SOME CRITERIA FOR UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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We derive some criteria for univalence of certain integral operators for analytic functions in the open unit disk.

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1. Introduction. Let $\mathcal{A}$ be the class of the functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Miller and Mocanu [1] have considered many integral operators for functions $f(z)$ belonging to the class $\mathcal{A}$. In this paper, we consider the integral operators

$$F_{\alpha}(z) = \left\{ \frac{1}{\alpha} \int_0^z (f(u))^{1/\alpha} u^{-1} du \right\}^{\alpha} \quad (z \in \mathbb{U}) \quad (1.1)$$

for $f(z) \in \mathcal{A}$ and for some $\alpha \in \mathbb{C}$. It is well known that $F_{\alpha}(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathbb{U}$.

2. Preliminary results. To discuss our integral operators, we need the following theorems.

THEOREM 2.1 [3]. *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad (2.1)$$

for all $z \in \mathbb{U}$, then the integral operator

$$G_{\alpha}(z) = \left\{ \alpha \int_0^z u^{\alpha-1} f'(u) du \right\}^{1/\alpha} \quad (2.2)$$

is in the class $\mathcal{S}$.

THEOREM 2.2 [4]. Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies (2.1) for all $z \in \mathbb{U}$, then, for any complex number β with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$, the integral operator

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{1/\beta} \quad (2.3)$$

is in the class $\mathcal{S}$.

EXAMPLE 2.3. Defining the function $f(z)$ by

$$f(z) = \int_0^z \left(\frac{1+u^{\operatorname{Re}(\alpha)}}{1-u^{\operatorname{Re}(\alpha)}} \right)^{1/2} du \quad (2.4)$$

with $\operatorname{Re}(\alpha) > 0$, we have that

$$\frac{1-z^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left(\frac{zf''(z)}{f'(z)} \right) = z^{\operatorname{Re}(\alpha)}. \quad (2.5)$$

Thus the function $f(z)$ satisfies the condition of Theorem 2.2. Therefore, for $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$,

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{1+u^{\operatorname{Re}(\alpha)}}{1-u^{\operatorname{Re}(\alpha)}} \right)^{1/2} du \right\}^{1/\beta} \quad (2.6)$$

is in the class $\mathcal{S}$.

THEOREM 2.4 [2]. If the function $g(z)$ is regular in $\mathbb{U}$, then, for all $\xi \in \mathbb{U}$ and $z \in \mathbb{U}$, $g(z)$ satisfies

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right|, \quad (2.7)$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}. \quad (2.8)$$

The equalities hold only in the case $g(z) = \varepsilon((z+u)/(1+\overline{u}z))$, where $|\varepsilon| = 1$ and $|u| < 1$.

REMARK 2.5 [2]. For $z = 0$, from inequality (2.7),

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \quad (2.9)$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \quad (2.10)$$

Considering $g(0) = a$ and $\xi = z$, we see that

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \quad (2.11)$$

for all $z \in \mathbb{U}$.

SCHWARZ LEMMA [2]. *If the function $g(z)$ is regular in $\mathbb{U}$, $g(0) = 0$, and $|g(z)| \leq 1$, for all $z \in \mathbb{U}$, then*

$$|g(z)| \leq |z| \quad (2.12)$$

for all $z \in \mathbb{U}$, and $|g'(0)| \leq 1$. The equality in (2.12) for $z \neq 0$ holds only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main results

THEOREM 3.1. *Let α be a complex number with $\operatorname{Re}(1/\alpha) = a > 0$ and the function $g(z) \in \mathcal{A}$ satisfying*

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.1)$$

Then, for

$$|\alpha| \geq \frac{2}{(2a+1)^{(2a+1)/2a}}, \quad (3.2)$$

the integral operator

$$F_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (g(u))^{1/\alpha} u^{-1} du \right\}^\alpha \quad (3.3)$$

is in the class $\mathcal{S}$.

PROOF. Let $1/\alpha = \beta$. Then we have

$$F_{1/\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u} \right)^\beta du \right\}^{1/\beta}. \quad (3.4)$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^\beta du. \quad (3.5)$$

Then the function

$$h(z) = \left(\frac{1}{|\beta|} \right) \frac{zf''(z)}{f'(z)} \quad (3.6)$$

is regular in $\mathbb{U}$ and the constant $|\beta|$ satisfies the inequality

$$|\beta| \leq \frac{(2a+1)^{(2a+1)/2a}}{2}. \quad (3.7)$$

From (3.5) and (3.6), we have that

$$h(z) = \frac{\beta}{|\beta|} \left(\frac{zg'(z)}{g(z)} - 1 \right). \quad (3.8)$$

Using (3.8) and (3.1), we obtain

$$|h(z)| \leq 1 \quad (z \in \mathbb{U}). \quad (3.9)$$

Noting that $h(0) = 0$ and applying the Schwarz lemma for $h(z)$, we get

$$\frac{1}{|\beta|} \left| \frac{zf''(z)}{f'(z)} \right| \leq |z| \quad (z \in \mathbb{U}) \quad (3.10)$$

and hence we obtain

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\beta| \left(\frac{1 - |z|^{2a}}{a} \right) |z| \quad (z \in \mathbb{U}). \quad (3.11)$$

Because

$$\max_{|z| \leq 1} \left(\frac{1 - |z|^{2a}}{a} |z| \right) = \frac{2}{(2a+1)^{(2a+1)/2a}}, \quad (3.12)$$

from (3.11) and (3.7), we have

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (3.13)$$

for $z \in \mathbb{U}$. From (3.13) and Theorem 2.1, it follows that (2.4) belongs to the class $\mathcal{G}$.

By means of (2.4) and (3.5), we have that the integral operator $F_{1/\beta}(z)$ is in the class $\mathcal{G}$, and hence we conclude that the integral operator $F_\alpha(z)$ is in the class $\mathcal{G}$. $\square$

EXAMPLE 3.2. If we take the function $g(z) = ze^z$ and $\alpha = 1/a > 0$, then

$$g(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad (3.14)$$

is analytic in $\mathbb{U}$ and

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| = |z| < 1 \quad (z \in \mathbb{U}). \quad (3.15)$$

Since the function $g(z)$ satisfies the condition of Theorem 3.1, we have

$$T_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z e^{u/\alpha} u^{1/\alpha-1} du \right\}^\alpha \in \mathcal{G}. \quad (3.16)$$

THEOREM 3.3. Let α, β be complex numbers with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha) > 0$ and the function $g(z) \in \mathcal{A}$ satisfying

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.17)$$

Then, for

$$|\alpha| \geq \max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\}, \quad (3.18)$$

the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z (g(u))^{1/\alpha} u^{\beta-1/\alpha-1} du \right\}^{1/\beta} \quad (3.19)$$

is in the class $\mathcal{S}$.

PROOF. We have

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u} \right)^{1/\alpha} du \right\}^{1/\beta}. \quad (3.20)$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{1/\alpha} du, \quad (3.21)$$

which is regular in $\mathbb{U}$. The function

$$p(z) = |\alpha| \frac{f''(z)}{f'(z)}, \quad (3.22)$$

where the constant $|\alpha|$ satisfies inequality (3.18), is regular in $\mathbb{U}$. From (3.22) and (3.21), we obtain

$$p(z) = \frac{|\alpha|}{\alpha} \left\{ \frac{zg'(z) - g(z)}{zg(z)} \right\} \quad (3.23)$$

and using (3.17), we have

$$|p(z)| < 1 \quad (z \in \mathbb{U}) \quad (3.24)$$

and $|p(0)| = |a_2|$. Applying Remark 2.5, we obtain

$$\left| \alpha \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + |a_2|}{1 + |a_2||z|} \quad (z \in \mathbb{U}). \quad (3.25)$$

It follows that

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq \left(\frac{1}{|\alpha|} \right) \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \quad (3.26)$$

for all $z \in \mathbb{U}$. We consider the function

$$Q(x) = \left(\frac{1 - x^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) x \left(\frac{x + |a_2|}{1 + |a_2|x} \right) \quad (x = |z|; x \in [0, 1]). \quad (3.27)$$

Because $Q(1/2) > 0$, $Q(x)$ satisfies

$$\max_{x \in [0,1]} Q(x) > 0. \quad (3.28)$$

Using this fact, (3.26) gives us that

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| z \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{|\alpha|} \max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\}. \quad (3.29)$$

From (3.29) and (3.18), we obtain (2.1). Using (2.1) and Theorem 2.2, we obtain that the integral operator (2.4) belongs to the class $\mathcal{G}$. Therefore, it follows from (2.4) and (3.21) that $F_{\alpha,\beta}(z)$ is in the class $\mathcal{G}$. $\square$

COROLLARY 3.4. *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and the function $g(z) \in \mathcal{A}$ satisfying (3.18). Then, for*

$$\max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\} \leq |\alpha| \leq 1, \quad (3.30)$$

the integral operator (3.3) is in the class $\mathcal{G}$.

PROOF. From Theorem 3.3 for $\beta = 1/\alpha$, the condition $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha) > 0$ is identical with $|\alpha| < 1$ and we have $F_{\alpha,\beta}(z) = F_{\alpha}(z)$. $\square$

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