PSEUDOINVERSION OF DEGENERATE METRICS

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Let (M, g) be a smooth manifold M endowed with a metric g . A large class of differential operators in differential geometry is intrinsically defined by means of the dual metric g^* on the dual bundle TM^* of 1-forms on M . If the metric g is (semi-)Riemannian, the metric g^* is just the inverse of g . This paper studies the definition of the above-mentioned geometric differential operators in the case of manifolds endowed with degenerate metrics for which g^* is not defined. We apply the theoretical results to Laplacian-type operator on a lightlike hypersurface to deduce a Takahashi-like theorem (Takahashi (1966)) for lightlike hypersurfaces in Lorentzian space $\mathbb{R}_1^{n+2}$.

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1. Introduction. The introduction of a Riemannian metric on a smooth manifold gives rise to a series of differential operators which reveal deep relationships between the geometry and the topology of the manifold. If Δ^k is the Laplacian on k -forms, it is well known that, on a compact manifold, its spectrum $\{\lambda_i^k\}$ contains topological and geometric information on the manifold. According to the Hodge decomposition theorem, the dimension of the kernel of Δ^k equals the k th Betti number so that the Laplacian determines Euler characteristic $\chi(M)$ of compact Riemannian manifolds (M, g) .

Among usual differential operators on (M, g) , the exterior derivative which takes k -forms to $(k+1)$ -forms is the only one defined in terms of the smooth structure of the manifold M . The others are defined by means of the metric g^* on the dual bundle, which is the inverse of the given metric g , in semi-Riemannian case.

Many situations arise in mathematical physics where the metric is degenerated and it is not possible to define the inverse g^* . A typical case is the one we have in the coupling of Einstein theory of gravity with both a quantum mechanics particle with spin and an electromagnetic field, that is, Einstein-Dirac-Maxwell (EDM) system. If, for instance, we consider Lorentz framed manifold $\bar{M}$ endowed with Eddington-Finkelstein metric (cf. Hawking and Ellis [12, page 150])

$$ds^2 = -\left(1 - \frac{2m}{r}\right)du^2 + 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.1)$$

given in a coordinate system (u, r, θ, ϕ) , where $u = t + \bar{r}$ is an advanced null coordinate with

$$\bar{r} = \int \frac{dr}{1 - 2m/r} = r + 2m \ln(r - 2m) \quad (r > 2m), \quad (1.2)$$

the hypersurface $M : u = \text{constant}$ is a degenerate hypersurface. Dirac operator can be written outside M and in its interior region. But it is not easy to match the two operators on M due to the fact that the inverse metric g^* cannot be defined on M . Normalization problems and the geometry of those manifolds are considered in several papers (see [1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14] and references therein).

In the following, we consider a smooth (semi)-Riemannian manifold (M, g) , its Levi-Civita connection ∇ , its tangent (resp., cotangent) bundle TM (resp., TM^*), and $\mathcal{F}(M)$ the space of smooth functions on M . For a vector bundle E , we denote by $\Gamma(E)$ the space of its smooth sections. For a smooth function f in $\mathcal{F}(M)$, the gradient of f is the vector field $\text{grad}^g f$ given by

$$g(\text{grad}^g f, X) = X(f) = df(X) \quad (1.3)$$

for any X in $\Gamma(TM)$.

In local coordinates $(x^1, \dots, x^n)$, the gradient of f is given by

$$\text{grad}^g f = g^{ij} f_i E_j, \quad (1.4)$$

where $f_i = \partial f / \partial x^i$, $E_j = \partial / \partial x^j$, and g^{ij} is the (i, j) -entry of the inverse g^{-1} of g .

The gradient can be defined using exterior differential d and the natural isomorphism $\#$ between the tangent bundle TM and its dual bundle TM^* , by the composition

$$\text{grad} : \mathcal{F}(M) \xrightarrow{d} TM^* \xrightarrow{\#} TM, \quad (1.5)$$

that is,

$$\text{grad} = \# \circ d, \quad (1.6)$$

where

$$\begin{aligned} \# : TM^* &\longrightarrow TM, \\ \omega &\longmapsto \omega^\# \end{aligned} \quad (1.7)$$

is such that, for all $Y \in \Gamma(TM)$,

$$g(\omega^\#, Y) = \omega(Y). \quad (1.8)$$

The inverse of $\#$ is the isomorphism $\flat = (\#)^{-1}$ defined by

$$\begin{aligned}\flat : TM &\longrightarrow TM^*, \\ X &\longmapsto X^\flat\end{aligned}\tag{1.9}$$

such that

$$X^\flat(Y) = g(X, Y)\tag{1.10}$$

for all Y in $\Gamma(TM)$.

The divergence of a vector field X is the codifferential of the dual 1-form $X^\flat$ defined by

$$\operatorname{div} X = \delta(X^\flat) = -\operatorname{trace}(Z \longmapsto \nabla_Z X).\tag{1.11}$$

The covariant differentiation ∇ defined on vector fields can be extended to 1-forms by duality using isomorphism $\#$:

$$\nabla \alpha(X, Y) = g(\nabla_X \alpha^\#, Y) \quad \forall \alpha \in T^*M.\tag{1.12}$$

The Laplace-Beltrami operator Δ^g , whose physical and mathematical importance is well known, is defined on smooth functions by

$$\Delta^g = \operatorname{div}(\operatorname{grad}) = \delta \circ \flat \circ \# \circ d,\tag{1.13}$$

that is, locally,

$$\Delta_g = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right\}.\tag{1.14}$$

From (1.4) and (1.14), it is obvious that the entries g^{ij} of the inverse of g_{ij} need to be computed in local coordinates. So the following question seems to be coming out in a natural way: how can we define those differential operators on a lightlike manifold, that is, on a submanifold with a degenerate (noninvertible) metric g ?

This question will be answered by [Proposition 3.1](#).

First, note that (dx^i) and $(\partial_i^\flat)$ are two local bases of T^*M and the decomposition

$$\partial_i^\flat = \lambda_{ij} dx^j, \quad 1 \leq i \leq n.\tag{1.15}$$

holds.

Hence,

$$g_{ik} = g(\partial_i, \partial_k) = \partial_i^\flat(\partial_k) = \lambda_{ij} dx^j(\partial_k) = \lambda_{ik},\tag{1.16}$$

and then,

$$\partial_i^b = g_{ik} dx^k, \quad 1 \leq i \leq n. \quad (1.17)$$

So inverting g , we get

$$dx^k = g^{ik} \partial_i^b, \quad 1 \leq k \leq n. \quad (1.18)$$

Thus, coefficients g^{ij} appear as the entries of the matrix of transition from basis (∂_i^b) to (dx^k) .

Then, an important step in answering the above question should be the construction of the isomorphisms $\#$ and $\flat$ adapted to the case of lightlike manifolds.

Below, we discuss lightlike submanifolds of semi-Riemannian manifolds. We are particularly interested in lightlike hypersurfaces. One may proceed similarly in any other codimension. The paper is organized as follows. [Section 2](#) deals with preliminaries and specifies our notations and some basic results. [Section 3](#) is devoted to the definition and construction of the pseudoinverse of a degenerate metric g , and a concept of *associate metric* with the degenerate metric g is also introduced. In [Section 4](#), the compatibility of this associate metric with respect to the connection induced by g is examined. The existence and the properties of the pseudoinverse allow the definition of a Laplacian Δ^g on smooth functions of (M, g) . Actually, this Δ^g is a d'Alembertian since g is nondefinite. However, for the sake of simplicity, we call it Laplacian and we denote it by Δ^g . It is formally defined in the same way as the Laplace operator in [Section 5](#). The Laplacian of lightlike hypersurfaces endowed with parallel screen distribution is then defined and its action on position vector field is expressed (see [Theorem 5.2](#)). Finally, in [Section 6](#), we give an application which oddly reminds us of Takahashi theorem [15]. It is our [Theorem 6.1](#).

2. Preliminaries on lightlike hypersurfaces. Let M be a hypersurface of an $(n+2)$ -dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. In the classical theory of nondegenerate hypersurface, the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of main geometric objects. In case of degenerate hypersurface, the situation is totally different. The normal bundle $TM^\perp$ is a rank-one distribution over $M : TM^\perp \subset TM$; hence, the metric g on M is degenerate, that is, noninvertible and of rank n . In fact, the following propositions is proved in [11]

PROPOSITION 2.1. *Let (M, g) be a hypersurface of an $(n+2)$ -dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

- (i) M is a lightlike hypersurface of $\bar{M}$;
- (ii) g has a constant rank n on M ;
- (iii) $TM^\perp = \cup_{x \in M} T_x M^\perp$ is a distribution on M .

A complementary bundle of $TM^\perp$ in TM is a rank- n nondegenerate distribution over M . It is called a screen distribution in the tangent bundle and is often denoted by $S(TM)$. In [11, Chapter 4], degenerate hypersurface (M, g) endowed with this specific distribution is denoted by $(M, g, S(TM))$ and is also called lightlike hypersurface. For the convenience of the reader, we summarize the properties of lightlike hypersurfaces which are appropriate to our purpose.

There exists a vector bundle $\text{tr}(TM)$ of rank one over M such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighbourhood $\mathcal{U} \subset M$, there is a unique section N of $\text{tr}(TM)$ defined on $\mathcal{U}$ such that

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0 \quad \forall W \in \Gamma(S(TM)|_{\mathcal{U}}). \quad (2.1)$$

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connection of $(\bar{M}, \bar{g})$ and an induced connection on $(M, g, S(TM))$, respectively. It is known that ∇ is not unique. It depends on both g and the considered screen distribution.

With the decompositions into orthogonal direct sums

$$TM = S(TM) \perp TM^\perp, \quad (2.2)$$

$$T\bar{M}|_M = S(TM) \perp (TM^\perp \oplus \text{tr}(TM)) = TM \oplus \text{tr}(TM), \quad (2.3)$$

we have Gauss and Weingarten formulae in the form

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \quad \forall X, Y \in \Gamma(TM), \\ \bar{\nabla}_V X &= -A_V X + \nabla_X^t V \quad \forall X \in \Gamma(TM), \quad \forall V \in \Gamma(\text{tr}(TM)), \end{aligned} \quad (2.4)$$

or equivalently,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2.5)$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad (2.6)$$

where we put

$$B(X, Y) = \bar{g}(h(X, Y), \xi) \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \quad (2.7)$$

$$\tau(X) = \bar{g}(\nabla_X^t N, \xi) \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \quad (2.8)$$

For $\nabla_X Y$ and $A_V X$ belonging to $\Gamma(TM)$ and h a $\Gamma(\text{tr}(TM))$ -valued symmetric $\mathcal{F}(M)$ -bilinear form on $\Gamma(TM)$, A_V is an $\mathcal{F}(M)$ -linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on the lightlike transverse vector bundle $\text{tr}(TM)$.

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.2). We have

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY) \quad \forall X, Y \in \Gamma(TM), \quad (2.9)$$

$$\nabla_X U = -A_U^* X + \nabla_X^{*t} U \quad \forall X \in \Gamma(TM), \quad \forall U \in \Gamma(TM^\perp), \quad (2.10)$$

where ∇_X^*PY and A_U^*X belong to $\Gamma(S(TM))$, ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $TM^\perp$, respectively, h^* is a $\Gamma(TM^\perp)$ -valued $\mathcal{F}(M)$ -bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, and A_U^* is a $\Gamma(S(TM))$ -valued $\mathcal{F}(\mathcal{U})$ -linear operator on $\Gamma(TM)$. They are the second fundamental form and the shape operator of the screen distribution $S(TM)$, respectively. Note that some authors called A_U^* (or A_N) the Burali-Forti affinor [3, 9].

Define the following also on $\mathcal{U}$:

$$C(X, PY) = \overline{g}(h^*(X, PY), N), \quad (2.11)$$

$$\varepsilon(X) = \overline{g}(\nabla_X^{*t}\xi, N). \quad (2.12)$$

One can verify that $\varepsilon(X) = -\tau(X)$. Thus, locally, one has

$$\nabla_X PY = \nabla_X^*PY + C(X, PY)\xi, \quad (2.13)$$

$$\nabla_X \xi = -A_\xi^*X - \tau(X)\xi, \quad \forall X \in \Gamma(TM). \quad (2.14)$$

DEFINITION 2.2. The screen distribution $S(TM)$ is said to be parallel with respect to the induced connection ∇ if $\nabla_X PY \in \Gamma(S(TM))$ for all $X, Y \in \Gamma(TM)$.

We will make use of the following theorem.

THEOREM 2.3 (Duggal and Bejancu [11, page 89]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g})$. Then the following assertions are equivalent:*

- (i) $S(TM)$ is parallel with respect to the induced connection ∇ ;
- (ii) h^* vanishes identically on M ;
- (iii) A_N vanishes identically on M .

Then it is easy to see that the following relations hold:

$$B(X, \xi) = 0 \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \quad (2.15)$$

$$B(X, Y) = g(A_\xi^*X, Y), \quad (2.16)$$

$$A_\xi^*\xi = 0. \quad (2.17)$$

3. Pseudoinverse of a degenerate metric. We start this section by the construction of the isomorphisms $\#$ and $\flat$ with respect to the noninvertible induced metric g .

First, we consider on M the 1-form η defined by

$$\eta(\cdot) = \overline{g}(N, \cdot), \quad (3.1)$$

where N is given by (2.1). Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ in (2.2). We have, for all $X \in \Gamma(TM)$,

$$X = PX + \eta(X)\xi \quad (3.2)$$

and from (2.1) and (3.1),

$$\eta(X) = 0 \iff X \in \Gamma(S(TM)). \quad (3.3)$$

Now, we define $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M), \\ X &\longmapsto X^\flat \end{aligned} \quad (3.4)$$

such that

$$X^\flat = g(PX, \cdot) + \eta(X)\eta(\cdot). \quad (3.5)$$

It is obvious that $\flat$ is linear.

As TM and T^*M have the same finite-constant rank, it suffices to show that $\flat$ is injective, that is, $X^\flat = 0 \iff X = 0$, to get the bijection of $\flat$. Consider $X \in \Gamma(TM)$ and assume $X^\flat = 0$. Then

$$X^\flat(Y) = 0 \quad \forall Y \in \Gamma(TM) \iff g(PX, Y) + \eta(X)\eta(Y) = 0, \quad \forall Y \in \Gamma(TM). \quad (3.6)$$

In particular, for any $Y \in \Gamma(S(TM))$, we have

$$g(PX, Y) = 0 \quad (3.7)$$

since $\eta(Y) = 0$.

As $\Gamma(S(TM))$ is nondegenerate and $PX \in \Gamma(S(TM))$, we have

$$PX = 0. \quad (3.8)$$

Therefore, X satisfies

$$\eta(X)\eta(Y) = 0 \quad \forall Y \in \Gamma(TM). \quad (3.9)$$

In particular, if $Y = \xi$, $\eta(X) = 0$. Hence, $X = PX + \eta(X)\xi = 0$ and $\flat$ is injective. We conclude that the so-defined $\flat$ is an isomorphism of $\Gamma(TM)$ on $\Gamma(T^*M)$. We denote by $\sharp$ its inverse isomorphism.

Note that the above construction generalizes the one of nondegenerate case for, in the latter case, $S(TM)$ coincides with TM . As a consequence, the 1-form η vanishes identically and P becomes the identity map on TM .

We now define the *associate metric* $\tilde{g}$ of g . Put

$$\tilde{g}(X, Y) = X^\flat(Y). \quad (3.10)$$

It is obvious that $\tilde{g}$ is bilinear in X and Y . Furthermore, from (3.5), we have that $X^\flat(Y) = Y^\flat(X)$, then $\tilde{g}$ is symmetric. Its inverse $\tilde{g}^{-1}$, which we denote by $g^{[\cdot, \cdot]}$, will be called the *pseudoinverse* of the degenerate metric g on M .

Note that in the case where g is nondegenerate, the associate metric $\tilde{g}$ coincides with g and the pseudoinverse $g^{[\cdot\cdot]}$ coincides with the dual g^* (or g^{-1}) of g .

In what follows, we use the following range of indices:

$$i, j = 1, \dots, n, \quad \alpha, \beta, \gamma = 0, 1, \dots, n, \quad A, B, C, \dots = 0, 1, \dots, n+1. \quad (3.11)$$

Now, consider a local coordinate system $(x^0, \dots, x^{n+1})$ adapted to decompositions (2.2) and (2.3), that is,

$$\left(X_i := \frac{\partial}{\partial x^i}, \quad \xi := \frac{\partial}{\partial x^0}, \quad N := \frac{\partial}{\partial x^{n+1}} \right) \quad (3.12)$$

is a local quasiorthogonal basis of $T\overline{M}|_M$ such that

$$\begin{aligned} TM^\perp &= \text{Span}\{\xi\}, & S(TM) &= \text{Span}\{X_1, \dots, X_n\}, \\ \text{tr}(TM) &= \text{Span}\{N\}. \end{aligned} \quad (3.13)$$

Using (3.1), (3.2), (3.3), and (3.5), we have a standard computation that gives

$$\begin{aligned} \left(\frac{\partial}{\partial x^0} \right)^b \left(\frac{\partial}{\partial x^0} \right) &= 0 + \eta \left(\frac{\partial}{\partial x^0} \right) \eta \left(\frac{\partial}{\partial x^0} \right) = 1 = dx^0 \left(\frac{\partial}{\partial x^0} \right), \\ \left(\frac{\partial}{\partial x^0} \right)^b (X_i) &= g(P\xi, X_i) + \eta(\xi)\eta(X_i) = 0 = dx^0 \left(\frac{\partial}{\partial x^i} \right), \end{aligned} \quad (3.14)$$

so

$$\left(\frac{\partial}{\partial x^0} \right)^b = dx^0, \quad (3.15)$$

and the first terms in (3.14) are $\tilde{g}_{00}$ and $\tilde{g}_{0i}$, respectively. Hence we have

$$\tilde{g}_{00} = 1, \quad \tilde{g}_{0i} = \tilde{g}_{i0} = 0, \quad i = 1, \dots, n. \quad (3.16)$$

We also have

$$\left(\frac{\partial}{\partial x^i} \right)^b \left(\frac{\partial}{\partial x^k} \right) = g(PX_i, X_k) + \eta(X_i)\eta(X_k) = g_{ik}. \quad (3.17)$$

Consequently,

$$\tilde{g}_{ij} = g_{ij}, \quad i, j = 1, \dots, n. \quad (3.18)$$

Thus, with respect to the quasiorthonormal basis $(\xi, X_1, \dots, X_n, N)$, the matrices of $\tilde{g}$ and $g^{[\cdot, \cdot]}$ are given by

$$\begin{aligned} \tilde{g} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g_{ij} & \\ 0 & & & \end{pmatrix}, \\ g^{[\cdot, \cdot]} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (g_{ij})^{-1} & \\ 0 & & & \end{pmatrix}. \end{aligned} \quad (3.19)$$

We have

$$g^{[\cdot, \cdot]} \cdot g = g \cdot g^{[\cdot, \cdot]} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}, \quad (3.20)$$

and this justifies the terminology pseudoinverse.

By (3.17) and (3.18), we have

$$\left(\frac{\partial}{\partial x^i} \right)^b \left(\frac{\partial}{\partial x^k} \right) = g_{ik} = \tilde{g}_{ij} dx^j \left(\frac{\partial}{\partial x^k} \right), \quad i, j = 1, \dots, n. \quad (3.21)$$

Hence

$$\left(\frac{\partial}{\partial x^i} \right)^b = \tilde{g}_{ij} dx^j. \quad (3.22)$$

Taking into account (3.15) and (3.22), we have

$$dx^\beta = g^{[\alpha\beta]} \left(\frac{\partial}{\partial x^\alpha} \right)^b, \quad \alpha, \beta = 0, \dots, n. \quad (3.23)$$

From (3.19) and (3.23), remark that if ρ is an endomorphism of TM (resp., a bilinear form on TM), its trace with respect to g is given by

$$\begin{aligned} \text{trace}_g \rho &= \sum_{\alpha=0}^n \tilde{g}(\rho(X_\alpha), X_\alpha), \\ \text{trace}_g \rho &= g^{[\alpha\beta]} \rho_{\alpha\beta}, \end{aligned} \quad (3.24)$$

respectively.

We use (3.19) and (3.23) to deduce that

$$\begin{aligned}\tilde{g}(\xi, \xi) &= 1, \\ \tilde{g}(X, Y) &= g(X, Y) \quad \forall X, Y \in S(TM), \\ \tilde{g}(\xi, X) &= \eta(X) \quad \forall X \in TM.\end{aligned}\tag{3.25}$$

Now, let $f : M \rightarrow \mathbb{R}$ be a smooth function on M . One defines intrinsically the gradient of f by $\text{grad}^g f = (df)^\#$. But

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha \Rightarrow (df)^\# = \frac{\partial f}{\partial x^\alpha} (dx^\alpha)^\#.\tag{3.26}$$

Hence, from (3.23), we infer that

$$\begin{aligned}\text{grad}^g f &= \frac{\partial f}{\partial x^\alpha} g^{[\alpha\beta]} \left(\left(\frac{\partial}{\partial x^\beta} \right)^\flat \right)^\# = \frac{\partial f}{\partial x^\alpha} g^{[\alpha\beta]} \frac{\partial}{\partial x^\beta}, \\ \text{grad}^g f &= g^{[\alpha\beta]} \frac{\partial f}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}.\end{aligned}\tag{3.27}$$

Let X be a smooth vector field defined on $\mathcal{U} \subset M$. The divergence $\text{div}^g X$ of X with respect to the degenerate metric g is intrinsically defined by

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha X_\alpha^\flat (\nabla_{X_\alpha} X).\tag{3.28}$$

Therefore, from (3.10), we have

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} X, X_\alpha)\tag{3.29}$$

with $\varepsilon_0 = 1$.

In summary, we have proved the following proposition.

PROPOSITION 3.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian $(n+2)$ -dimensional manifold $(\bar{M}, \tilde{g})$. There exists an associate metric $\tilde{g}$ and a pseudoinverse $g^{[\cdot]}$ of g on M such that locally on $\mathcal{U} \subset M$, the following holds:*

(i) *for any smooth function $f : \mathcal{U} \subset M \rightarrow \mathbb{R}$,*

$$\text{grad}^g f = g^{[\alpha\beta]} f_\alpha \partial_\beta,\tag{3.30}$$

where $f_\alpha = \partial f / \partial x^\alpha$, $\partial_\beta = \partial / \partial x^\beta$, $\alpha, \beta = 0, \dots, n$;

(ii) *for any vector field X on $\mathcal{U} \subset M$,*

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} X, X_\alpha), \quad \varepsilon_0 = 1;\tag{3.31}$$

(iii) for a smooth function f defined on $\mathcal{U} \subset M$,

$$\Delta^g f = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} \text{grad}^g f, X_\alpha). \quad (3.32)$$

Here Δ^g is the d 'Alembertian with respect to g on $\mathcal{U}$.

Note that if the section ξ is globally defined on M , then our result is also global.

4. Compatibility of $\tilde{g}$ with respect to the connection ∇ induced by g . It is well known that the induced metric g is not compatible with the induced connection ∇ in general, and this compatibility arises if and only if the lightlike hypersurface M is totally geodesic in $\bar{M}$. This section deals with the compatibility of the associate metric $\tilde{g}$ with respect to ∇ .

For $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} & (\nabla_X \tilde{g})(Y, Z) \\ &= X \cdot \tilde{g}(Y, Z) - \tilde{g}(\nabla_X Y, Z) - \tilde{g}(Y, \nabla_X Z) \\ &= X \cdot \{ \tilde{g}(PY, PZ) + \tilde{g}(PY, \eta(Z)\xi) + \tilde{g}(\eta(Y)\xi, PZ) + \eta(Y)\eta(Z)\tilde{g}(\xi, \xi) \} \\ &\quad - \tilde{g}(\nabla_X(PY + \eta(Y)\xi), PZ + \eta(Z)\xi) - \tilde{g}(PY + \eta(Y)\xi, \nabla_X(PZ + \eta(Z)\xi)) \\ &= X \cdot \{ g(PY, PZ) + \eta(Z)\eta(PY) + \eta(Y)\eta(PZ) + \eta(Y)\eta(Z) \} \\ &\quad - \tilde{g}(\nabla_X PY + \nabla_X(\eta(Y)\xi), PZ) - \tilde{g}(\nabla_X PY + \nabla_X(\eta(Y)\xi), \eta(Z)\xi) \\ &\quad - \tilde{g}(PY, \nabla_X PZ + \nabla_X(\eta(Z)\xi)) - \tilde{g}(\eta(Y)\xi, \nabla_X PZ + \nabla_X(\eta(Z)\xi)) \\ &= X \cdot \{ g(PY, PZ) + \eta(Y)\eta(Z) \} - \tilde{g}(\nabla_X PY, PZ) - \tilde{g}(\nabla_X(\eta(Y)\xi), PZ) \\ &\quad - \tilde{g}(\nabla_X PY, \eta(Z)\xi) - \tilde{g}(\nabla_X(\eta(Y)\xi), \eta(Z)\xi) - \tilde{g}(PY, \nabla_X PZ) \\ &\quad - \tilde{g}(PY, \nabla_X(\eta(Z)\xi)) - \tilde{g}(\eta(Y)\xi, \nabla_X PZ) - \tilde{g}(\eta(Y)\xi, \nabla_X(\eta(Z)\xi)). \end{aligned} \quad (4.1)$$

Now use (2.9), (2.11), and (3.10) to obtain

$$\begin{aligned} (\nabla_X \tilde{g})(Y, Z) &= X \cdot \{ g(PY, PZ) \} + \eta(Y)X \cdot \eta(Z) + \eta(Z)X \cdot \eta(Y) \\ &\quad - \tilde{g}(\overset{*}{\nabla}_X PY, PZ) - \tilde{g}(C(X, PY)\xi, PZ) \\ &\quad - g(\overset{*}{\nabla}_X PY, PZ) - \eta(PZ)C(X, PY) - (X \cdot \eta(Y))\eta(PZ) \\ &\quad - \eta(Y)\tilde{g}(-A_\xi^*, PZ) - \eta(Y)\tilde{g}(-\tau(X)\xi, PZ) \\ &\quad - \eta(Z)\eta(\overset{*}{\nabla}_X PY) - \eta(Z)C(X, PY) - \eta(Z)X \cdot \eta(Y) \end{aligned}$$

$$\begin{aligned}
& -\eta(Y)\eta(Z)\tilde{g}\left(-\overset{*}{A}_\xi X, \xi\right) - \eta(Y)\eta(Z)\tilde{g}(-\tau(X)\xi, \xi) \\
& -g\left(PY, \overset{*}{\nabla}_X PZ\right) - C(X, PZ)\eta(PY) - \eta(PY)X \cdot \eta(Z) \\
& -\eta(Z)\tilde{g}\left(PY, \overset{*}{A}_\xi X\right) - \eta(Z)\tilde{g}(-PY, -\tau(X)\xi) \\
& -\eta(Y)\eta\left(\overset{*}{\nabla}_X PZ\right) - \eta(Y)C(X, PZ) - \eta(Y)X \cdot (\eta(Z)) \\
& -\eta(Z)\eta(Y)\tilde{g}\left(\xi, -\overset{*}{A}_\xi X\right) - \eta(Z)\eta(Y)\tilde{g}(\xi, -\tau(X)\xi).
\end{aligned} \tag{4.2}$$

But from (3.3), we know that $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Using (2.16) and (3.2) leads to

$$\begin{aligned}
(\nabla_X \tilde{g})(Y, Z) &= X \cdot \{g(PY, PZ)\} + \eta(Y)X \cdot \eta(Z) + \eta(Z)X \cdot \eta(Y) \\
& - \tilde{g}\left(\overset{*}{\nabla}_X PY, PZ\right) + \eta(Y)B(X, PZ) - \eta(Z)C(X, PY) \\
& - \eta(Z)X \cdot (\eta(Y)) + \tau(X)\eta(Y)\eta(Z) - g\left(PY, \overset{*}{\nabla}_X PZ\right) \\
& + \eta(Z)B(X, PY) - \eta(Y)C(X, PZ) - \eta(Y)X \cdot (\eta(Z)) \\
& + \tau(X)\eta(Y)\eta(Z).
\end{aligned} \tag{4.3}$$

The above expression reduces to

$$\begin{aligned}
(\nabla_X \tilde{g})(Y, Z) &= \left\{X \cdot g(PY, PZ) - g\left(\overset{*}{\nabla}_X PY, PZ\right) - g\left(PY, \overset{*}{\nabla}_X PZ\right)\right\} \\
& + \eta(Y)B(X, PZ) + \eta(Z)B(X, PY) - \eta(Z)C(X, PY) \\
& - \eta(Y)C(X, PZ) + 2\tau(X)\eta(Y)\eta(Z).
\end{aligned} \tag{4.4}$$

Now, recall that since the connection $\overset{*}{\nabla}$ is compatible with g , the expression between brackets is zero, and therefore we have

$$\begin{aligned}
(\nabla_X \tilde{g})(Y, Z) &= \eta(Y)(B(X, PZ) - C(X, PZ)) \\
& + \eta(Z)(B(X, PY) - C(X, PY)) \\
& + 2\tau(X)\eta(Y)\eta(Z) \quad \forall X, Y, Z \in \Gamma(TM).
\end{aligned} \tag{4.5}$$

Then we get the following theorem.

THEOREM 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The induced connection ∇ by g on M is compatible with the associate metric $\tilde{g}$ of g if and only if, for all $X, Z \in \Gamma(TM)$,*

$$B(X, PZ) = C(X, PZ), \quad \tau(X) = 0. \tag{4.6}$$

PROOF. It is well known that ∇ is compatible with $\tilde{g}$ if and only if $(\nabla_X \tilde{g})(Y, Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$. Putting $Y = Z = \xi$ in (4.5) gives $\tau(X) = 0$ for all $X \in \Gamma(TM)$, and setting $Y = \xi$ yields $B(X, PZ) = C(X, PZ)$ for all $X, Z \in \Gamma(TM)$. $\square$

We derive from (4.5) the main covariant derivative formula

$$\begin{aligned} X \cdot \tilde{g}(X, Z) &= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) + \eta(Y)(B(X, PZ) - C(X, PZ)) \\ &\quad + \eta(Z)(B(X, PY) - C(X, PY)) + 2\tau(X)\eta(Y)\eta(Z). \end{aligned} \quad (4.7)$$

We discuss here the effect of the change of the screen distribution on the compatibility of $\tilde{g}$ with g . The first equation in (4.6) indicates that only some privileged changes in the screen distribution maintain the compatibility property of the associate metric $\tilde{g}$ with g . For two screen distributions $S(TM)$ and $S(TM)'$ on (M, g) , with two orthogonal frame fields $(W_i)_{i=1, \dots, n}$ and $(W'_i)_{i=1, \dots, n}$, respectively, the relationship between the second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, is given by (see [4])

$$C'(X, PY) = C(X, PY) - \frac{1}{2} \langle W, W \rangle B(X, Y) + g(\nabla_X PY, W), \quad (4.8)$$

where $W = \sum_{i=1}^n c_i W_i$ is the characteristic vector field of the screen change.

Therefore, a change in screen distribution leaves the compatibility of $\tilde{g}$ with g invariant if and only if

$$\omega\left(\nabla_X PY - \frac{1}{2}B(X, Y)W\right) = 0 \quad \forall X, Y \in \Gamma(TM), \quad (4.9)$$

where ω is the dual 1-form of W defined by $\omega(\cdot) = g(W, \cdot)$.

From (3.1) and (3.10), if $\tilde{g}$ is associated to g on $S(TM)$ and $\tilde{g}'$ is associated to g on $S(TM)'$, we have

$$\tilde{g}'(X, Y) = \tilde{g}(X, Y) + D(X, Y), \quad (4.10)$$

where D is the bilinear form defined by

$$D(X, Y) = \eta(X)\omega(Y) + \eta(Y)\omega(X) + \omega(X)\omega(Y). \quad (4.11)$$

Before starting the next section, it is important to stress that the 1-form τ of (4.6), defined in (2.6) and (2.8), depends on the normalization of the rank-one subbundle $\text{Rad}(TM)$, that is, on the choice of $\xi \in \Gamma(TM^\perp|_{\mathcal{U}})$. It has been proved in [11, page 99] that there exists a pair (a normalization) $\{\xi, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes identically.

From now on we use this normalization. Then, relations (2.6) and (2.14) become

$$\bar{\nabla}_X N = -A_N X, \quad (4.12)$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X, \quad (4.13)$$

for all $X \in \Gamma(TM)$

5. Laplacian of lightlike hypersurfaces endowed with parallel screen distribution. Let $(M^{n+1}, g, S(TM))$ be a lightlike hypersurface of the Lorentzian space $\mathbb{R}_1^{n+2}$. The metric on $\mathbb{R}_1^{n+2}$ and the position vector field of M^{n+1} will be denoted by $\langle \cdot, \cdot \rangle$ and x , respectively.

We have

$$\begin{aligned} TM &= \text{Rad}(TM) \perp S(TM) = TM^\perp \perp S(TM), \\ T\mathbb{R}_1^{n+2}|_M &= (\text{Rad}(TM) \oplus \text{tr}(TM)) \perp S(TM), \end{aligned} \quad (5.1)$$

and $(X_0 := \xi, N, X_1, \dots, X_n)$ is a local quasiorthonormal frame field on $\mathbb{R}_1^{n+2}$ such that at each point $x \in M$, $(X_1, \dots, X_n)$ represents a basis of $S(T_x M)$. As the plane spanned by the pair $\{\xi_x, N_x\}$ is a hyperbolic plane at any $x \in M$ and $\mathbb{R}_1^{n+2}$ has index $q = 1$, it follows that the screen distribution on $M \subset \mathbb{R}_1^{n+2}$ is Riemannian, that is, the induced metric on $S(TM)$ is positive definite. As a consequence, the signature ε_k of X_k , $k = 1, \dots, n$, is $\varepsilon_k = 1$.

Define on M the smooth functions

$$f_0(x) = \langle x, N \rangle, \quad f_{n+1}(x) = \langle x, \xi \rangle, \quad f_k(x) = \langle x, X_k \rangle, \quad k = 1, \dots, n, \quad (5.2)$$

where x stands for the position vector field of M in $\mathbb{R}_1^{n+2}$. We have

$$U_A := \text{grad}^g f_A = g^{[\alpha\gamma]} f_{A;\alpha} X_\gamma, \quad (5.3)$$

where $f_{A;\alpha} := X_\alpha \cdot f_A$.

The position vector field x can be written in $\mathbb{R}_1^{n+2}$ as

$$x = (\langle x, N \rangle, \langle x, \xi \rangle, \varepsilon_1 \langle x, X_1 \rangle, \dots, \varepsilon_n \langle x, X_n \rangle) \quad (5.4)$$

with respect to $(\xi, N, X_1, \dots, X_n)$. As $\varepsilon_k = 1$, $k = 1, \dots, n$, we have

$$x = (f_0(x), f_{n+1}(x), f_1(x), \dots, f_n(x)). \quad (5.5)$$

Let Δ^g denote the Laplace operator on functions with respect to the degenerate metric g . Then

$$\Delta^g x = (\Delta^g f_0(x), \Delta^g f_N(x), \Delta^g f_1(x), \dots, \Delta^g f_{n+1}(x)). \quad (5.6)$$

But

$$\Delta^g f_A = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha). \quad (5.7)$$

According to the above remark on the signature and (2.1), we have here $\varepsilon_\alpha = 1$ for all α .

Using (4.7) yields

$$\begin{aligned} \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha) &= X_\alpha \cdot \tilde{g}(U_A, X_\alpha) - \tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha) \\ &\quad - \eta(U_A)(B(X_\alpha, PX_\alpha) - C(X_\alpha, PX_\alpha)) \\ &\quad - \eta(X_\alpha)(B(X_\alpha, PU_A) - C(X_\alpha, PU_A)) \\ &\quad - 2\tau(X_\alpha)\eta(X_\alpha)\eta(U_A). \end{aligned} \quad (5.8)$$

In this section, we assume $S(TM)$ parallel. From Theorem 2.3 and (2.11), we have $C = 0$ and $A_N = 0$ on M . Moreover, we can write, without loss of generality, $\nabla_X X_i = 0$ for all $X \in \Gamma(TM)$, where the X_i 's are vectors of a local basis of $\Gamma(S(TM))$. Finally, as $\tau = 0$, we have

$$\begin{aligned} \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha) &= X_\alpha \cdot \tilde{g}(U_A, X_\alpha) - \tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha) \\ &\quad - \eta(U_A)B(X_\alpha, PX_\alpha) - \eta(X_\alpha)B(X_\alpha, PU_A), \quad \alpha = 0, \dots, n. \end{aligned} \quad (5.9)$$

But

$$\begin{aligned} X_\alpha \cdot \tilde{g}(U_A, X_\alpha) &= X_\alpha \cdot \{\tilde{g}(g^{[\varepsilon_Y]} f_{A;\varepsilon} X_Y, X_\alpha)\} \\ &= X_\alpha \cdot \{f_{A;\varepsilon} g^{[\varepsilon_Y]} \tilde{g}_{Y\alpha}\} = X_\alpha \cdot \{\delta_{\alpha\varepsilon}^\varepsilon f_{A;\varepsilon}\}. \end{aligned} \quad (5.10)$$

Hence

$$X_\alpha \cdot \tilde{g}(U_A, X_\alpha) = X_\alpha \cdot \{f_{A;\alpha}\} =: f_{A;\alpha\alpha}. \quad (5.11)$$

Now we compute $f_{A;\alpha}$ and $f_{A;\alpha\alpha}$ for $A = 0, \dots, n+1$ and $\alpha = 0, \dots, n$:

$$f_0(x) = \langle x, N \rangle \implies f_{0;\alpha} = X_\alpha \cdot \langle x, N \rangle = \langle \bar{\nabla}_{X_\alpha} x, N \rangle + \langle x, \bar{\nabla}_{X_\alpha} N \rangle. \quad (5.12)$$

Note that as $\tau = 0$ and $S(TM)$ is parallel, N is a parallel vector field. Indeed, $\bar{\nabla}_X N = -A_N X + \tau(X)N$. It follows from (4.12) and Theorem 2.3 that

$$\bar{\nabla}_X N = 0 \quad \forall X \in \Gamma(TM). \quad (5.13)$$

Hence

$$\begin{aligned} f_{0;\alpha} &= \langle X_\alpha, N \rangle, \\ f_{0;\alpha\alpha} &= X_\alpha \cdot \langle X_\alpha, N \rangle \\ &= \langle \bar{\nabla}_{X_\alpha} X_\alpha, N \rangle + \langle X_\alpha, \bar{\nabla}_{X_\alpha} N \rangle \\ &= \langle \nabla_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha)N, N \rangle. \end{aligned} \quad (5.14)$$

Thus

$$\begin{aligned} f_{0;\alpha\alpha} &= \langle \nabla_{X_\alpha} X_\alpha, N \rangle, \\ f_{n+1}(\mathcal{X}) &= \langle \mathcal{X}, \xi \rangle \Rightarrow f_{n+1;\alpha} = \langle X_\alpha, \xi \rangle + \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \xi \rangle. \end{aligned} \quad (5.15)$$

Therefore,

$$\begin{aligned} f_{n+1;\alpha} &= \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \xi \rangle, \\ f_{n+1;\alpha\alpha} &= X_\alpha \cdot \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \xi \rangle \\ &= \langle X_\alpha, \bar{\nabla}_{X_\alpha} \xi \rangle + \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle \\ &= -B(X_\alpha, X_\alpha) + \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle. \end{aligned} \quad (5.16)$$

But a straightforward computation leads to

$$\bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi = -\nabla_{X_\alpha}^* \bar{A}_\xi X_\alpha + B(\nabla_{X_\alpha} \xi, X_\alpha) N. \quad (5.17)$$

We also obtain

$$\begin{aligned} \langle -\nabla_{X_\alpha}^* \bar{A}_\xi X_\alpha, X_i \rangle &= -X_\alpha \cdot \langle \bar{A}_\xi X_\alpha, X_i \rangle + B(X_\alpha, \bar{\nabla}_{X_\alpha} X_i) \\ &= -X_\alpha \cdot B(X_\alpha, X_i), \end{aligned} \quad (5.18)$$

where we have used

$$\bar{\nabla}_{X_\alpha}^* X_i = \nabla_{X_\alpha} X_i - C(X_\alpha, X_i) = \nabla_{X_\alpha} X_i = 0. \quad (5.19)$$

Therefore,

$$\begin{aligned} \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle &= B(\nabla_{X_\alpha} \xi, X_\alpha) \langle \mathcal{X}, N \rangle - \sum_{i=1}^n \langle \mathcal{X}, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i) \\ &= -B(\bar{A}_\xi X_\alpha, X_\alpha) \langle \mathcal{X}, N \rangle - \sum_{i=1}^n \langle \mathcal{X}, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i), \end{aligned} \quad (5.20)$$

so

$$\begin{aligned} f_{n+1;\alpha\alpha} &= -B(X_\alpha, X_\alpha) - B(\bar{A}_\xi X_\alpha, X_\alpha) \langle \mathcal{X}, N \rangle \\ &\quad - \sum_{i=1}^n \langle \mathcal{X}, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i), \\ f_k(\mathcal{X}) &= \langle \mathcal{X}, X_k \rangle \Rightarrow f_{k;\alpha} = \langle X_\alpha, X_k \rangle + \langle \mathcal{X}, \bar{\nabla}_{X_\alpha} X_k \rangle, \\ f_{k;\alpha} &= \langle X_\alpha, X_k \rangle + \langle \mathcal{X}, B(X_\alpha, X_k) N \rangle, \\ f_{k;\alpha\alpha} &= \langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle + \langle X_\alpha, \bar{\nabla}_{X_\alpha} X_k \rangle \\ &\quad + X_\alpha \cdot B(X_\alpha, X_k) \langle \mathcal{X}, N \rangle + B(X_\alpha, X_k) \langle X_\alpha, N \rangle. \end{aligned} \quad (5.21)$$

Moreover,

$$\langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle = \langle \nabla_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha)N, X_k \rangle = \langle \nabla_{X_\alpha} X_\alpha, X_k \rangle. \quad (5.22)$$

But $\nabla_{X_\alpha} X_\alpha = 0$ for all α . Indeed, if $\alpha = 0$, from (2.14) and (2.17), $\nabla_\xi \xi = 0$. For $\alpha = i \neq 0$, $\nabla_{X_i} X_i = 0$ from the condition of parallel screen distribution. So $\langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle = 0$. We also have that

$$\langle X_\alpha, \bar{\nabla}_{X_\alpha} X_k \rangle = 0, \quad B(X_\alpha, X_k) \langle X_\alpha, N \rangle = 0 \quad (5.23)$$

for at least one of $B(X_\alpha, X_k)$ or $\langle X_\alpha, N \rangle$ being zero according to either $\alpha = 0$ or $\alpha \neq 0$, so

$$f_{k;\alpha\alpha} = [X_\alpha \cdot B(X_\alpha, X_k)] \langle x, N \rangle. \quad (5.24)$$

From $\nabla_{X_\alpha} X_\alpha = 0$, the term $\tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha)$ in expression (5.9) is zero.

Next we compute the term $\tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha)$ in the left-hand side of (5.9) for $A = 0, \dots, n+1$ and $\alpha = 0, \dots, n$:

$$\tilde{g}(\nabla_\xi U_0, \xi) = f_{0;00} - \tilde{g}(U_0, \nabla_\xi \xi) - \eta(U_0 B(\xi, P\xi)) - \eta(\xi) B(\xi, P U_0) = 0. \quad (5.25)$$

Hence

$$\tilde{g}(\nabla_\xi U_0, \xi) = 0. \quad (5.26)$$

Similar computations give the other terms as follows:

$$\begin{aligned} \tilde{g}(\nabla_{X_k} U_0, X_k) &= -B(X_k, X_k), & \tilde{g}(\nabla_\xi U_k, \xi) &= 0, \\ \tilde{g}(\nabla_{X_l} U_k, X_l) &= [X_l \cdot B(X_l, X_k)] \langle x, N \rangle, & \tilde{g}(\nabla_\xi U_{n+1}, \xi) &= 0, \\ \tilde{g}(\nabla_{X_l} U_{n+1}, X_l) &= -B(X_l, X_l) - B(A_\xi^* X_l, X_l) \langle x, N \rangle \\ &\quad - \sum_{i=1}^n \langle x, X_i \rangle X_l \cdot B(X_l, X_i). \end{aligned} \quad (5.27)$$

We will make use of the following lemma.

LEMMA 5.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\mathbb{R}_1^{n+2}$. Set $B_{jk;i} := X_i \cdot B(X_j, X_k)$. If $S(TM)$ is parallel and $\tau = 0$, then $B_{jk;i}$ is symmetric with respect to the indices i, j , and k .*

PROOF. Clearly, $B_{jk;i}$ is symmetric with respect to j and k . It suffices to show its symmetry with respect to i and j . First,

$$(\nabla_{X_i} B)(X_j, X_k) = X_i \cdot B(X_j, X_k) - B(\nabla_{X_i} X_j, X_k) - B(X_j, \nabla_{X_i} X_k). \quad (5.28)$$

Since $S(TM)$ is parallel, the two last terms of this equality are zero. Hence, for all $i, j, k = 1, \dots, n$,

$$(\nabla_{X_i} B)(X_j, X_k) = X_i \cdot B(X_j, X_k). \quad (5.29)$$

Next, Gauss-Codazzi equation gives

$$\begin{aligned}\tilde{\mathcal{G}}(\tilde{R}(X_i, X_j)X_k, \xi) &= (\nabla_{X_i}B)(X_j, X_k) - (\nabla_{X_j}B)(X_i, X_k) \\ &\quad + B(X_j, X_k)\tau(X_i) - B(X_i, X_k)\tau(X_j).\end{aligned}\quad (5.30)$$

The ambient space is $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$, hence $\tilde{R} = 0$. As $\tau = 0$, we get

$$(\nabla_{X_i}B)(X_j, X_k) = (\nabla_{X_j}B)(X_i, X_k), \quad (5.31)$$

that is,

$$X_i \cdot B(X_j, X_k) = X_j \cdot B(X_i, X_k), \quad (5.32)$$

which shows that $B_{jk;i}$ is symmetric with respect to i and j . This achieves the proof. $\square$

Now we are in a position to compute each term $\Delta^\theta f_A$ in (5.7) for $A = 0, \dots, n+1$. We have

$$\begin{aligned}\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{\mathcal{G}}(\nabla_{X_\alpha} U_0, X_\alpha) &= \tilde{\mathcal{G}}(\nabla_\xi U_0, \xi) + \sum_{k=1}^n \varepsilon_k \tilde{\mathcal{G}}(\nabla_{X_k} U_0, X_k) \\ &= - \sum_{k=1}^n B(X_k, X_k).\end{aligned}\quad (5.33)$$

Hence, using (2.15), we infer that

$$\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{\mathcal{G}}(\nabla_{X_\alpha} U_0, X_\alpha) = - \sum_{\alpha=0}^n B(X_\alpha, X_\alpha). \quad (5.34)$$

We also compute similar sums for U_{n+1} and U_k :

$$\begin{aligned}\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{\mathcal{G}}(\nabla_{X_\alpha} U_{n+1}, X_\alpha) &= \tilde{\mathcal{G}}(\nabla_\xi U_{n+1}, \xi) + \sum_{k=1}^n \varepsilon_k \tilde{\mathcal{G}}(\nabla_{X_k} U_{n+1}, X_k) \\ &= - \sum_{k=1}^n B(X_k, B(X_k)) - \left[\sum_{k=1}^n B(A_\xi^* X_k, X_k) \right] \langle x, N \rangle \\ &\quad - \sum_{k=1}^n \sum_{i=1}^n \langle x, X_i \rangle X_k \cdot B(X_k, X_i).\end{aligned}\quad (5.35)$$

By using Lemma 5.1 and (2.15), we obtain

$$\begin{aligned}\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{\mathcal{G}}(\nabla_{X_\alpha} U_{n+1}, X_\alpha) &= - \sum_{k=1}^n B(X_k, X_k) - \left[\sum_{k=1}^n B(A_\xi^* X_k, X_k) \right] \langle x, N \rangle \\ &\quad - \sum_{i=1}^n \left[X_i \cdot \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] \langle x, X_i \rangle.\end{aligned}\quad (5.36)$$

Finally,

$$\begin{aligned} \sum_{\alpha=0}^n \varepsilon_{\alpha} \tilde{g}(\nabla_{X_{\alpha}} U_k, X_{\alpha}) &= \tilde{g}(\nabla_{\xi} U_k, \xi) + \sum_{l=1}^n \varepsilon_l \tilde{g}(\nabla_{X_l} U_l, X_k) \\ &= \sum_{l=1}^n X_l \cdot B(X_l, X_k) \langle x, N \rangle, \end{aligned} \quad (5.37)$$

and using [Lemma 5.1](#) and (2.15), we also obtain

$$\sum_{\alpha=0}^n \varepsilon_{\alpha} \tilde{g}(\nabla_{X_{\alpha}} U_k, X_{\alpha}) = \left[X_k \cdot \sum_{\alpha=0}^n B(X_{\alpha}, X_{\alpha}) \right] \langle x, N \rangle. \quad (5.38)$$

Put

$$H = \frac{1}{n+1} \left[\sum_{\alpha=0}^n B(X_{\alpha}, X_{\alpha}) \right] (\xi + N) \quad (5.39)$$

to obtain

$$\begin{aligned} \Delta^g x &= (n+1)H - \sum_{k=1}^n \langle x, N \rangle X_k \cdot \langle (n+1)H, N \rangle X_k \\ &\quad + \left\{ \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) \langle x, N \rangle - \sum_{i=1}^n X_i \cdot \sum_{\alpha=0}^n B(X_{\alpha}, X_{\alpha}) \langle x, X_i \rangle \right\} N. \end{aligned} \quad (5.40)$$

If we now put

$$K = \left[\sum_{\alpha=0}^n B(X_{\alpha}, X_{\alpha}) \right], \quad (5.41)$$

then

$$\begin{aligned} \Delta^g x &= (n+1)H - \sum_{k=1}^n (\langle x, N \rangle X_k \cdot K) X_k \\ &\quad + \left\{ \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) \langle x, N \rangle - \sum_{k=1}^n \langle x, X_k \rangle X_k \cdot K \right\} N. \end{aligned} \quad (5.42)$$

Thus, we have established the following theorem.

THEOREM 5.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of the Lorentzian space $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$, endowed with a parallel screen distribution $S(TM)$ and with symmetric Ricci tensor. Then its Laplacian with respect to the degenerate*

metric g is given by

$$\begin{aligned} \Delta^g x = (n+1)H - \sum_{k=1}^n (\langle x, N \rangle X_k \cdot K) X_k \\ + \left\{ \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) \langle x, N \rangle - \sum_{k=1}^n \langle x, X_k \rangle X_k \cdot K \right\} N. \end{aligned} \quad (5.43)$$

DEFINITION 5.3. On the lightlike hypersurface $(M, g, S(TM))$, the smooth function σ defined by

$$\sigma = \frac{1}{\sqrt{2}(n+1)} K, \quad K = \sum_{\alpha}^n B(X_{\alpha}, X_{\alpha}), \quad (5.44)$$

is called the mean curvature function of M .

6. An application. It is well known that minimal immersions of smooth manifolds M into Euclidean spheres are those immersions whose coordinate functions in the ambient Euclidian space are eigenfunctions of the Laplacian in the induced metric with eigenvalue $\lambda = -\dim(M)$. Then the following general problem comes out in a natural way: can we classify lightlike submanifolds by means of some Laplacian differential equation involving the isometric immersion?

Our purpose in this section is to study lightlike hypersurfaces satisfying the eigenvalue equation $\Delta^g x = \lambda x$, where λ is a constant and x the position vector field. Our result here stands as follows.

THEOREM 6.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of the Lorentzian space $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$, endowed with a parallel screen distribution $S(TM)$ and with symmetric Ricci tensor. If $(M, g, S(TM))$ satisfies the eigenvalue equation $\Delta^g x = \lambda x$, then*

- (1) $\lambda = 0$;
- (2) the mean curvature function σ vanishes identically on M ;
- (3) the shape operator A_{ξ}^* of the screen distribution is trace-free with respect to the second fundamental form B of M , that is,

$$\sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) = 0. \quad (6.1)$$

PROOF. If M is an eigenhypersurface of Δ^g with eigenvalue λ , then we have, for all $x \in M$,

$$\lambda \langle x, N \rangle = K, \quad (6.2)$$

$$\lambda \langle x, \xi \rangle = K + \langle x, N \rangle \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) - \sum_{k=1}^n \langle x, X_k \rangle (X_k \cdot K), \quad (6.3)$$

$$\lambda \langle x, X_k \rangle = \langle x, N \rangle X_k \cdot K. \quad (6.4)$$

Equation (6.2) leads to

$$\xi \cdot K = \lambda \langle \xi, N \rangle = \lambda, \quad (6.5)$$

but we have

$$\begin{aligned} \xi \cdot K &= \xi \cdot \left[\sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] = \sum_{k=1}^n \xi \cdot B(X_k, X_k) \\ &= \sum_{k=1}^n \{ (\nabla_\xi B)(X_k, X_k) + 2B(\nabla_\xi X_k, X_k) \} \\ &= \sum_{k=1}^n \{ (\nabla_\xi B)(X_k, X_k) \}. \end{aligned} \quad (6.6)$$

From Gauss-Codazzi equation, we have

$$(\nabla_\xi B)(X_k, X_k) = (\nabla_{X_k} B)(\xi, X_k). \quad (6.7)$$

This leads to

$$(\nabla_\xi B)(X_k, X_k) = B(A_\xi^* X_k, X_k), \quad (6.8)$$

and using this and (2.15), we deduce that

$$\xi \cdot K = \lambda = \sum_{\alpha=0}^n B(A_\xi^* X_\alpha, X_\alpha). \quad (6.9)$$

Then, if $\lambda = 0$, the first claim in Theorem 6.1 is obvious. The second one comes from (6.2) and Definition 5.3 of σ . Finally, the third assertion follows from relation (6.9).

Now assume $\lambda \neq 0$. The differentiation of equation (6.2) with respect to X_k , $k = 1, \dots, n$, leads to

$$X_k \cdot K = \lambda \langle X_k, N \rangle = 0, \quad (6.10)$$

where we use $\bar{\nabla}_X N = 0$ for all $X \in \Gamma(TM)$ and (2.1). Hence we get

$$X_k \cdot K = 0, \quad k = 1, \dots, n. \quad (6.11)$$

Substitute (6.11) in (6.4) and $\lambda \neq 0$ to get

$$\langle x, X_k \rangle = 0, \quad k = 1, \dots, n. \quad (6.12)$$

Hence, there exist two smooth functions r and l such that the position vector field along M is given by

$$x = r(x)\xi + l(x)N. \quad (6.13)$$

Using (6.11), (6.9), and (6.2) in (6.3) leads to

$$K + K - 0 = \lambda \langle x, \xi \rangle = \lambda l(x). \quad (6.14)$$

Hence $l(x) = 2K/\lambda$. On the other hand, we get from (6.2) that $K = \lambda r(x)$ so that $r(x) = K/\lambda$. Thus the position vector field x along M is given by

$$x = \frac{K}{\lambda}(\xi + 2N). \quad (6.15)$$

Therefore, we obtain

$$\xi = \bar{\nabla}_\xi x = \frac{\xi \cdot K}{\lambda}(\xi + 2N) + \frac{K}{\lambda}(\bar{\nabla}_\xi \xi + \bar{\nabla}_\xi N). \quad (6.16)$$

The last term in this equality is zero ($\bar{\nabla}_\xi \xi = 0$, $\bar{\nabla}_\xi N = 0$). Then use (6.5) to conclude that $\xi = \xi + 2N$, that is, $N \equiv 0$, which is a contradiction. Hence $\lambda = 0$, and the proof is complete. $\square$

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