since $V_{BH}(E)$ is in $\mathcal{F}$, $B (= B^{H} \cdot V_{B}(B^{H}))$ is a projective Hirata separable extension of $V_{BH}(E)$ and contains $V_{BH}(E)$ as a direct summand as a $V_{BH}(E)$-bimodule by the transitivity property of projective Hirata separable extensions and the direct summand conditions. Thus, $V_{BH}(E)$ satisfies the double centralizer property in $B$, that is, $V_{B}(V_{B}(V_{BH}(E))) = V_{BH}(E)$. This implies that the centers of $V_{BH}(E)$ and $V_{B}(V_{BH}(E))$ are the same. Therefore, $D$ and $E$ have the same center $Z$. Noting that $D$ and $E$ are separable $C$-algebras by Theorem 4.2, we conclude that $E (= D^{H})$ is an Azumaya subalgebra of $D$ over $Z$; and so $D \cong E \otimes_{Z} V_{D}(E)$ as Azumaya $Z$-algebras (see [1, Theorem 4.3, page 57]).

**Remark 4.4.** When $B$ is an $H^{*}$-Galois Azumaya extension of $B^{H}$, the correspondence $A \rightarrow V_{B}(A)$ as given in Lemma 4.1 recovers the one-to-one correspondence between the set of separable subalgebras of $B^{H}$ and the set of $H^{*}$-Galois extensions in $B$ containing $V_{B}(B^{H})$ as given in [3].

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